

# SCHLOSS ELMAU

## Topological Hochschild homology

$K$  commutative ring (possibly a field)

$A$  flat  $K$ -algebra

$$HH_*^K(A) = \text{Tor}_{*}^{A \otimes_R A^{\text{op}}}(A, A) \quad \text{we'll be interested in } A \text{ commutative} \rightarrow \text{Tor}_{*}^{A \otimes_K A}(A, A).$$

If  $A$  is a smooth  $K$ -algebra

$$HH_*^K(A) = \Omega_{A/K}^* \rightarrow \text{relationship between HH \& de Rham coh.} \\ \text{(i.e. here you're matching graded pieces...)}$$

Specialize to  $K = \mathbb{F}_p$ .

Question: Can the crystalline coh of  $A$  be recovered in terms of Hochschild homology?

$$W(K) \rightarrow K \rightarrow A$$

$$CM_*^K(A) = A \underset{A \otimes_K A}{\overset{L}{\otimes}} A \quad \leftarrow \text{we'll derive both tensor products.}$$

chain complex computing  $HH_*^K(A)$ .

Example:  $A = \mathbb{F}_p$ ,  $K = \mathbb{Z}_p$ . (no  $W(K)$  isom before)

$$A = \mathbb{F}_p \quad \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \quad \underset{\mathbb{Z}_p}{\otimes} \mathbb{F}_p \\ \downarrow \quad \downarrow \quad \downarrow \\ 0 \rightarrow \mathbb{F}_p$$

$$\text{also } \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \quad \underset{\mathbb{Z}_p}{\otimes} \mathbb{F}_p \rightsquigarrow \mathbb{F}_p[\epsilon]/(\epsilon^2) \\ \downarrow \quad \downarrow \quad \downarrow \\ \mathbb{Z}_p[\epsilon]/\epsilon^2 \quad \text{from degree 1.} \quad d\epsilon = p. \\ d\epsilon = 0.$$

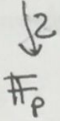
$$\Rightarrow CM_*^K(A) = A \underset{A \otimes_K A}{\overset{L}{\otimes}} A = \mathbb{F}_p \underset{\mathbb{F}_p[\epsilon]}{\overset{L}{\otimes}} \mathbb{F}_p$$



# SCHLOSS ELMAU

Consider:

$$\mathbb{F}_p[\varepsilon, \frac{S^n}{n!}]_{st.} \quad \deg \varepsilon = 1, \quad \deg S = 2.$$



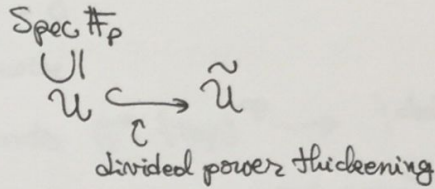
$$d \frac{S^n}{n!} = \frac{S^{n-1}}{(n-1)!} \varepsilon.$$

resolution by free modules over  $\mathbb{F}_p[\varepsilon]$ .

$$CH_*^K(A) = \dots = \mathbb{F}_p[\varepsilon, \frac{S^n}{n!}] \otimes_{\mathbb{F}_p[\varepsilon]} \mathbb{F}_p = \mathbb{F}_p[\frac{S^n}{n!}]$$

divided power algebra.

Recall crystalline coh: crys site :



Objects of the crystalline site are:

algebras  $A$

$$\varepsilon: A \rightarrow \mathbb{F}_p$$

$\ker(\varepsilon)$  is pd nilpotent

+ divided powers on  $\ker(\varepsilon)$

(which are compatible w/ usual divided powers on  $(p) \subset \mathbb{Z}_p$  ↑)

"fine print"

- if you add fine print get crystalline coh of  $\mathbb{F}_p$  to be  $\mathbb{Z}_p$
- without it, answer is  $\mathbb{Z}_p$  [divided powers on  $p$ ] suitably completed

Can we modify theory of HH in order to get right answer?

## SCHLOSS ELMAU

Eilenberg-  
Steenrod  
axioms

Recall: If  $X$  is a top space, have singular cohs  $H^*(X; A)$

1).  $H^*$  is a functor  $\{\text{top}\}^{\text{op}} \rightarrow \text{graded abelian groups}$ .

2). Homotopy invariant

$$3). H^*(\sqcup X_\alpha) \simeq \prod H^*(X_\alpha)$$

$$4). H^*_{\text{red}}(X) \simeq H^*_{\text{red}}(\Sigma X)$$

$$5). H^*(X/Y) \rightarrow H^*(X) \rightarrow H^*(Y)$$

6) Dimension Axiom:

$$H^*(pt) = \begin{cases} A & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Def: A cohomology theory is a functor  $E^*: \{\text{top}\}^{\text{op}} \rightarrow \{\text{abelian grps}\}$

$$+ \text{ isomorphisms } E^*_{\text{red}}(X) \simeq E^*_{\text{red}}(\Sigma X)$$

satisfying 2) through 5).

