J. Lurie (1)

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Topological Hochschild homology

k commutative ring (possibly a field)

A flat k-algebra

$HH_*^k(A) = \mathrm{Tor}_*^{A \otimes_k A^{op}}(A, A)$ we'll be interested in A commutative $\rightsquigarrow \mathrm{Tor}_*^{A \otimes_k A}(A, A)$.

If A is a smooth k-algebras

$$HH_*^k(A) = \Omega^*_{A/k}$$ $\rightsquigarrow$ relationship between HH & deRham coh.
(i.e. here you're matching graded pieces...)

Specialize to $k = \mathbb{F}_p$.

<u>Question</u>: Can the crystalline coh of A be recovered in terms of Hochschild homology?

$W(k) \longrightarrow k \longrightarrow A$

$$\underset{\substack{\cap \\ \text{chain complex computing } HH_*^k(A).}}{CH_*^k(A)} = A \underset{A \otimes_k^L A}{\overset{L}{\otimes}} A$$ we'll derive both tensor products.

<u>Example</u>: $A = \mathbb{F}_p$, $k = \mathbb{Z}_p$ (so $W(k)$ from before)

$A = \mathbb{F}_p$ $\qquad$ $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p$ $\qquad \otimes_{\mathbb{Z}_p} \mathbb{F}_p$
$\qquad\qquad\qquad\qquad \wr\!\wr$
$\qquad\qquad\qquad 0 \longrightarrow \mathbb{F}_p$

also $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p$
$\qquad\qquad \wr\!\wr$ $\leftarrow$ from degree 1.
$\mathbb{Z}_p[\varepsilon]/\varepsilon^2 \quad d\varepsilon = p.$ $\qquad \otimes_{\mathbb{Z}_p} \mathbb{F}_p \rightsquigarrow \mathbb{F}_p[\varepsilon]/(\varepsilon^2)$, $d\varepsilon = 0$.

so $CH_*^k(A) = A \underset{A \otimes_k^L A}{\overset{L}{\otimes}} A = \mathbb{F}_p \underset{\mathbb{F}_p[\varepsilon]}{\overset{L}{\otimes}} \mathbb{F}_p$

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Consider:

$$\mathbb{F}_p\left[\varepsilon, \frac{\delta^n}{n!}\right], \text{s.t. } \deg \varepsilon = 1, \ \deg \delta = 2.$$

$$d \frac{\delta^n}{n!} = \frac{\delta^{n-1}}{(n-1)!}\varepsilon.$$

$$\downarrow\wr$$

$\mathbb{F}_p$ resolution by free modules over $\mathbb{F}_p[\varepsilon]$.

$$CH^K_*(A) = \cdots = \mathbb{F}_p\left[\varepsilon, \frac{\delta^n}{n!}\right] \otimes_{\mathbb{F}_p[\varepsilon]} \mathbb{F}_p = \mathbb{F}_p\left[\frac{\delta^n}{n!}\right]$$

divided power algebra.

- Recall crystalline coh: crys site:

$$\begin{array}{c} \operatorname{Spec} \mathbb{F}_p \\ \cup\!| \\ \mathcal{U} \xhookrightarrow[C]{} \tilde{\mathcal{U}} \end{array}$$

divided power thickening

- Objects of the crystalline site are:

algebras A

$\varepsilon : A \to \mathbb{F}_p$ $\ker(\varepsilon)$ is pd nilpotent

+ divided powers on $\ker(\varepsilon)$

(which are compatible w usual divided powers on $(p) \subset \mathbb{Z}_p$)

"fine print"

- if you add fine print get crystalline coh of $\mathbb{F}_p$ to be $\mathbb{Z}_p$
- without it, answer is $\mathbb{Z}_p\left[\begin{array}{l}\text{divided} \\ \text{powers on } p\end{array}\right]$ suitably completed

Can we modify theory of HH in order to get right answer?

THE LEADING HOTELS OF THE WORLD

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Recall: If X is a top space have singular coh $H^*(X;A)$

Eilenberg-Steenrod axioms

1). H^* is a functor $\{top\}^{op} \longrightarrow$ graded abelian groups.

2). Homotopy invariant

3). $H^*(\coprod X_\alpha) \simeq \prod H^*(X_\alpha)$

4). $H^*_{red}(X) \simeq H^{*+1}_{red}(\Sigma X)$

5). $$H^*_{red}(X/Y) \to H^*(X) \longrightarrow H^*(Y)$$

6) Dimension Axiom:

$$H^*(pt) = \begin{cases} A & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Def: A cohomology theory is a functor $E^* : \{top\}^{op} \longrightarrow \{\text{abelian gps}\}$

+ isomorphisms $E^*_{red}(X) \simeq E^{*+1}_{red}(\Sigma X)$

satisfying 1) through 5).

Thm (Brown): If E^* is a coh theory there exist spaces

$\{E(n)\}_{n\in\mathbb{Z}}$ st.

$$E^n(X) \simeq [X, E(n)]$$

$$\| $$

$$\{\text{continuous maps } X \to E(n)\}/\text{homotopy}.$$

$$E^n_{red}(X) \simeq [X, E(n)]_* \leftarrow \text{pointed maps}$$

$$[X, E(n)]_* \simeq [\Sigma X, E(n+1)] \text{ so}$$

$$E(n) \simeq \Omega E(n+1) = \{ p : [0,1] \longrightarrow E(n+1) \mid p(0) = p(1) = * \}$$

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Def.: A spectrum is a collection of pointed spaces $\{E(n)\}_{n \in \mathbb{Z}^+}$

homotopy equivalences $E(n) \xrightarrow{\sim} \Omega E(n+1)$.

