Hochschild homology, cyclic homology, and de Rham cohomology

Setup: k is a commutative ring, A is an associative k-algebra. Assume that A is flat over k.

The Hochschild homology of A/k, denoted $\mathrm{HH}(A/k)$ is defined by the following diagram:

$$A \otimes_k A \otimes_k A \Rrightarrow A \otimes_k A \rightrightarrows A;$$

(the triple arrows are $a \otimes b \otimes c \mapsto ab \otimes c, a \otimes bc, ca \otimes b$; the double arrows are $a \otimes b \mapsto ab, ba$); this is a cyclic object in k-modules. This means roughly that there is a $\mathbb{Z}/n\mathbb{Z}$-action on the n-th term (cycling the factors), plus degeneracy maps which do *not* commute with the cyclic actions (e.g., $a \mapsto a \otimes 1$ or $a \mapsto 1 \otimes a$), but satisfying various compatibilities.

One can extract from this a simplicial object, thus a complex by Dold-Kan: $\mathrm{HH}(A/k) \in D(k)$ with extra structure.

$$\cdots \to A \otimes_k A \otimes_k A \to A \otimes_k A \to A$$

$$a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc + c \otimes ab, \qquad a \otimes b \mapsto ab - ba.$$

Two parts:

(1) discuss combinatorial index categories (simplicial, cyclic) and construct $\mathrm{HH}(A/k)$ as a cyclic object;
(2) explain what you get from a cyclic object.

Let Δ be the category of nonempty finite totally ordered sets with nondecreasing maps. The objects $[n]_\Delta = \{0, 1, \dots, n\} \in \Delta$ represent all isomorphism classes. A *simplicial object* in a category C is a functor $\Delta^{\mathrm{op}} \to C$.

Let Λ_∞ be the category of totally ordered nonempty sets S plus a strictly increasing automorphism $\sigma : S \to S$ such that $S/\sigma^{\mathbb{Z}}$ is finite (and *archimedean*: for any $x, y \in S$ there is an integer n such that $\sigma^n(x) \geq y$). Morphisms are nondecreasing maps commuting with σ. Up to isomorphism, every object has the form $[n]_{\Lambda_\infty} = \frac{1}{n}\mathbb{Z}$ with $\sigma : x \mapsto x + 1$. Also,

$$\mathrm{Hom}_{\Lambda_\infty}([m]_{\Lambda_\infty}, [n]_{\Lambda_\infty}) = \{f : \tfrac{1}{m}\mathbb{Z} \to \tfrac{1}{n}\mathbb{Z} \text{ such that } f(x+1) = f(x) + 1\}.$$

A *paracyclic object* is a functor $\Lambda_\infty^{\mathrm{op}} \to \mathcal{C}$. Note that $\Lambda_\infty \cong \Lambda_\infty^{\mathrm{op}}$: fix objects, and if $f : [m]_{\Lambda_\infty} \to [n]_{\Lambda_\infty}$, take $f^\circ : [n]_{\Lambda_\infty} \to [m]_{\Lambda_\infty}$ where $f^\circ(x) = \min\{y : f(y) \geq x\}$.

Key lemma: The functor $\Delta^{\mathrm{op}} \to \Lambda_\infty^{\mathrm{op}}$ taking S to $\mathbb{Z} \times S$ ordered lexicographically (with σ translating on $\mathbb{Z}$) is cofinal (even as ∞-categories). In other words, for all paracyclic objects $F : \Lambda_\infty^{\mathrm{op}} \to C$, $\mathrm{colim}_{\Lambda_\infty^{\mathrm{op}}} F \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} F|_{\Delta^{\mathrm{op}}}$.

Corollary: the geometric realization (nerve) of $|\Lambda_\infty| \simeq |\Delta| \simeq$ point is contractible.

Next, define Λ and cyclic objects. Note: there is a natural free action of $\mathbb{Z}$ on $\mathrm{Hom}_{\Lambda_\infty}(S, S')$ given by $f \mapsto f \circ \sigma_S = \sigma_{S'} \circ f$. Equivalently, there is an action of the classifying stack $\mathrm{B}\mathbb{Z}$ on Λ_∞ (and the self-duality is $\mathrm{B}\mathbb{Z}$-equivariant).

Define $\Lambda := \Lambda_\infty/\mathrm{B}\mathbb{Z}$. That is, objects are as before, but morphism sets are quotiented by the $\mathbb{Z}$-action. (This is also a quotient in ∞-categories.) Morphism sets in Λ are finite. Example: $\mathrm{Aut}_\Lambda([n]_\Lambda) = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Corollary: $|\Lambda| \cong |\Lambda_\infty|/\mathrm{B}\mathbb{Z} \cong \mathrm{B}S^1 \cong \mathbb{CP}^\infty$.