Thm (Brown): If  $E^*$  is a coh theory there exist spaces

$\{E(n)\}_{n \in \mathbb{Z}}$  st.

$$E^n(X) \simeq [X, E(n)]$$

$\{ \text{continuous maps } X \rightarrow E(n) \} / \text{homotopy}$ .

$$E^n_{\text{red}}(X) \simeq [X, E(n)]_* \leftarrow \text{pointed maps}$$

$$[X, E(n)]_* \simeq [\Sigma X, E(n+1)]_*$$

$$E(n) \simeq \Omega E(n+1) = \{ p: [0,1] \rightarrow E(n+1) \mid p(0) = p(1) = * \}$$

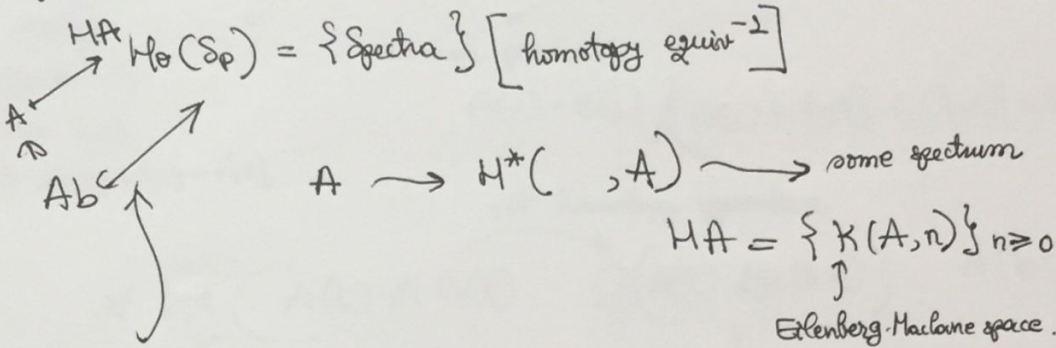
# SCHLOSS ELMAU

J. Lurie ①

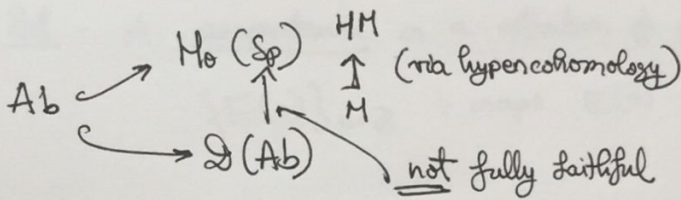
Def: A spectrum is a collection of pointed spaces  $\{E(n)\}_{n \in \mathbb{Z}^+}$

homotopy equivalences  $E(n) \xrightarrow{\sim} \Omega E(n+1)$ .

A spectrum determines a coh theory using Brown rep thm.



$Ho(Sp)$  enlargement of  $Ab$  (can say more, e.g. triangulated cat,  $\otimes$  structure st-  
 $Ab$  is the heart)



because:

$$M, N \text{ abelian groups} \rightarrow \text{Ext}_{Ab}^n(M, N) = \text{Hom}_{D(Ab)}(M, N[n])$$

$$\rightarrow \text{Hom}_{Ho(Sp)}(HM, HN[n])$$

isom in degrees 0, 1

but not in general

$A, B$  abelian groups, can consider

$$A \times B \rightarrow A \otimes B$$

bilinear

$A, B$  coh theories; want to study  
 "bilinear" maps  $A \times B \rightarrow C$ , i.e.

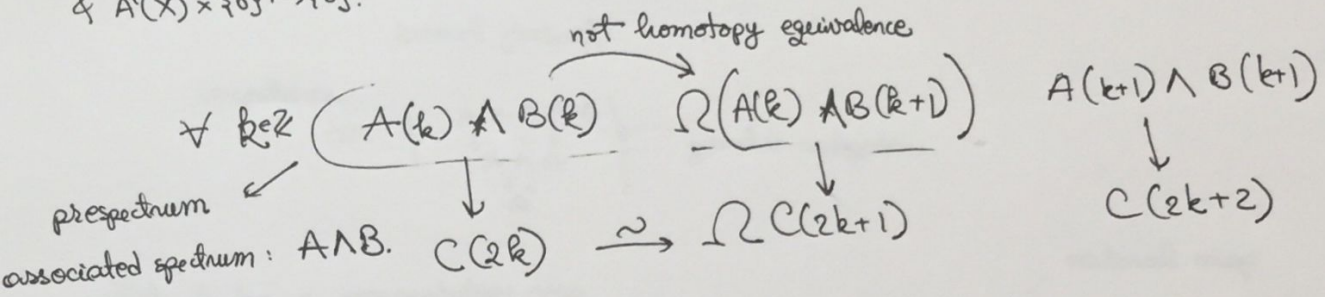
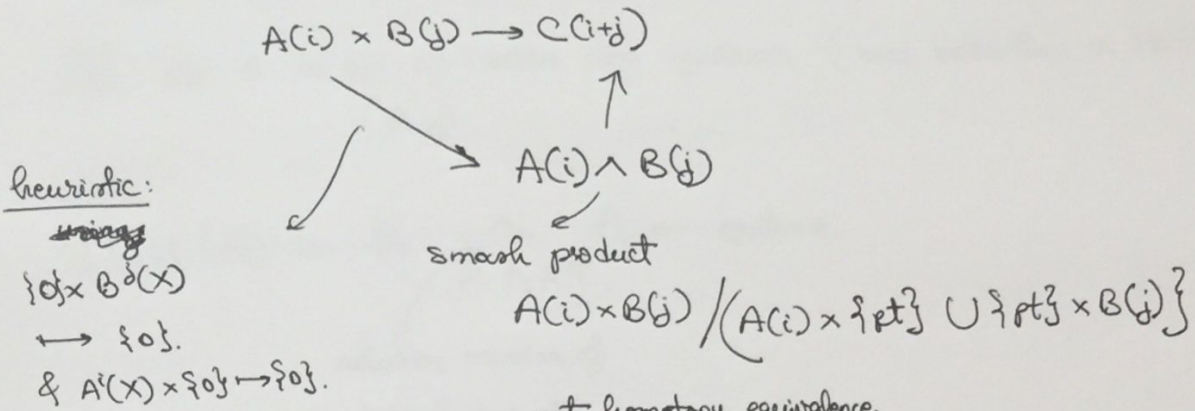
$$A^i(X) \times B^d(X) \rightarrow C^{i+d}(X)$$