A spectrum determines a coh theory using Brown rep thm.

$$\mathrm{Ho}(\mathrm{Sp}) = \{\text{Spectra}\}\left[\text{homotopy equiv}^{-1}\right]$$

$$\begin{tikzcd} & HA \arrow[r, phantom, "\in"] & \mathrm{Ho}(\mathrm{Sp}) \\ A \arrow[ur, mapsto] \arrow[r, phantom, "\in"] & \mathrm{Ab} \arrow[ur, hook] \end{tikzcd}$$

fully faithful.

$A \longrightarrow H^*(\ \ , A) \longrightarrow$ some spectrum

$HA = \{K(A,n)\}_{n \geq 0}$

Eilenberg-Maclane space.

+contractible in deg < 0.

$\mathrm{Ho}(\mathrm{Sp})$ enlargement of Ab (can say more, e.g ~~triangulated~~ triangulated cat, $\exists$ t-structure st. Ab is the heart)

$$\begin{tikzcd} & \mathrm{Ho}(\mathrm{Sp}) \\ \mathrm{Ab} \arrow[ur, hook] \arrow[r, hook] & \mathcal{D}(\mathrm{Ab}) \arrow[u] \end{tikzcd}$$

$HM \mapsfrom M$ (via hypercohomology)

<u>not</u> fully faithful

Because:

M, N abelian groups $\rightsquigarrow$ $\mathrm{Ext}^n_{\mathrm{Ab}}(M, N) = \mathrm{Hom}_{\mathcal{D}(\mathrm{Ab})}(M, N[n])$

$\longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp})}(HM, HN[n])$

Isom in degrees 0, 1

but not in general

A, B abelian groups, can consider

$$A \times B \xrightarrow[\text{bilinear}]{} A \otimes B$$

A, B coh theories; want to study "bilinear" maps $A \times B \longrightarrow C$, i.e.

$$A^i(X) \times B^j(X) \longrightarrow C^{i+j}(X)$$

suitably compatible w suspension isoms.

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at the level of spectra:

$$\begin{tikzcd} A(i) \times B(j) \arrow[r] \arrow[dr] & C(i+j) \\ & A(i) \wedge B(j) \arrow[u] \end{tikzcd}$$

smash product

$$A(i) \times B(j) \Big/ \big(A(i) \times \{pt\} \cup \{pt\} \times B(j)\big)$$

<u>heuristic:</u>

~~using~~

$\{0\} \times B^j(X) \longmapsto \{0\}$.

& $A^i(X) \times \{0\} \mapsto \{0\}$.

not homotopy equivalence

$$\forall k \in \mathbb{Z} \quad \begin{tikzcd} A(k) \wedge B(k) \arrow[r, bend left] \arrow[d] & \Omega\big(A(k) \wedge B(k+1)\big) \arrow[d] & A(k+1) \wedge B(k+1) \arrow[d] \\ C(2k) \arrow[r, "\sim"] & \Omega C(2k+1) & C(2k+2) \end{tikzcd}$$

prespectrum

associated spectrum: $A \wedge B$.

<u>Def</u>: A <u>prespectrum</u> is a collection of pointed spaces $\{E(n)\}_{n \in \mathbb{Z}}$ + maps $E(n) \to \Omega E(n+1)$.

<u>Note</u>: If $E(n)$ is a prespectrum, can form associated spectrum

$$E(n) \rightsquigarrow \varinjlim \Omega^k E(n+k).$$

not obviously commutative or associative, but

$\triangle$ $Ho(Sp)$ is a symmetric monoidal category under $\wedge$.

$\hookrightarrow$ not quite good enough, but we'll pretend it is...

Let M, N be abelian groups (or chain complexes)

$$\rightsquigarrow \begin{tikzcd} HM \wedge HN \arrow[r] \arrow[dr] & H(M \otimes N) \\ & H(M \overset{L}{\otimes} N) \arrow[u] \end{tikzcd}$$

J. Lurie

If R is a ring,

$$HR \wedge HR \longrightarrow H(R \otimes R) \xrightarrow{m} HR \quad \text{(ring spectrum)}$$

Def: Let A be an associative ring spectrum (need better than in $Ho(Sp)$)
(A_∞)

$$THH(A) := A \underset{A \wedge A^{op}}{\wedge} A \longleftarrow \text{spectrum.}$$

relative version of smash product

computed via

$$\simeq \left| \begin{array}{c} A \wedge A \wedge A \\ \downarrow \;\; \downarrow \downarrow \\ A \wedge A \\ \downarrow \downarrow \\ A \end{array} \right| \longleftarrow \text{geometric realization.}$$

Let A be a commutative ring.

natural map

$$THH(A) := THH(HA) = HA \underset{HA \wedge HA}{\wedge} HA \longrightarrow H\left(A \overset{L}{\underset{A \overset{L}{\otimes} A}{\otimes}} A\right)$$

Def: Let E be a spectrum

$$\pi_n E := E^{-n}(pt) \simeq \pi_{n+m} E(m) \quad \text{as soon as this makes sense}$$

Ex.: If M is a chain complex $\pi_* HM \simeq H_*(M)$

$$\rightsquigarrow \quad \pi_* THH(A) \longrightarrow HH_*^{\mathbb{Z}}(A)$$

Go back to $\mathbb{F}_p$ example: $\pi_* THH(\mathbb{F}_p) \longrightarrow HH_*^{\mathbb{Z}}(\mathbb{F}_p) \simeq$ divided power algebra over $\mathbb{F}_p$ on one gen. δ in degree 2.

Thm (Bökstedt): $\pi_* THH(\mathbb{F}_p) \simeq \mathbb{F}_p[\delta] \longrightarrow \mathbb{F}_p\left[\frac{\delta^n}{n!}\right]$