Hochschild homology, cyclic homology, and the relationship to de Rham cohomology

Setup: k commutative ring

A associative k-algebra, unital

Assume: A is flat$/k$.

Hochschild homology of A/k defined by the following diagram

$$\begin{array}{ccccc}
\circlearrowright \mathbb{Z}/3\mathbb{Z} & & \circlearrowright \mathbb{Z}/2\mathbb{Z} \quad 1\otimes a \leftarrow\!\!\shortmid a & & \\
\cdots A \otimes_k A \otimes_k A & \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} & A \otimes_k A & \begin{array}{c}\longrightarrow\\ \longleftarrow \\ \longrightarrow\end{array} & A \\
& & & a\otimes 1 \leftarrow\!\!\shortmid a & \\
a\otimes b\otimes c & \longmapsto & ab\otimes c & & \\
& \longmapsto & a\otimes bc & & \\
& \longmapsto & ca\otimes b & & \\
& & a\otimes b & \longmapsto & ab \\
& & & \longmapsto & ba
\end{array}$$

$HH_k(A)$ or $HH(A/k)$

this will be a cyclic object in k-modules.

One can extract from this a simplicial object, thus a complex by Dold-Kan:

$$A \underset{k}{\otimes} A \underset{k}{\otimes} A \longrightarrow A \underset{k}{\otimes} A \longrightarrow A$$

$$a\otimes b \longmapsto ab - ba$$

$$a\otimes b\otimes c \longmapsto ab\otimes c - a\otimes bc + ca\otimes b$$

$\rightsquigarrow HH(A/k) \in D(k)$ with extra structure.

2 parts: 1) Discuss combinatorial index categories (simplicial, cyclic) + construct $HH(A/k)$ as a cyclic object.

2). Explain what you get from a cyclic object

1). <u>Def'n</u>: $\Delta = \left\{ \begin{array}{c} \text{non-empty finite totally ordered sets,} \\ \text{non-decreasing maps} \end{array} \right\}$

objects $[n]_\Delta = \{0, 1, \dots, n\} \in \Delta$, $n \geq 0$.

A <u>simplicial object</u> in category $\mathcal{C}$ is a functor:

$$\Delta^{op} \longrightarrow \mathcal{C}$$

<u>Def'n</u>: $\Lambda_\infty = \{$ nonempty totally ordered sets S + (strictly) increasing autom $\sigma : S \xrightarrow{\sim} S$

s.t. $S/_{\sigma^{\mathbb{Z}}}$ is finite.

& s.t. $S = \frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{Z}_{>0}$.

maps: nondecreasing maps commuting w σ $\}$

objects:

$$[n]_{\Lambda_\infty} = \tfrac{1}{n}\mathbb{Z} \in \Lambda_\infty$$

$\sigma: x \mapsto x+1$

$$\mathrm{Hom}_{\Lambda_\infty}([m]_{\Lambda_\infty}, [n]_{\Lambda_\infty}) = \left\{ \begin{array}{l} \text{non-decr. } f: \tfrac{1}{m}\mathbb{Z} \longrightarrow \tfrac{1}{n}\mathbb{Z} \\ \text{s.t. } f(x+1) = f(x)+1 \end{array} \right\}$$

A <u>paracyclic object</u> is a functor

$$\Lambda_\infty^{op} \longrightarrow \mathcal{C}.$$

<u>Remark</u>: Λ_∞ is self-dual $\Lambda_\infty \simeq \Lambda_\infty^{op}$

$$[n]_{\Lambda_\infty} \longmapsto [n]_{\Lambda_\infty}$$

$$f: [m]_{\Lambda_\infty} \longrightarrow [n]_{\Lambda_\infty}$$

$$\downarrow$$

$f^o : [n]_{\Lambda_\infty} \longrightarrow [m]_{\Lambda_\infty}$ def by $f^o(x) = \min\{y \mid f(y) \geq x\}$.

