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SCHLOSS ELMAU

Cohomology of p-adic Stein spaces

(joint w P. Colmez, G. Dospinescu)

p prime, $k \twoheadleftarrow \mathcal{O}_K \subset K$ char $(0,p)$, k perfect, $F = \mathrm{Frac}(W(k))$, $C = \widehat{\overline{K}}$, $G_K = \mathrm{Gal}(\overline{K}/K)$.

__Goal__: p-adic (pro)-étale coh of p-adic symmetric spaces, coverings of Ω^1_{Dr}, bdd elts in $H_{proét}$

proetale $(\mathbb{Q}_p)$ → étale $(\mathbb{Q}_p)$ "up to small denominators"

proetale $(\mathbb{Q}_p)$

$\Uparrow$

rational p-adic HT

$\Uparrow$

$R\Gamma_{dR}$, $R\Gamma_{HK}$

Stein property

étale $(\mathbb{Q}_p)$

$\Uparrow$

reasonable integral p HT crystalline

$\Uparrow$

integral $\left\{ R\Gamma_{cr}/F^r \;\; R\Gamma_{HK},\ \wr\!\wr,\ R\Gamma_{dR}^{\#}/F^r \right.$

Ω^d_{Dr} ok. by Grosse-Klönne

Stein $\Rightarrow$ "integral Stein" ($\Leftarrow$ not in general)

__Would like more:__

étale coh $(\mathbb{Z}_p)$

e.g. Bloch-Kato, Hyodo, Colengo → $H^*_{ét}(\mathbb{Z}_p) \simeq H^*_{ét}(W\Omega^\bullet_{log})$? Stein ordinary variety.

__Stein spaces__: a rigid analytic space with an admissible affinoid covering

$$X = \bigcup_{i \in \mathbb{N}} \mathcal{U}_i$$

s.t. $\mathcal{U}_i \Subset \mathcal{U}_{i+1}$

__key properties__: X Stein

(1) $H^i(X, \mathcal{F}) = 0$ $i > 0$, $\mathcal{F}$-coherent

this is our assumption ← (2) $\exists\, \mathfrak{X}$ semistable model of X, all the irreducible components are proper & smooth + (tubes of certain open subsets corr to $\mathcal{U}_i$ --)

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<u>Thm 1</u>: If X is as above $/K$ $\Rightarrow$ $\exists$ G_K-equivariant diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{r-1}_C/\ker d \arrow[r] & H^r_{\text{proét}}(X_C, \mathbb{Q}_p(r)) \arrow[r] \arrow[d, "\beta"] \arrow[r, phantom, shift right=6, "\text{strict + w closed images}"] & \left(H^r_{HK}(\mathfrak{X}_0)\widehat{\otimes}_F B^+_{st}\right)^{N=0,\ \varphi=p^r} \arrow[r] \arrow[d, "\theta\circ\iota_{HK}"] & 0 \\
0 \arrow[r] & \Omega^{r-1}_C/\ker d \arrow[r, "\text{strictly}"] & \Omega^r_{X_C,\, d=0} \arrow[r] & H^r_{dR}(X_C) \arrow[r] & 0
\end{tikzcd}$$

The rows are exact

$$\ker(\beta) = \ker(\theta\circ\iota_{HK}) = \left(H^r_{HK}(\mathfrak{X}_0)\widehat{\otimes}_F B^+_{st}\right)^{N=0,\ \varphi=p^r}$$

Here (1) $H^r_{HK}(\mathfrak{X}_0) := \overset{\varphi,\, N}{H^r}\, R\Gamma_{HK}(\mathfrak{X}_0/\mathcal{O}_F^0)$ — log str. — F-vector space

$\wr$ Grosse-Klönne $\qquad$ X is qc $\Rightarrow$ f.dim v-sp $/F$.

Hyodo-Kato map: $\exists\ \iota_{HK}: R\Gamma_{rig}(\mathfrak{X}_0/\mathcal{O}_F^0) \to R\Gamma_{rig}(\mathfrak{X}_0/\mathcal{O}_K^\times)$ - K-vector space.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R\Gamma_{dR}(\mathfrak{X}_K)$

parallel transport

between $\log 0 \longleftrightarrow \log \pi$
$0 \xleftarrow{T} \mathcal{R}(T) \xrightarrow{T\mapsto \pi} \pi$

(2). <u>Topology</u>: Locally convex spaces $/\mathbb{Q}_p$ $\leftarrow$ spherically complete $\rightsquigarrow$ tensor products behave well

quasi-abelian ($\sim$ filtered vector space)

(Fréchet $= \varprojlim_{\text{countable}}$ finite v.sp (Banach spaces))

<u>Examples</u>: ① $\overset{r\geq 1}{H^r_{HK}}(\mathfrak{X}_0) = 0$ $\xrightarrow{\text{Thm 1}}$ $\Omega^{r-1}_{X_C}/\ker d \xrightarrow{\sim} H^r_{\text{proét}}(\mathfrak{X}_C, \mathbb{Q}_p(r))$

e.g. $X = \mathbb{A}^d_K$ $\rightsquigarrow$ $H^r_{HK}\otimes K \simeq H^r_{dR} = 0$

