 D. Kaledin (1)

## Co-periodic cyclic homology

- Two types of derived de Rham cohomology (Beilinson, Bhatt)
- $k$ ring, $A/k$ commutative ring $\rightsquigarrow$ $A_\bullet$ simplicial resolution
- Product total

$$\begin{tikzcd}
A_0 \arrow[r] & \Omega^1(A_0) \arrow[r] & \cdots \\
A_1 \arrow[r] \arrow[u] & \Omega^1(A_1) \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

$$\Lambda^i \Omega^1(A) \Rightarrow H^\bullet_{DR}(\operatorname{Spec} A) \qquad \to B_{dR}$$

- If $\mathbb{Q} \in A$ trivial
- If $pk = 0$ $\quad \Lambda^i \Omega^1(A)^{(p)} \Rightarrow \overline{H}^\bullet_{dR}(\operatorname{Spec} A)$

$B_{cris}$

A associative (unital) $/k$ $\quad A_\bullet$ DG algebra.

$$\overbrace{\begin{tikzcd}
A_\bullet \arrow[r, "1-\sigma"] & A_\bullet \\
A_\bullet^{\otimes 2} \arrow[r, "1-\sigma"'] \arrow[u] & A_\bullet^{\otimes 2} \arrow[u] \\
A^{\otimes 3} \arrow[r, "1-\sigma"] \arrow[u] & A^{\otimes 3} \arrow[from=u]
\end{tikzcd}}^{CH_\bullet(A_\bullet)}$$

$\sigma : A_\bullet^{\otimes n}$ order $n$ permutation $(-1)^{n+1}$

$HH_\bullet(A_\bullet)$ is quasi-iso invariant

derived Morita invariant

$D(A^\bullet) \simeq D(B^\bullet) \Rightarrow HH_\bullet(A_\bullet) \simeq HH_\bullet(B_\bullet)$.

$X$ alg variety $\quad D(A^\bullet) \simeq D(X)$.

$B : CH_\bullet(A) \longrightarrow CH_{\bullet+1}(A)$ Connes-Tsygan differential

discovered by G. Rinehart 1963

Let $CP_\bullet(A_\bullet) := \langle CH_\bullet(A_\bullet)((u)), \beta + uB \rangle$

$\deg u = 2$ cohomological

Then $HP_\bullet(A_\bullet)$ is periodic cyclic homology of $A$.

$HH_\bullet(A)((u)) \Longrightarrow HP_\bullet(A)$ so $HP_\bullet$ is derived Morita invariant.

<u>Def.</u> $\overline{CP}_\bullet(A_\bullet) = \langle CH(A)((u^{-1})), \beta + uB)$

$\overline{HP}_\bullet(A_\bullet)$ is co-periodic cyclic homology of $A$.

If $k \ni \mathbb{Q}$, $\overline{HP}_\bullet(A) = 0$.

<u>Thm 1:</u> $pk = 0$. Then we have a sp sequence

$$HH_\bullet(A_\bullet^{(p)})((u^{-1})) \Rightarrow \overline{HP}(A_\bullet).$$

<u>Cor</u>: $\overline{HP}_\bullet(A_\bullet)$ is derived Morita invariant.

<u>Thm</u>: If $A_\bullet$ is smooth and proper

$$\overline{HP}_\bullet(A_\bullet) \longrightarrow HP_\bullet(A_\bullet) \to HP_\bullet(A_\bullet) \longrightarrow HP_\bullet(A_\bullet) \otimes \mathbb{Q} \to \dots$$

<u>How this works</u>:

1. Define everything in terms of cyclic objects:

$$E_\bullet : \Lambda \longrightarrow C_\bullet(k)$$

2. Use filtrations

Assume that $E_\bullet$ is filtered

$$\begin{tikzcd}
F^{i+1} E_\bullet([1]) \arrow[r, "1-\sigma"] & E^{i+1}([1]) \\
F^{i+2} E_\bullet([2]) \arrow[u] \arrow[r, "1-\sigma"] & E^{i+2}([2]) \arrow[u] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

If we equip $E_\bullet$ with the stupid filtration then $\overline{CP}_\bullet(E)$ is the filtered completion $cp_\bullet(E_\bullet) = CH_\bullet(E_\bullet)[u, u^{-1}]$

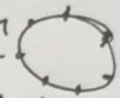
$F^i M$

$\varinjlim_{i<0} \varprojlim_{i>0} F^i M$

$\Lambda = \Lambda_\infty / \sigma$

$\Lambda_n = \Lambda_\infty / \sigma^n$

$$\begin{array}{c} \Lambda_n \\ \pi \downarrow \\ \Lambda \end{array}$$ Grothendieck fibration $B\,\mathbb{Z}/n\mathbb{Z}$.

$i_n : \Lambda_n \to \Lambda$ edgewise subdivision $\tau$ 

$\tau^n = id.$

$i_\infty : \Lambda_\infty \to A_\infty$

$S \longmapsto [n] \times S$

For any filtered $(M, F^\bullet)$ let $M^{[n]}$ be $M$ with $F^i M^{[n]} = F^{in} M$

$i_\infty : \Lambda_n \to \Lambda$ is cofinal

In particular, $HP_\bullet(i_n^* E_\bullet) \longrightarrow HP_\bullet(E_\bullet)$ is an isom.

<u>Prop</u>: $p \cdot k = 0$, $p$ is a prime. Take $E_\bullet : \Lambda \longrightarrow C_\bullet(k)$

equip it w stupid filtration $\longrightarrow$ « bête »

Then we have a filtered quasi-isomorphism:

$$cp_\bullet(i_p^* E_\bullet^{[p]}) \simeq cp(E_\bullet)^{[p]}$$

<u>Thm</u> (Hochschild-Kostant-Rosenberg).

If $A$ smooth $k$-algebra

$$\begin{array}{ccc} HH_i(A/k) & \simeq & \Omega^i \\ \downarrow d & \circlearrowleft & \downarrow \\ HH_{i+1}(A/k) & \xrightarrow{\sim} & \Omega^{i+1} \end{array}$$

$HH_\bullet(A)[[u]]$

$\Downarrow$ ~~In general~~,

$HP_\bullet(A_\bullet)$

$H^\bullet_{DR}(\operatorname{Spec} A)[[u]] \Rightarrow HP_\bullet(A_\bullet)$ on $A^q$

where $A^q : \Lambda \to \mathcal{C}_\bullet(k)$

D Kaledin (4)

Fontaine-Laffaille modules

$$\langle M, F^\bullet, \varphi \rangle \quad \varphi^i : F^iM \to M$$

$$\varphi^i|_{F^{i+1}} = p\,\varphi^{i+1}$$

These can be packaged nicely in terms of $\Delta$, $\Delta_p$.

$(M, F^\bullet, \overline{M}, V_\bullet)$ conj filtration, $\varphi$ interpolates between these

$$\overline{M} \to M$$

$$B_{cris} \to B_{dR}.$$