This talk is an overview/dictionary to "Integral p-adic Hodge theory" (Bhatt–Morrow–Scholze). The goal of this talk: to define main objects of the paper, state the main properties. The goal of the overall series: to introduce a new definition using topological Hochschild homology (THH).

1. Notation: $\mathbf{A}_{\inf}$, μ, ξ, etc.

Let C be a perfectoid field of characteristic 0 containing all p-power roots of unity; fix a compatible sequence $1, \zeta_p, \zeta_{p^2}, \dots \in C$. Let $\mathcal{O}$ be the ring of integers of C. Let k be the residue field of C. Form the tilts $C^\flat$ and $\mathcal{O}^\flat = \varprojlim_\varphi \mathcal{O}/p\mathcal{O}$. Define the ring $\mathbf{A}_{\inf} = W(\mathcal{O}^\flat)$.

Define

$$\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat$$

$$\xi = \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1}$$

. Recall that ξ generates the kernel of Fontaine's surjective map $\theta : \mathbf{A}_{\inf} \to \mathcal{O}$. More generally, we have maps $\theta_r : \mathbf{A}_{\inf} \to W_r(\mathcal{O})$ sending $[\alpha]$ to $[\theta([\alpha])]$; the kernel is generated by $\xi_r := ([\epsilon] - 1)/([\epsilon^{1/p^r}] - 1)$. (Note: W_r is the p-typical Witt vectors of length r.)

Also put $\tilde{\theta}_r := \theta_r \circ \varphi^{-r} : \mathbf{A}_{\inf} \to W_r(\mathcal{O})$ with kernel generated by $\tilde{\xi}_r := \varphi^r(\xi_r) := ([\epsilon^{p^r}] - 1)/([\epsilon] - 1)$.

2. Proétale site

For any rigid analytic space or Noetherian adic space X over C, we have the "old' proétale site X_{proet}.

- The objects of X_{proet} are proobjects of X_{et} (i.e., formal inverse limit over some small cofiltered category) in which each morphism is finite étale.
- The morphisms of X_{proet} are morphisms are morphisms of proobjects.
- Write $\left|\text{“}\varprojlim\text{”}_{i \in I} U_i\right| := \varprojlim_{i \in I} |U_i|$ for the *underlying topological space* of an object of X_{proet}.
- Coverings in X_{proet} are families which are coverings at the level of underlying topological spaces, plus some technical condition on each individual map.

Some of our favourite sheaves on X_{proet}:

- $\widehat{\mathcal{O}}_X^+$: the p-adic completion of the sheaf

$$\text{“}\varprojlim_i\text{''} U_i \mapsto \varinjlim_i \Gamma(U_i, \mathcal{O}_{U_i}^+).$$

- $W_r(\widehat{\mathcal{O}}_X^+)$: take Witt vectors of the previous sheaf;
- $\mathcal{O}_{X^\flat}^+ := \varprojlim_\varphi \widehat{\mathcal{O}}_X^+/p\widehat{\mathcal{O}}_X^+ \cong \lim_{x \mapsto x^p} \widehat{\mathcal{O}}_X^+$;
- $\mathbf{A}_{\inf,X} := W(\mathcal{O}_{X^\flat}^+$.

Note: X_{proet} is locally perfectoid in the following sense: there exists a basis $\mathcal{U}$ of proaffinoids such that $A = \widehat{\mathcal{O}}_X^+(U)$ is a perfectoid ring and $\mathcal{O}_{X^\flat}^+ = A^\flat$ and $\mathbf{A}_{\inf,X}(U) = W(A^\flat)$.

Everything carries over at this level. In particular, we have a map $\tilde{\theta}_r : \mathbf{A}_{\inf,X}/\tilde{\xi}_r \cong W_r(\widehat{\mathcal{O}}_X^+)$.

3. Décalage functor

If A is a ring and $f \in A$ is a nonzerodivisor, then for any complex $\mathcal{C}$ of f-torsion-free A-modules, define the subcomplex $\eta_f \mathcal{C} \subseteq \mathcal{C}[f^{-1}]$ by

$$(\eta_f \mathcal{C})^n = \{x \in f^n \mathcal{C}^n : dx \in f^{n+1}\mathcal{C}^{n+1}\} \qquad (n \in \mathbb{Z}).$$

(There is a general construction of Deligne extending this. See the final chapter of Berthelot-Ogus.)

More generally, for any complex D of A-modules, one can well-define $\mathbb{L}\eta_f D := \eta_f C$ where $\mathcal{C}$ is quasi-isomorphic to D as above (e.g., a projective resolution).

Relation to the Bockstein construction: there is a natural quasi-isomorphism

$$(\mathbb{L}\eta_f D) \otimes_A A/fA \cong [\cdots \to H^n(D \otimes^{\mathbb{L}}_A A/fA) \to H^{n+1}(D \otimes^{\mathbb{L}}_A A/fA) \to \cdots]$$

where the boundary maps on the right are the Bockstein operators.

