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## THH and cyclotomic spectra

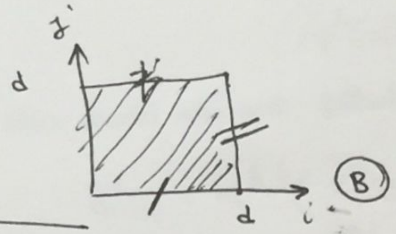
$A/k \rightsquigarrow HH(A/k) \circlearrowleft S^1$ $A/k$ smooth $HH_*(A/k) \xleftarrow{\sim} \Omega^*_{A/k}$ $\circlearrowleft d$.

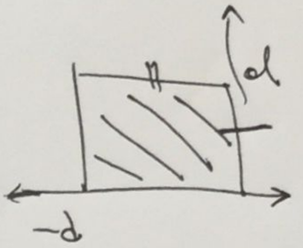
$X$ scheme $/k$ $\rightarrow$ $H_{dR}$ = hypercoh of de Rham complex

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{dR}(X/k) \qquad Ⓑ$$

Assume smooth $\rightarrow$

$$E^2_{i,j} = H^{-i}(X, \Omega^j_{X/k}) \Rightarrow HH_{i+j}(X/k) \qquad Ⓐ$$





Ⓐ

Now we come to circle action

$HH(X/k) \circlearrowleft \mathbb{T}$ ~~$HH_*(X/k) \circlearrowleft d$.~~

(Peter's talk) $\rightarrow$ $HC^-(X/k)$

$$\begin{tikzcd} HP(X/k) \arrow[r, phantom, "="] & HH(X/k)^{t\mathbb{T}} & \text{cone.} \arrow[l] \\ HC^-(X/k) \arrow[u] \arrow[r, phantom, "="] & HH(X/k)^{h\mathbb{T}} \arrow[u] & \\ \Sigma HC(X/k) \arrow[u] \arrow[r, phantom, "="] & \Sigma HH(X/k)_{h\mathbb{T}} \arrow[u, "N"'] & \text{"norm map"} \arrow[lu] \end{tikzcd}$$

general picture:

$$\begin{tikzcd} X^{tG} \\ X^{hG} \arrow[u] \\ (X \otimes_{\mathbb{S}} \mathbb{S}^{\mathfrak{g}})_{hG} \arrow[u, "N"] \end{tikzcd}$$

(Jacob's talk)

$$\begin{tikzcd} \mathbb{Z} \arrow[d, dash] & D(\mathrm{Mod}_{\mathbb{Z}}) \arrow[d, "\text{forgetful functor}"] & H \arrow[d, mapsto] \\ \mathbb{S} & D(\mathrm{Mod}_{\mathbb{S}}) & HH \end{tikzcd}$$

$$THH(X) := THH(X/\mathbb{S}) := HH(X/\mathbb{S}) \circlearrowleft \mathbb{T}$$

$$TP(X) := HP(X/\mathbb{S}) \hookleftarrow \text{"De Rham coh of } X/\mathbb{S}\text{"}$$

Yesterday ⤳

$$\begin{tikzcd} HH_*(\mathbb{F}_p/\mathbb{Z}) \arrow[r, phantom, "="] & \Gamma_{\mathbb{F}_p}\{\delta\} \\ HH_*(\mathbb{F}_p/\mathbb{S}) \arrow[u] \arrow[r, phantom, "="] & S_{\mathbb{F}_p}\{\delta\} \arrow[u] \end{tikzcd}$$

( first miracle that happens when you descend from $\mathbb{Z}$ to $\mathbb{S}$ )

Bockstedt periodicity

$$E^2_{i,j} = H^{j-i}(X, \mathbb{Z}(j)) \Rightarrow K^{-i-j}(X)$$

"weight spectral sequence"

$$\psi^k(x) = k^j \cdot x$$

there should also exist filtration for alg K-theory:

$$\begin{tikzcd} E^2_{i,j} = H^{j-i}(X, \mathbb{Z}(j)) \arrow[r, Rightarrow] \arrow[d, "\text{map doesn't exist yet}"'] & K_{i+j}(X) \arrow[d, "\text{Chern character}"] \\ E^2_{i,j} = H^{j-i}(X, W(j)) \arrow[r, Rightarrow] & TP_{i+j}(X) \end{tikzcd}$$

"motivic cohomology"

(cyclotomic trace)

there should also exist:

If $X/\mathbb{F}_p$ is smooth then $W(j) \simeq W\Omega^{\cdot}_X$

if $X/\mathcal{O}_C$ smooth then $W(j) \simeq A\Omega^{\cdot}_X$

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$$A \longrightarrow \Omega^{\cdot}_{A/k}$$

universal

map of graded k-algebras

$$A \longrightarrow THH(A)$$

A commutative $E_\infty$-ring.

map of $E_\infty$-rings

$E_\infty$-ring w $\mathbb{T}$-action

map is universal for maps A → $E_\infty$-rings w $\mathbb{T}$-action.

