Logarithmic $\mathcal{O}\mathbb{B}_{dR}$ (joint w X. Zhu)

§1. Introduction:

$[K:\mathbb{Q}_p]<\infty$ X connected smooth alg variety $/k$

$\mathbb{L}$: $\mathbb{Q}_p$-local systems on $X_{\acute{e}t}$.

de Rham rigidity: If $\mathbb{L}_x$ is de Rham for some $x\in X$ (x,y closed points)

then $\mathbb{L}_y$ — " — $y\in Y$.

In fact, $\mathbb{L}^{an}$ is de Rham local system on X^{an}_{et}

$\exists$ filtered vector bundle M on X^{an}_{et} with IC + transversality st.

$$\mathbb{L}^{an}\otimes \mathcal{O}\mathbb{B}_{dR} \simeq M\otimes \mathcal{O}\mathbb{B}_{dR}$$

$$\text{“}\mathcal{O}_X \hat{\otimes} \mathbb{B}_{dR}\text{”}.$$

Theorem A: $M = D_{dR}(\mathbb{L}^{an})$ algebraic

$\rightsquigarrow D_{dR}(\mathbb{L})$ V.b. IC, trans.

§ log adic space (H. Diao & F. Tan)

Def'n: X noetherian adic space $/k$

- pre-log structure $\alpha: M \longrightarrow \mathcal{O}_{X_{\acute{e}t}}$.
- log structure $\alpha^{-1}(\mathcal{O}^{\times}_{X_{\acute{e}t}}) \longrightarrow \mathcal{O}^{\times}_{X_{\acute{e}t}}$ isom.

Associate the log structure

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & M^a \\ \uparrow & & \uparrow \\ \alpha^{-1}(\mathcal{O}^{\times}_X) & \longrightarrow & M \end{array}$$

Ex: $\mathbb{A}^r = \mathrm{Spa}(k\langle T_1,\dots,T_r\rangle, \mathcal{O}_k\langle T_1,\dots,T_r\rangle, \mathbb{N}^r)$.

<u>Def'n</u>: $f: X \to Y$ map of fs log adic spaces is called

<u>Kummer étale</u> if étale locally $\exists$ chart $P \to Q$ of f

s.t. 1). $P \to Q$ is Kummer

2). $X \to Y \times_{\mathrm{Spa}(K[P], \mathcal{O}_K[P])} \mathrm{Spa}(K[Q], \mathcal{O}_K[Q])$ is

étale.

Define "finite Kummer étale covers" similarly.

$\rightsquigarrow X_{két}$, X_{proket} ("old" pro-étale site)

§. $\mathcal{O}\mathbb{B}_{dR}^{log}$: heuristic " $d \log T_i = \frac{dT_i}{T_i} = \frac{d \log T_i}{[T_i^{\flat}]}$ "

Let $\mathcal{U} = \varprojlim \mathcal{U}_i = \varprojlim (\mathrm{Spa}(R_i, R_i^+), M_i, \alpha_i)$ $\hat{\mathcal{U}} = \mathrm{Spa}(R, R^+)$

$(M, \alpha) = \varinjlim (M_i, \alpha_i)$, $M^{\flat} := \varprojlim_p M$

$\alpha^{\flat}: M^{\flat} \to R^{\flat}$

$(m_0, m_1, \dots) \mapsto (\alpha(m_0), \alpha(m_1) \dots)$.

Let $\theta_i: S_i := R_i^+ \otimes_{W(K^{\flat})} A_{inf}(R, R^+)\left[\frac{1}{p}, \left\{\frac{\theta([\alpha^{\flat}(m)])}{[\alpha^{\flat}(m)]}\right\}_{m \in R_i^+}\right] \to R$

<u>Def'n</u> Let $\mathcal{O}\mathbb{B}_{dR}^{+log}$:= sheafification of the presheaf $\mathcal{U} \mapsto \varinjlim \widehat{S_i}_{\ker \theta_i}$

<u>Example</u>: $f: X \to \widetilde{\mathbb{A}}^r = \varprojlim \mathbb{A}_n^r = \left(\mathrm{Spa}(K_n\langle T_1^{1/n}, \dots, T_r^{1/n}\rangle, \mathcal{O}_{K_n}\langle T_1^{1/n}, \dots, T_r^{1/n}\rangle)\right)$

