M. Morrow

An Overview of integral p-adic Hodge theory

Goal: define main objects in IpHT, state main properties.

In Peter's 2nd talk: new definition via topological Hochschild homology.

§1. Notation: A_{inf}, μ, ξ, etc...

§2. Pro-étale site

§3. Décalage functor

§4. Def'ns of main objects $A\Omega_{R/\mathcal{O}}$ and $W_r\tilde{\Omega}_{R/\mathcal{O}}$

§5. Main properties

§1. · C perfectoid field of char 0 with all p power roots of unity (fix $1, \zeta_p, \zeta_{p^2}, \ldots \in C$).

· Ring of integers $\mathcal{O} \subset C$, res field k.

· Tilts $C^\flat \supset \mathcal{O}^\flat = \varprojlim_{\varphi} \mathcal{O}/p\mathcal{O}$

· Period ring $A_{inf} = W(\mathcal{O}^\flat)$.

Elements: · $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}^\flat$; $\mu = [\varepsilon] - 1$

· $\xi = \dfrac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} \in A_{inf}$ generates kernel of $\theta: A_{inf} \twoheadrightarrow \mathcal{O}$.

· More generally, have maps $\theta_r: A_{inf} \to W_r(\mathcal{O})$ $(\subseteq \mathcal{O}^r)$

$$[\alpha] \longmapsto [\theta([\alpha])]$$
$$\alpha \in \mathcal{O}^\flat$$

with kernel generated by

$$\xi_r = [\varepsilon]-1 \Big/ [\varepsilon^{1/p^r}]-1.$$

· Frobenius twists $\tilde{\theta}_r = \theta_r \circ \varphi^{-r}: A_{inf} \to W_r(\mathcal{O})$

with kernel generated by $\tilde{\xi}_r = \varphi^r(\xi_r) = \dfrac{[\varepsilon]^{p^r}-1}{[\varepsilon]-1}$

§2. For any rigid analytic space, or Noetherian adic space X over C, have proétale site $X_{\text{proét}}$.

· Objects are pro-objects of $X_{\text{ét}}$ i.e. $\text{“}\varprojlim_{i\in I}\text{”}\,\mathcal{U}_i$, where

I cofiltered cat $\to X_{\text{ét}}$ st.

$\mathcal{U}_j \to \mathcal{U}_i$ finite étale cover for all $i \to j$.

· Morphisms are morphisms of pro-objects

· Write $|\text{“}\varprojlim_i\text{”}\,\mathcal{U}_i| = \varprojlim_{i\in I} |\mathcal{U}_i|$ thought of as adic spaces.

underlying top space

· Coverings in $X_{\text{proét}}$ are families $\{\mathcal{U}_\lambda \to \mathcal{U}\}_{\lambda\in\Lambda}$ st.

$\{|\mathcal{U}_\lambda| \to |\mathcal{U}|\}$ is a covering of top. spaces, + technical condition

on each $\mathcal{U}_\lambda \to \mathcal{U}$

Favorite sheaves on $X_{\text{proét}}$

· $\widehat{\mathcal{O}}_X^+ :=$ p-adic completion of the sheaf

$$\text{“}\varprojlim_i\text{”}\,\mathcal{U}_i \longmapsto \varinjlim_i \Gamma(\mathcal{U}_i, \mathcal{O}^+_{\mathcal{U}_i})$$

· $W_r(\widehat{\mathcal{O}}_X^+)$

· $\widehat{\mathcal{O}}^+_{X^\flat} := \varprojlim_\varphi \widehat{\mathcal{O}}_X^+/p\,\widehat{\mathcal{O}}_X^+ \simeq \varprojlim_{x\mapsto x^p} \widehat{\mathcal{O}}_X^+$

· $\mathbb{A}_{\inf,X} = W(\mathcal{O}^+_{X^\flat})$

NB: $X_{\text{proét}}$ is locally perfectoid in the sense that $\exists$ basis $\mathcal{U}$ s.t. each $\widehat{\mathcal{O}}_X^+(\mathcal{U}) = A^+$ is a perfectoid ring and $\widehat{\mathcal{O}}^+_{X^\flat}(\mathcal{U}) = A^{\flat+}$ and $\mathbb{A}_{\inf,X}(\mathcal{U}) = W(A^{\flat+})$. Since §1 remains valid for A in place of $\mathcal{O}$, deduce

$$\tilde{\theta}_r : \mathbb{A}_{\inf,X}/\tilde{\xi}_r \simeq W_r(\hat{\mathcal{O}}_X^+)$$

§3. If A is a ring and $f \in A$ is a non-zero divisor, then for any complex C of f-torsion-free A-modules, define subcomplex (Deligne, Berthelot-Ogus)

$$\eta_f C \subseteq C[1/f] \text{ by:}$$

$$(\eta_f C)^n := \{x \in f^n C^n : dx \in f^{n+1} C^{n+1}\}, \quad n \in \mathbb{Z}$$

