1. Geometric Langlands on GL_1

Let X be a proper smooth curve over a field. We want to prove that $\pi_1(X)^{\mathrm{ab}} \cong \pi_1(\mathrm{Jac}_X)$. We do this by showing $\pi_1(X)^{\mathrm{ab}} \cong \pi_1(\mathrm{Div}^d)$ where $\mathrm{Div}^d = X^d/\sigma_d$ for $d > 1$; then use the Abel–Jacobi morphism $\pi_1(\mathrm{Div}^d) \overset{\pi_1(\mathrm{AJ}^d)}{\to} \pi_1(\mathrm{Pic}^d)$, which is an étale locally trivial fibration in $\mathbb{P}^{d-g}$ when $d > 2g-2$ by Riemann-Roch where g is the genus of X. The point is that the fibers are simply connected. We then get the result via $\pi_1(\mathrm{Pic}^d) \xrightarrow{\sim} \pi_1(\mathrm{Pic}^0)$ using the group structure on the Picard moduli scheme.

In the dual setting of étale local systems, let $\mathcal{E}$ be a $\overline{\mathbb{Q}}_\ell$-rank 1 étale local system on X. Then consider

$$\Sigma^d : (Div^1)^d = X^d \to X^d/\sigma_d = \mathrm{Div}^d$$

the "sum of d degree 1 divisors" morphism and set

$$\mathcal{E}^{(d)} := (\Sigma_*^d \mathcal{E}^{\boxtimes d})^{\sigma_d}.$$

This is a rank 1 $\overline{\mathbb{Q}}_\ell$-rank étale local system on Div^d (in general this would only give a perverse sheaf, but in the rank 1 case we again get a local system). For $d \gg 0$, $\mathcal{E}^{(d)}$ descends along AJ^d to $\mathcal{F}^{(d)}$ on Pic^d. The family $(\mathcal{F}^{(d)})_{d\gg 0}$ extends to all d using the group structure on Pic again.

In the previous discussion, we may replace the coarse moduli space with the moduli stack because the map is a $\mathbb{G}_m$-gerbe. But later, we need to use the stack!

2. Content of Fargues's conjecture

Let E be either $\mathbb{F}_q((\pi))$ or a finite extension of $\mathbb{Q}_p$ with $\mathbb{F}_q = \mathcal{O}_E/\pi$. Consider the category $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ of perfectoid spaces over $\overline{\mathbb{F}}_q$ with the pro-étale topology.

$$\mathrm{Bun}_{\mathrm{GL}_1} = \mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d$$

We have an isomorphism

$$\mathrm{Pic}^d \cong [\bullet/\underline{E}^\times]$$

(classifying stack of pro-étale $\underline{E}^\times$-torsors) given by

$$\mathcal{L} \to \mathrm{Isom}(\mathcal{O}(d), \mathcal{L})$$

where $\mathcal{O}(d)$ is the "canonical" degree line bundle on the (relative) curve.

Div^1 is the sheaf of degree-1 effective line bundles on the curve. One has the isomorphism

$$\mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \cong \mathrm{Div}^1.$$

This morphism is defined by the following construction. For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, $X_S \to \mathrm{Spa}(E)$ is the "family of curves parametrized by S". If $S^\sharp$ is an untilt of S over E, we obtain a Cartier divisor $S^\sharp \hookrightarrow X_S$ over $\mathrm{Spa}(E)$ (defined by $\theta = 0$ where θ is Fontaine's theta).

Thus, we have two different pictures

$$\begin{array}{ccc} X_S^\diamond = (S \times \mathrm{Spa}(E)^\diamond)/\varphi_S^{\mathbb{Z}} & \qquad & \mathrm{Div}_S^1 = (S \times \mathrm{Spa}(E))^\diamond/\varphi_E^{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \mathrm{Spa}(E)^\diamond & & S \end{array}$$

where by contrast with the classical case, the two things on top are not isomorphic (but have the same étale topos).

Warning: Div^1 is a non-spatial diamond (it is not quasiseparated). That said, $\mathrm{Div}^1 \to \mathrm{Spa}(\overline{\mathbb{F}}_q)$ is spatial, and even smooth proper.

For $d \geq 1$,

$$\mathrm{Div}^d(S) = \{(\mathcal{L}, u)\}// \sim$$

where $\mathcal{L}$ is a degree d line bundle on X_S and $u \in H^0(X_S, \mathcal{L})$ is fiberwise on S nonzero.

Theorem: $(\mathrm{Div}^1)^d/\sigma_d \cong (\mathrm{Div}^d)$ as pro-étale sheaves. (The quotient on the left is pro-étale, not étale!) Moreover, the symmetrization morphism $(\mathrm{Div}^1)^d \to (\mathrm{Div}^1)^d/\sigma_d$ is quasi-pro-étale with finite fibers.