Example: If S is a smooth k-algebra, Berthelot-Ogus prove that φ induces

$$\mathbb{R}\Gamma_{\mathrm{crys}}(S/W) \xrightarrow{\sim} \mathbb{L}\eta_p \mathbb{R}\Gamma_{\mathrm{crys}}(S/W)$$

($\mathbb{L}\eta$ with respect to p) and by reduction mod p,

$$\mathbb{R}\Gamma_{\mathrm{dR}}(S/k) \cong [\cdots \to H^n_{\mathrm{dR}}(S/k) \xrightarrow{\mathrm{Bock}_p} H^n_{\mathrm{dR}}(S/k) \to \cdots]$$

(the map being the Cartier isomorphism).

4. Definitions of the main objects

Let R be the p-adic completion of a smooth $\mathcal{O}$-algebra and put $X := \mathrm{Spa}(R[\frac{1}{p}], R)$. We then obtain $\mathbb{R}\Gamma_{\mathrm{proet}}(X, \mathbf{A}_{\mathrm{inf},X})$, and then apply L_η (ring $\mathbf{A}_{\mathrm{inf}}$, element $\mu := [\epsilon] - 1$) to obtain

$$L\eta_\mu \mathbb{R}\Gamma_{\mathrm{proet}}(X, \mathbf{A}_{\mathrm{inf},X}) =: \mathbb{A}\Omega_R.$$

Similarly,

$$L\eta_{[\zeta_{p^r}]-1} \mathbb{R}\Gamma_{\mathrm{proet}}(X, W_r(\widehat{\mathcal{O}}^+_X)) =: \widetilde{W_r\Omega}_R.$$

(where $[\zeta_{p^r}] - 1 \in W_r(\mathcal{O})$).

The map $\tilde{\theta}_r : \mathbf{A}_{\mathrm{inf}} \to W_r(\mathcal{O})$ sends μ to $[\zeta_{p^r}] - 1$, so there is an induced map

$$\mathbb{A}\Omega_R \otimes^{\mathbb{L}}_{\mathbf{A}_{\mathrm{inf}}} \mathbf{A}_{\mathrm{inf}}/\tilde{\xi}_r \to \widetilde{W_r\Omega}_R.$$

This is a quasi-isomorphism (on the nose, not "almost").

5. Main properties

We have a "Cartier isomorphism": there exist natural isomorphisms

$$\widehat{W_r\Omega}^n_{R/\mathcal{O}} \xrightarrow{\sim} H^n(\widetilde{W_r\Omega}_R)\{n\}$$

where $\{n\}$ means $\otimes_{W_r(\mathcal{O})}(\tilde{\xi}_r \mathbf{A}_{\mathrm{inf}}/\tilde{\xi}_r^2 \mathbf{A}_{\mathrm{inf}})^{\otimes n}$ and the object on the left is Langer–Zink's relative de Rham-Witt complex for $\mathcal{O} \to R$ (then p-adically complete). This is set up so that the differential on the left corresponds to the Bockstein on the right for $\tilde{\xi}_r$ (see the previous quasi-isomorphism).

To relate this to crystalline and de Rham cohomology,: from "Cartier isomorphism" and Bockstein property of L_η and the previous quasi-isomorphism, one formally gets

$$\mathbb{A}\Omega_R \otimes^{\mathbb{L}}_{\mathbf{A}_{\inf}} \mathbf{A}_{\inf}/\xi_r \cong \widehat{W_r\Omega}^{\bullet}_{R/\mathcal{O}}.$$

If one sets $r = 1$:

$$\mathbb{A}\Omega_R \otimes^{\mathbb{L}}_{\mathbf{A}_{\inf},\theta} \mathcal{O} \cong \widehat{\Omega}^{\bullet}_{R/\mathcal{O}}.$$

If one instead tensors to $W(k)$ and take $\varprojlim_r$:

$$\mathbb{A}\Omega_R \otimes^{\mathbb{L}}_{\mathbf{A}_{\inf}} W(k) \cong W\Omega^{\bullet}_{R\otimes_{\mathcal{O}}K/k}.$$

More generally, for $\mathfrak{X}$ smooth p-adic formal scheme over $\mathcal{O}$, may sheafify the construction to get $\mathbb{A}\Omega_{\mathfrak{X}}$. If $\mathfrak{X}$ is also proper, the "primitive comparison theorem" of Scholze implies

$$\mathbb{R}\Gamma_{\mathrm{Zar}}(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}})[\frac{1}{\mu}] \cong \mathbb{R}\Gamma_{\mathrm{et}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbf{A}_{\inf}[\frac{1}{\mu}].$$