A *cyclic object* is a functor $\Lambda^{\mathrm{op}} \to C$. Again, $\Lambda \cong \Lambda^{\mathrm{op}}$ because the previous duality is $\mathrm{B}\mathbb{Z}$-equivariant.

Recall: the *associative operad* is the category $\mathrm{Ass}^{\otimes}$ consisting of finite (possibly empty) sets S, morphisms $S' \to S$ being maps $f : S' \to S$ plus a total ordering on $f^{-1}(s)$ for each $s \in S$.

Note: a unital, associative k-algebra A gives a functor $A^{\otimes} : \mathrm{Ass}^{\otimes} \to k - mod$ taking S to $\otimes_{s \in S} A$ (empty set goes to k); this is functorial because of total orderings on preimages.

The construction: there is a natural functor $\Lambda \to \mathrm{Ass}^{\otimes}$ taking S to $S/\sigma^{\mathbb{Z}}$. We now define

$$\mathrm{HH}(A/k) : \Lambda^{\mathrm{op}} \cong \Lambda \to \mathrm{Ass}^{\otimes} \xrightarrow{A^{\otimes}} k - mod.$$

Remark: there is a functor from Λ to the category of finite nonempty *cyclically ordered* sets, but this is not an equivalence. For example, $[n]_{\Lambda}$ goes to $0 < 1 < \cdots < n-1 < 0$; but $\mathrm{Hom}_{\Lambda}([n]_{\Lambda}, [1]_{\Lambda})$ has n-elements, whereas on the target there is only one morphism.

From cyclic objects to S^1-equivariant objects: Let $\mathcal{C}$ be any ∞-category (sense of Lurie), e.g., $\mathcal{D}(k)$ the derived ∞-category of k-modules. Then there is a natural functor

$$\mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{B}S^1, \mathcal{C})$$

(the target being "S^1-equivariant objects in $\mathcal{C}$"; viewing S^1 as a Kan complex, and hence an ∞-category). On the level of underlying objects, this is $\mathrm{colim}_{\Delta^{\mathrm{op}}} F|_{\Delta^{\mathrm{op}}}$. Construction: the object on the left is $\mathrm{Fun}^{\mathrm{B}\mathbb{Z}-\mathrm{equiv}}(\Lambda_{\infty}^{\mathrm{op}}, \mathcal{C})$. Taking $\mathrm{colim}_{\Lambda_{\infty}^{\mathrm{op}}}$, get

$$\mathrm{Fun}^{\mathrm{B}\mathbb{Z}-\mathrm{equiv}}(pt, \mathcal{C}) = \mathrm{Fun}^{S^1-\mathrm{equiv}}(pt, \mathcal{C}) = \mathrm{Fun}(\mathrm{B}S^1, \mathcal{C}).$$

In this way, regard $\mathrm{HH}(A/k)$ as an S^1-equivariant object in $\mathcal{D}(k)$.

Some concrete structures that this induces:

$$X \in \mathcal{D}(k)^{S^1} = \mathrm{Fun}(\mathrm{B}S^1, \mathcal{D}(k)) = \mathrm{Fun}(\mathbb{CP}^{\infty}, \mathcal{D}(k))$$

This maps to $\mathrm{Fun}(S^2, \mathcal{D}(k))$; i.e., $Y \in \mathcal{D}(k)$ plus map $S^1 \to \mathrm{End}(Y)$; i.e., Y plus $Y \to Y[-1]$.

On homology, get maps $\mathrm{HH}_i(A/k) \xrightarrow{d} \mathrm{HH}_{i+1}(A/k)$. ("Connes differential", often called B). Truncating less gives more structures...

Theorem (Hochschild–Kostant–Rosenberg, Connes): if A is smooth commutative k-algebra, can identify $\mathrm{HH}_i(A/k)$ with $\Omega^i_{A/k}$ such that the extra differential corresponds to the exterior differential on forms.

Define cyclic homology $\mathrm{HC}(A/k) = \mathrm{HH}(A/k)_{hS^1}$ meaning $\mathrm{colim}_{\mathrm{B}S^1} \mathrm{HH}(A/k)$, as an object of $\mathcal{D}(k)$. Also negative cyclic homology $\mathrm{HC}^-(A/k) := \mathrm{HH}(A/k)^{hS^1}$ meaning $\lim_{\mathrm{B}S^1} \mathrm{HH}(A/k)$, again as an object of $\mathcal{D}(k)$. In the smooth algebra case, get an E_2-spectral sequence

(not drawn here)

and this resembles de Rham cohomology.