suitably compatible w suspension isoms.



# SCHLOSS ELMAU

at the level of spectra:

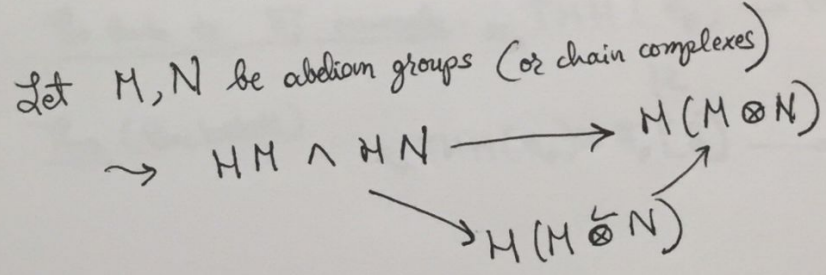


Def: A prespectrum is a collection of pointed spaces  $\{E(n)\}_{n \in \mathbb{Z}}$  + maps  $E(n) \rightarrow \Omega E(n+1)$ .

Note: If  $E(n)$  is a prespectrum, can form associated spectrum  $E(n) \xrightarrow{\sim} \varinjlim \Omega^k E(n+k)$ .

not obviously commutative or associative, but

$\mathcal{H}_0(\text{Sp})$  is a symmetric monoidal category under  $\wedge$ .  
 (not quite good enough, but we'll pretend it is...)



# SCHLOSS ELMAU

J. Lurie ©

If  $R$  is a ring,

$$MR \wedge NR \rightarrow H(R \otimes R) \xrightarrow{m} HR \quad (\text{ring spectrum})$$

Def: Let  $A$  be an associative ring spectrum (need better than in  $Ho(Sp)$ )  
 $(A_{\infty})$

$$TMM(A) := \left( A \overset{\wedge}{\underset{A \wedge A^{\text{op}}}{\longrightarrow}} A \right) \leftarrow \text{spectrum.}$$

relative version of  
smash product

computed via

$$\cong \left| \begin{array}{c} A \wedge A \wedge A \\ \downarrow \downarrow \downarrow \\ A \wedge A \\ \downarrow \downarrow \\ A \end{array} \right| \leftarrow \text{geometric realization.}$$

Let  $A$  be a commutative ring.

$$TMM(A) := TMM(HA) = HA \wedge HA \underset{HA \wedge HA}{\longrightarrow} H \left( \begin{array}{c} A \overset{\wedge}{\underset{A}{\longrightarrow}} A \\ A \overset{\wedge}{\underset{A}{\longrightarrow}} A \end{array} \right)$$

natural map

Def: Let  $E$  be a spectrum

$$\pi_n E := E^{-n}(pt) \cong \pi_{n+m} E(m) \quad \leftarrow \text{as soon as this makes sense}$$

Ex: If  $M$  is a chain complex  $\pi_* MM \cong M_*(M)$

$$\leadsto \pi_* TMM(A) \longrightarrow MM_*^{\mathbb{Z}}(A)$$

Go back to  $\mathbb{F}_p$  example:  $\pi_* TMM(\mathbb{F}_p) \rightarrow MM_*^{\mathbb{Z}}(\mathbb{F}_p) \cong$  divided power algebra over  $\mathbb{F}_p$  on one gen.  $S$  in degree 2.

Thm (Bockstedt):  $\pi_* TMM(\mathbb{F}_p) \cong \mathbb{F}_p[S] \xrightarrow{\quad} \mathbb{F}_p \left[ \frac{S^{\mathbb{Z}}}{n!} \right]$