Kummer étale as map of log adic spaces + étale on underlying adic spaces

$\widetilde{X} := X \times_{\mathbb{A}^r} \widetilde{\mathbb{A}}^r$

$\mathcal{O}\mathbb{B}_{dR}^{+,log}|_{\widetilde{X}} \simeq \mathbb{B}_{dR}(\widetilde{X})[[X_1, \dots, X_r]]$

$\frac{T_i}{[T_i^{\flat}]} - 1 \leftarrow X_i$

$$dX_i = \frac{T_i}{[T_i^\flat]}, \quad \frac{dT_i}{T_i} = \underbrace{(1+X_i)}_{\text{invertible}} \, d\log T_i$$

⇝ Poincaré lemma holds

⇝ $gr^i \mathcal{O}\mathbb{B}_{dR}^{\log} \simeq t^i \widehat{\mathcal{O}}_X \left[\left\{ \log \frac{T_i}{[T_i^\flat]} \right\}_{1 \le i \le r} \right]$

Let $\mathbb{L}$ be a d-dimensional $\mathbb{Q}_p$-local system on $X_{k\acute{e}t}$ ← log smooth

$K = \widehat{K_\infty}$

Consider the diagram:

$$\begin{array}{ccc} X_{prok\acute{e}t} & \longleftarrow & X_{prok\acute{e}t}/X_K \\ \downarrow \nu & & \downarrow \nu' \\ X_{k\acute{e}t} & \xleftarrow{\mu} & (X_K)_{k\acute{e}t} \end{array}$$

$$RH^{\log}(\mathbb{L}) := R\nu'_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{dR}^{\log})$$

<u>Theorem B</u>: 1). $RH^{\log}(\mathbb{L}) = \nu'_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{dR}^{\log})$ filtered vector bundle on $X_{\mathbb{B}_{dR}}$

is a d-dim'l filtered vector bundle on $X_{\mathbb{B}_{dR}}$ with ILC + trans + $Gal(\overline{K}/K)$-action

2). $D_{dR}^{\log}(\mathbb{L}) := \mu_* RH^{\log}(\mathbb{L})$

filtered cokernel + ILC (integrable log connection)

+ transversality

$X_{\mathbb{B}_{dR}}$: ringed space (topos) $\left((X_K)_{k\acute{e}t}, \ \mathcal{O}_X \hat{\otimes} \mathbb{B}_{dR} \right)$

$$\mathcal{O}_X \hat{\otimes} \mathbb{B}_{dR} = \left(\varprojlim \left(\mathcal{O}_X \otimes \mathbb{B}_{dR}/t^k \right) \right) \left[\tfrac{1}{t} \right]$$

Now, let $X = \overline{X} \setminus D$ be a quasi-proj variety
D normal crossings divisor.

$\mathbb{L}$ on $X_{\acute{e}t}$

$Y = (\overline{X}, D)$ log scheme, $i : X_{\acute{e}t} \longrightarrow Y_{k\acute{e}t}$

$\rightsquigarrow D_{dR}^{\log}(i_*^{an}\mathbb{L}^{an})$ on $Y_{k\acute{e}t}^{an}$

\+ GAGA $\Rightarrow D_{dR}(\mathbb{L}^{an})$ is algebraic.

(In progress) $\mathbb{L}' = i_*^{an}\mathbb{L}^{an}$

1). $D_{dR}^{\log}(\mathbb{L}')$ is a vector bundle

& $\mu^*(D_{dR}^{\log}(\mathbb{L}')) \simeq \underbrace{RH^{\log}(\mathbb{L}^{an})}_{\text{on } X_{\mathbb{B}_{dR}}}$

· residues of $RH^{\log}(\mathbb{L}^{an})$ along divisors $\in \mathbb{B}_{dR}^{Gal(\overline{K}/K)} = K$ extend $D_{dR}(\mathbb{L})$

· can extend $D_{dR}(\mathbb{L})$ to $\overline{X}$ using the residues.

2). Comparison

$H^*_{\acute{e}t}(X, \mathbb{L}) \simeq H^*_{\acute{e}t}(X^{an}, \mathbb{L}^{an}) \simeq H^*(Y_{k\acute{e}t}, i_*^{an}\mathbb{L}^{an})$

$H^*_{dR}(X, D_{dR}(\mathbb{L})) \simeq H^*_{dR}(\overline{X}, D_{dR}^{\log}(\mathbb{L})) \simeq H^*_{dR}(Y^{an}, D_{dR}^{\log}(Y^{an}))$