More generally, for D any complex of A-modules, may define

$$\mathbb{L}\eta_f D := \eta_f C \quad \text{where } C \xrightarrow{\sim} D \text{ is a complex as above.}$$

(this ends up being well-defined)

<u>Bockstein property</u>: $\exists$ natural quasi-isomorphism

$$(\mathbb{L}\eta_f D) \otimes^{\mathbb{L}}_A A/fA \simeq \left[\cdots \to H^n(D \otimes^{\mathbb{L}}_A A/fA) \xrightarrow{\mathrm{Bock}_f} H^{n+1}(D \otimes^{\mathbb{L}}_A A/fA) \to \cdots \right]$$

<u>Eg.</u>: If S is a smooth k-algebra, then

Berthelot-Ogus proved that ϕ induces a quasi-isomorphism:

$$R\Gamma_{crys}(S/W(k)) \xrightarrow{\sim} \underset{C}{\mathbb{L}\eta_p}\, R\Gamma_{crys}(S/W(k))$$

we apply $\mathbb{L}\eta$ w.r.t. $p \in W(k)$

$$\overset{\text{mod } p}{\Rightarrow} \quad R\Gamma_{dR}(S/k) \underset{\text{Cartier isomorphism}}{\xrightarrow{\sim}} \left[\to H^n_{dR}(S/k) \xrightarrow{\mathrm{Bock}_p} H^{n+1}_{dR}(S/k) \to \cdots \right]$$

M. Morrow ④

§4. Let R = p-adic completion of a smooth $\mathcal{O}$-alg and set:

$$X = \mathrm{Spa}(R[1/p], R)$$

$$\rightsquigarrow R\Gamma_{\text{proét}}(X, \mathbb{A}_{\inf,X})$$

$$\rightsquigarrow L\eta_{\mu} R\Gamma_{\text{proét}}(X, \mathbb{A}_{\inf,X}) =: \mathbb{A}\Omega_R.$$

($L\eta$ w.r.t. $\mu \in \mathbb{A}_{\inf}$, $\mu = [\varepsilon]-1$)

Similarly,

$$\underbrace{L\eta_{[\zeta_{p^r}]-1}}_{\in W_r(\mathcal{O})} R\Gamma_{\text{proét}}(X, W_r(\widehat{\mathcal{O}_X^+})) =: \widetilde{W_r\Omega_R}$$

$\tilde{\theta}_r : \mathbb{A}_{\inf} \longrightarrow W_r(\mathcal{O})$ sends μ to $[\zeta_{p^r}]-1$, so it induces a map:

$$\mathbb{A}\Omega_R \overset{\mathbb{L}}{\otimes}_{\mathbb{A}_{\inf}} \mathbb{A}_{\inf}/\tilde{\xi}_r \longrightarrow \widetilde{W_r\Omega_R} \quad (*)$$

this is a quasi-isomorphism.

§5. "Cartier isomorphism": $\exists$ natural isomorphisms:

$$\underbrace{\widehat{W_r\Omega^n_{R/\mathcal{O}}}}_{\text{Langer-Zink's relative de Rham-Witt complex for } \mathcal{O}\to R} \xrightarrow{\ \cong\ } H^n(\widetilde{W_r\Omega_R})\underbrace{\{n\}}_{-\otimes_{W_r(\mathcal{O})}(\tilde{\xi}_r\mathbb{A}_{\inf}/\tilde{\xi}_r^2\mathbb{A}_{\inf})^{\otimes n}}$$

p-adic completion

s.t. d (differential in dRW complex) $\longleftrightarrow$ Bock$_{\tilde{\xi}_r}$ (cf (*))

Crystalline & de Rham

From "Cartier isom" and "Bockstein property of $L\eta$ on $(*)$"
one formally gets

$$A\Omega_R \overset{\mathbb{L}}{\underset{A_{inf}}{\otimes}} A_{inf}/\xi_r \simeq W_r\Omega^\bullet_{R/\mathcal{O}}.$$

• (set $r=1$) $$A\Omega_R \overset{\mathbb{L}}{\underset{A_{inf},\,\theta}{\otimes}} \mathcal{O} \simeq \hat{\Omega}^\bullet_{R/\mathcal{O}}. \quad (dR)$$

• ($\otimes W_r(k)$ and $\varprojlim_r$) $$A\Omega_R \overset{\mathbb{L}}{\underset{A_{inf}}{\otimes}} W(k) \simeq W\Omega^\bullet_{R\otimes_{\mathcal{O}} k/k}. \quad (crys)$$

More generally, for $\mathfrak{X}$ smooth formal scheme $/\mathcal{O}$ may sheafify construction to get $A\Omega_{\mathfrak{X}}$. Then, for $\mathfrak{X}$ proper,

the "primitive comparison theorem" implies

$$R\Gamma_{Zar}(\mathfrak{X}, A\Omega_{\mathfrak{X}})[1/\mu] \simeq R\Gamma_{et}(X,\mathbb{Z}_p)\overset{\mathbb{L}}{\underset{\mathbb{Z}_p}{\otimes}} A_{inf}[1/\mu].$$