(This is a consequence of the Fargues–Fontaine theorem on factorization of primitive elements.)

3. Abel–Jacobi

$$\mathrm{AJ}^d : \mathrm{Div}^d \to \mathrm{Pic}^d, \qquad D \mapsto \mathcal{O}(D)$$

Let $\mathbb{B}$ be the period ring, as a pro-étale sheaf with action of φ, where $\mathbb{B}(S) = \mathcal{O}(Y_S)$ where $X_S = Y_S/\varphi^{\mathbb{Z}}$. Then

$$H^0(X_S, \mathcal{O}(d)) = \mathbb{B}(S)^{\varphi=\pi^d}$$

where $\mathbb{B}^{\varphi=\pi^d}$ is an *absolute Banach–Colmez space* in the sense that its pullback to any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ is a BC space.

Proposition: $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$ is a diamond. Moreover, the quotient of this by $\underline{E}^\times$ may be identified with Div^d, with the composition with AJ^d giving the obvious map to $[\bullet/\underline{E}^\times]$.

Upshot (and the main point): AJ^d is a pro-étale locally trivial fibration with fibers of the form $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$.

4. Geometric local class field theory

Via $\mathrm{Div}^1 = \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}}$, $\overline{\mathbb{Q}}_\ell$-local systems on Div^1 correspond to $\overline{\mathbb{Q}}_\ell$-representations of the Weil group W_E. That is, "$\pi_1(\mathrm{Div}^1) = W_E$."

$$\begin{tikzcd} W_E \arrow[r, equal] \arrow[d] & \pi_1(\mathrm{Div}^1) \arrow[r, "\pi_1(\mathrm{AJ}^d)"] & \pi_1(\mathrm{Pic}^1 = [\bullet/\underline{E}^\times]) \arrow[r, equal] & E^\times \\ W_E^{\mathrm{ab}} \arrow[urrr, "Art_E^{-1}"'] & & & \end{tikzcd}$$

where Art_E is Artin reciprocity map. A character $\chi : W_E \to \overline{\mathbb{Q}}_\ell^\times$ gives rise to $\mathcal{E}_\chi^{(1)}$, a rank 1 $\overline{\mathbb{Q}}_\ell$-local system on Div^1. We wish to prove it descends to Pic^1 via AJ^1, by following the model from geometric Langlands. That is, for $d \gg 1$, form

$$\mathcal{E}_\chi^{(d)} = \left(\Sigma_*^d \mathcal{E}_\chi^{(1)\boxtimes d}\right)^{\sigma_d}$$

(this is rank 1 over Div^d; some work required to check this).

Theorem : For $d > 1$, $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$ is simply connected. That is, any finite étale cover splits. (Not claiming that the de Jong fundamental group is trivial! Also, the pullback to a perfectoid algebraically closed field is not simply connected! we really need to work "absolutely" over $\overline{\mathbb{F}}_q$)

From the theorem, we see that for $d > 1$, $\mathcal{E}_\chi^{(d)} = (\mathrm{AJ}^d)^* \mathcal{F}_\chi^{(d)}$ where $\mathcal{F}_\chi^{(d)}$ is a local system on $\mathrm{Pic}^d = [\bullet / \underline{E}^\times]$; and this last corresponds to the character $E^\times \to \overline{\mathbb{Q}}_\ell^\times$ that we wanted.

5. Notes on the proof of the theorem

Start with the equal-characteristic case $E = \mathbb{F}_q((\pi))$. In this case, $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$ is a perfectoid space, namely $\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]])$ minus the nonanalytic point $V(x_1, \ldots, x_d)$. Then use Elkik plus GAGA to reduce to showing that $\mathrm{Spec}(\overline{\mathbb{F}}_q[[x_1, \ldots, x_d]]) \setminus V(x_1, \ldots, x_d)$ is simply connected for $d > 1$ (Zariski-Nagata purity).

In the mixed-characteristic case (much harder):

(1) Reduce to proving purity for $\mathbb{B}^{\varphi=\pi^d} \setminus \{0\} \to \mathbb{B}^{\varphi=\pi^d}$.
(2) Dévissage to a purity statement in rigid geometry over $\mathbb{Q}_p$.
(3) Let $X/\mathbb{Q}_p$ be a rigid analytic space smooth of pure dimension $d > 1$. Then finite étale covers of X are the same as over $X \setminus X(\mathbb{Q}_p)$. (For $d = 2$ this is due to Gabber. But $d > 2$ is easier.)