② finite covers of $\Omega^1_{Dr,K}$, $r=1$

③ Drinfeld half-space $\Omega^d_{Dr,K}\left(= \mathbb{P}^d_K \setminus \bigcup_{H\in\mathcal{H}_K} H\right)$

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Stein space: $\exists$ canonical (weak) formal model $\widetilde{\Omega}^d_{Dr}$

Everything is equivariant for $GL_{d+1}(K)$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{r-1}_C/\ker d \arrow[r] \arrow[d, equal] & H^r_{proét}(X_C, \mathbb{Q}_p(r)) \arrow[r] \arrow[d, hook, "\beta"] & H^r_{HK}(\widetilde{\Omega}_{Dr,0}) \arrow[r] \arrow[d, hook, "\theta\circ\iota_{HK}"] & 0 \\
0 \arrow[r] & \Omega^{r-1}_C/\ker \arrow[r] & \Omega^r_{X_C,\, d=0} \arrow[r] & H^r_{dR}(X_C) \arrow[r] & 0.
\end{tikzcd}$$

$K = \mathbb{Q}_p$

Diagram:

$(\ast)$

$$\begin{tikzcd}
R\Gamma_{syn}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_p(r)) \arrow[r] \arrow[d, dashed] & (R\Gamma_{HK}(\mathfrak{X}_0)\hat{\otimes}_F B^+_{st})^{N=0,\varphi=p^r} \arrow[r] \arrow[d, "\iota_{HK}\circ\iota"] & (R\Gamma_{dR}(\mathfrak{X}_K)\hat{\otimes}_K B^+_{dR})/F^r \arrow[d, equal] \\
F^r(\quad) \arrow[r] \arrow[d, "\theta"] & R\Gamma_{dR}(\mathfrak{X}_K)\hat{\otimes} B^+_{dR} \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & (\quad)/F^r \arrow[d] \\
F^r R\Gamma_{dR}(\mathfrak{X}_K)\hat{\otimes} C \arrow[r] & R\Gamma_{dR}(\mathfrak{X}_K)\hat{\otimes}_K C \arrow[r] & R\Gamma_{dR}(\mathfrak{X}_K)\hat{\otimes} C/F^r.
\end{tikzcd}$$

Rks: (1). Thm 1 $\rightsquigarrow$ overconvergent affinoids

$\rightsquigarrow$ "general" rigid-analytic spaces.

(2) $\exists$ comparison thm:

$H^*_{proét} \longleftarrow R\Gamma_{dR},\ R\Gamma_{HK}, + \dots$

Pf of Thm 1

Step 1: $\exists$ diagram $H_{proét} \rightsquigarrow H_{syn}$

X/K - variety $\overset{\text{assume}}{\rightsquigarrow} \exists\ \mathfrak{X}/\mathcal{O}_K$ semistable

$$R\Gamma_{syn}(X_{\overline{K}}, \mathbb{Q}_p(r)) := \Big[(R\Gamma_{HK}(X_{\overline{K}})\otimes_{F^{nr}} B^+_{st})^{N=0,\varphi=p^r} \xrightarrow{\iota_{HK}} (R\Gamma_{dR}(X_{\overline{K}})\otimes_{\overline{K}} B^+_{dR})/F^r \Big]$$

"Deligne coh"

$\exists$ map $R\Gamma_{syn}(X_{\overline{K}}, \mathbb{Q}_p(r)) \to R\Gamma_{ét}(X_{\overline{K}}, \mathbb{Q}_p(r))$

this map is

$\simeq$ after $\tau_{\leq r}$

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Apply H^r to diagram (*):

$$0 \to \underset{\substack{\wr \\ H^0(\mathcal{X}_{\bar{K}},\, \Omega^r)\hat{\otimes} C}}{F^r} \longrightarrow H^r_{dR}(\mathcal{X}_K)\hat{\otimes} C \longrightarrow H^r_{dR}/F^r \longrightarrow 0$$

(bottom row of (*))

$\rightsquigarrow$ $\mathcal{X}$ stein $\rightsquigarrow$ replace everything by overconvergent

$$F^r(R\Gamma_{dR}(\mathcal{X}_K)\otimes_K B^+_{dR}) = \mathcal{O}\hat{\otimes} F^r B^+_{dR} \to \Omega^1 \hat{\otimes} F^{r-1} B^+_{dR} \to \dots$$

$$(\qquad)/F^r = \mathcal{O}\hat{\otimes} B^+_{dR} \to \Omega^1 \hat{\otimes} B^+_{dR}/F^{r-1} \to \dots \to \Omega^{r-1}\otimes (B^+_{dR}/F^1)$$

<u>Bottom rows</u>: distinguished Δ

$$\sigma_{\geq r}\Omega^\bullet \hat{\otimes} C[r] \to \Omega^\bullet_{\mathcal{X}_K}\hat{\otimes} C \to \sigma_{\leq r-1}\Omega \hat{\otimes} C$$

$$H^{r-1}_{dR}\hat{\otimes} C \to \Omega^{r-1}\otimes C \to \Omega^r \hat{\otimes} C_{d=0} \to H^r_{dR}(\mathcal{X}_K)\hat{\otimes} C \to 0.$$

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