Key Lemma: The functor

$$\Delta^{op} \longrightarrow \Lambda_\infty^{op}$$

$$S \longmapsto \underset{\sigma : n \mapsto n+1}{\overset{\circlearrowleft}{\mathbb{Z}}} \times S \quad \text{lexicographic ordering}$$

$$([n]_\Delta \longrightarrow [n+1]_{\Lambda_\infty})$$

is cofinal (even as ∞-categories).

Equivalently: $\forall$ paracyclic objects $F : \Lambda_\infty^{op} \to C$

$$\operatorname*{colim}_{\Lambda_\infty^{op}} F \simeq \operatorname*{colim}_{\Delta^{op}} F|_{\Delta^{op}}$$

(true for C category but even if C is an ∞-category)

Cor.: The geometric realization (= nerve)

$$|\Lambda_\infty| \simeq |\Delta| \simeq pt \text{ is contractible.}$$

Next, define Λ, cyclic objects

Note: There is a natural free action of $\mathbb{Z}$ on $\mathrm{Hom}_{\Lambda_\infty}(S, S') \ \forall S, S' \in \Lambda_\infty$

given by $f \longmapsto f \circ \sigma_S = \sigma_{S'} \circ f$.

Equiv, there is an action of $B\mathbb{Z}$ (classifying category) on Λ_∞.

(self-duality is $B\mathbb{Z}$-equivariant)

Defn: $\Lambda = \Lambda_\infty / B\mathbb{Z}$

Equiv, obj. $[n]_\Lambda \in \Lambda$, $n \geq 1$

$$\mathrm{Hom}_\Lambda([m]_\Lambda, [n]_\Lambda) = \mathrm{Hom}_{\Lambda_\infty}([m]_{\Lambda_\infty}, [n]_{\Lambda_\infty})/\mathbb{Z}$$

is finite.

<u>Cor</u>: $|\Lambda| = |\Lambda_\infty| / \underbrace{B\mathbb{Z}}_{= S^1} \simeq BS^1 \simeq \mathbb{CP}^\infty$.

<u>Example</u>: $\mathrm{Aut}_\Lambda([n]_\Lambda) = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

<u>Def'n</u>: A <u>cyclic object</u> is a functor

$$\Lambda^{op} \longrightarrow C$$

<u>Rmk</u>: $\Lambda \simeq \Lambda^{op}$ (as $\Lambda_\infty \simeq \Lambda_\infty^{op}$ is $B\mathbb{Z}$-equivariant)

<u>Recall</u>: The <u>associative operad</u> is the category

$$Ass^\otimes = \Big\{ \text{finite (possibly empty) sets } S, \ \text{maps } \underbrace{\{ f : S' \longrightarrow S + \text{total ordering on } f^{-1}(s)\ \forall s \in S \}}_{\mathrm{Hom}(S', S)} \Big\}$$

<u>Note</u>: A unital, associative k-algebra A gives a functor:

$$A^\otimes : Ass^\otimes \longrightarrow k\text{-mod}$$
$$S' \longmapsto \bigotimes_{s \in S} A.$$

This is functorial because of the total orderings on preimages.

<u>Construction</u>: There is a natural functor

$$\Lambda_\infty / B\mathbb{Z} = \Lambda \longrightarrow Ass^\otimes$$
$$S \longmapsto S/\sigma^{\mathbb{Z}}$$

<u>Def'n</u>: $HH(A/k)$: $\Lambda^{op} \simeq \Lambda \rightarrow Ass^\otimes \xrightarrow{A^\otimes} k\text{-modules}$

<u>Remark</u>: There is a functor

$$\Delta \longrightarrow \{\text{finite non-empty } \underline{\text{cyclically}} \text{ ordered sets}\}$$

$$[n]_\Delta \longmapsto \{0 < 1 \ldots\ldots < n-1 < 0\}$$

This is <u>not</u> an equivalence.

$$\mathrm{Hom}_\Delta([n]_\Delta, [1]_\Delta) \quad \text{but} \quad \text{one element Hom set.}$$

has n elements

e.g.

$$\left(\begin{tikzcd} A \otimes_k A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[l, bend right, "2 \text{ elements}"'] \arrow[l, bend left] \end{tikzcd} \right)$$

<u>From cyclic objects to S^1-equiv objects.</u>

Let $\mathcal{C}$ be any ∞-category (e.g. $\mathcal{C} = \mathcal{D}(k)$)

<u>Construction</u>: There is a natural functor

$$\mathrm{Fun}(\Lambda^{op}, \mathcal{C}) \longrightarrow \mathrm{Fun}(BS^1, \mathcal{C})$$

"S^1-equiv objects in $\mathcal{C}$".