Also have $A \longrightarrow HH(A/k)$

object in $\infty$-category of complexes of k-modules w circle action

; But what is special when going from $\mathbb{Z}$ to $\mathbb{S}$ is:

$$X \xrightarrow{\Delta_p} \big(\underbrace{X \otimes_{\mathbb{S}} \cdots\cdots \otimes_{\mathbb{S}} X}_{p \text{ times, } p \text{ prime}}\big)^{tC_p}$$

$C_p \subset \mathbb{T}$
cyclic gp of order $p$.

Tate diagonal map
of spectra.

$X \mapsto (X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}$ is an exact functor
to define $\Delta_p$ enough to do it for $\mathbb{S}$

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$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\Delta_p"'] & \mathrm{THH}(A) \arrow[d, dashed, "\varphi_p\ \exists!"] \\
(A \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} A)^{tC_p} \arrow[r] & \mathrm{THH}(A)^{tC_p}
\end{tikzcd}$$

By universal property

$C_p$-equivariant

— still have an action of $\mathbb{T}/C_p$

$$\begin{tikzcd}
\mathbb{T} \arrow[d] & \ni z \arrow[d, leftrightarrow] \\
\mathbb{T}/C_p & z^{1/p}C_p
\end{tikzcd}$$

$\varphi_p$ = "Frobenius map" on THH Only exists as a map of spectra.
(2nd miracle)

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$X$ spectrum w $C_{p^2}$-action

$$\begin{tikzcd}
(X_{hC_p})_{h(C_{p^2}/C_p)} \arrow[r, "\simeq"] \arrow[d] & X_{hC_{p^2}} \arrow[r, "N_{C_{p^2}}"'] & X^{hC_{p^2}} \arrow[r] \arrow[d, "\wr"] & X^{tC_{p^2}} \arrow[d, "\wr"] \\
(X_{hC_p})^{h(C_{p^2}/C_p)} \arrow[rr] \arrow[d] & & (X^{hC_p})^{h(C_{p^2}/C_p)} \arrow[r] & (X^{tC_p})^{h(C_{p^2}/C_p)} \\
(X_{hC_p})^{t(C_{p^2}/C_p)} & & &
\end{tikzcd}$$

$\simeq 0$ if $X$ is bounded below ("Tate lemma"
Nikolaus-Scholze)
(homotopy gps
become 0 in low enough degree)

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can build tower:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] \\[-20pt]
\mathrm{TR}^3(X;p) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
\mathrm{TR}^2(X;p) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
\mathrm{TR}^1(X;p) := \mathrm{THH}(X)
\end{tikzcd}$$

define this

$X = \operatorname{Spec} A$ ...

$$\begin{tikzcd}
\cdots \arrow[d, shift right] \arrow[d, shift left] & \cdots \arrow[d, shift right] \arrow[d, shift left] \\
\mathrm{TR}^2_0(A;p) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \arrow[r, phantom, "="] & W_2(A) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
\mathrm{TR}^1_0(A;p) \arrow[r, phantom, "="] & A
\end{tikzcd}$$

Level 2:

$$\begin{tikzcd}
\mathrm{TR}^2(X;p) \arrow[r, "f"] \arrow[d, "R"] \arrow[rr, bend left, "F"] \arrow[dr, phantom, "\mathrm{Cart}"] & \mathrm{THH}(X)^{hC_p} \arrow[r] \arrow[d] & \mathrm{THH}(X) \arrow[d, equal] \\
\mathrm{TR}^1(X;p) \arrow[r, "\varphi_p"] \arrow[d, equal] & \mathrm{THH}(X)^{tC_p} & \mathrm{TR}^1(X;p). \\
\mathrm{THH}(X) & &
\end{tikzcd}$$

Level 3

$$\begin{tikzcd}
 & \mathrm{TR}^3(X;p) \arrow[rr] \arrow[dd, "R"] \arrow[dl, squiggly, no head] & & \mathrm{THH}(X)^{hC_{p^2}} \arrow[d] \\
W_3\Omega_X & & \mathrm{Cart} & \mathrm{THH}(X)^{tC_{p^2}} \arrow[d, "\wr"', "\text{Tate Lemma}"] \\
 & \mathrm{TR}^2(X;p) \arrow[r, "f"'] & \mathrm{THH}(X)^{hC_p} \arrow[r, "\varphi_p^{hC_p}"'] & \left(\mathrm{THH}(X)^{tC_p}\right)^{h(C_{p^2}/C_p)}
\end{tikzcd}$$