- underlying object $= \operatorname{colim}_{\Delta^{op}} F|_{\Delta^{op}}$

<u>Functor defined by</u>:

$$\mathrm{Fun}(\Lambda^{op}, \mathcal{C}) = \mathrm{Fun}^{B\mathbb{Z}\text{-equiv}}(\Lambda_\infty^{op}, \mathcal{C})$$

$$\Big\downarrow \operatorname{colim}_{\Lambda_\infty^{op}} \; (= \operatorname{colim} \Delta^{op})$$

$$\mathrm{Fun}(BS^1, \mathcal{C}) \qquad \mathrm{Fun}^{B\mathbb{Z}\text{-equiv}}(pt, \mathcal{C})$$

$$= \mathrm{Fun}^{S^1\text{-equiv}}(pt, \mathcal{C}) =$$

$$\rightsquigarrow HH^{\cdot}(A/k) \in \mathcal{D}(k)^{S^1} \qquad S^1\text{-equiv obj in } \mathcal{D}(k)$$

Some concrete structure this induces:

$X \in \mathcal{D}(k)^{S^1} = \mathrm{Fun}(BS^1, \mathcal{D}(k)) = \mathrm{Fun}(\mathbb{CP}^\infty, \mathcal{D}(k))$

$\downarrow$

$\{Y \in \mathcal{D}(k) + \underset{\text{map}}{} \; S^1 \to \mathrm{End}(Y) \;\; = \mathrm{Fun}(S^2, \mathcal{D}(k))$

$\mathrm{End}(Y) \cong \tau_{\geq 0} R\mathrm{Hom}(Y, Y)$ Dold-Kan

$= \{Y + \text{map } Y \to Y[-1]\}$

on homology, get maps

$$HH_i(A/k) \xrightarrow{d} HH_{i+1}(A/k)$$

"Connes's differential" (often called B)

<u>Theorem</u> (Hochschild-Kostant-Rosenberg, Connes)

If A smooth k-algebra,

$$\begin{array}{ccc} HH_i(A/k) & \simeq & \Omega^i_{A/k} \\ \downarrow d & \circlearrowleft & \downarrow d \\ HH_{i+1}(A/k) & \simeq & \Omega^{i+1}_{A/k} \end{array}$$

<u>Def'n</u>. $HC(A/k) := HH(A/k)_{hS^1} \in \mathcal{D}(k)$ <u>cyclic homology</u>.

$$\Vert$$

$$\operatorname*{colim}_{BS^1} HH(A/k)$$

$HC^-(A/k) := HH(A/k)^{hS^1} \in \mathcal{D}(k)$ <u>negative cyclic homology</u>

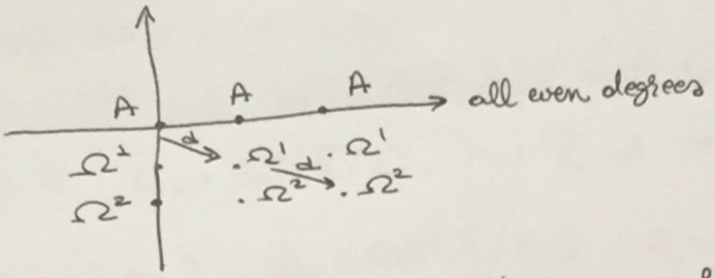
$$\Vert$$

$$\lim_{BS^1} HH(A/k)$$

$\cong$

de Rham cohomology

HC^- : E_2-spectral sequence



<u>Thm</u>: If $k/\mathbb{Q}$, this degenerates after E_2-page + filtration on E_∞-page splits.