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Cohomology of p-adic curves (w G. Dospinescu & W. Niziol)

$C = \widehat{\overline{\mathbb{Q}}_p}$, $[L:\mathbb{Q}_p] < +\infty$

$(\mathcal{M}_n)_{n \in \mathbb{N}}$ Drinfeld tower of coverings of $\mathbb{P}^1_C - \mathbb{P}^1(\mathbb{Q}_p)$

Thm (CDN). Let V be an abs irred L-rep'n of $G_{\mathbb{Q}_p}$, $\dim V \geq 2$.

via WD, then JL.

p-adic local Langlands

dual

$$\mathrm{Hom}_{L[W_{\mathbb{Q}_p}]}\Big(V, \varinjlim_n H^1_{\acute{e}t}(\mathcal{M}_n, L(1))\Big) = \begin{cases} JL(V) \otimes \Pi(V)^* & \text{if } V \text{ is de Rham, 2-dim'l, HT wts } \{0,1\}, \\ & D_{pst}(V) \text{ irreducible as WD rep'n.} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: can hope $\mathrm{Hom}(V, H^1_{\acute{e}t}(\widehat{\mathcal{M}}_\infty, L(1))$

$$= \underset{\substack{\uparrow \\ \text{Scholze's functor}}}{JL_p}(\Pi(V)) \widehat{\otimes} \Pi(V)^*$$

for V 2 dim'l, abs irreducible (no dR etc restrictions...)

Proof uses:
- can recover $H^1_{\acute{e}t}$ inside $H^1_{pro\acute{e}t}$
- $H^1_{pro\acute{e}t}$ in terms of H^1_{HK} + de Rham complex

focus on → ✱ describe

- description of de Rham complex in terms of p-adic local Langlands correspondence (Dospinescu - Le Bras)
- $H^1_{HK} \simeq H^1_{pro\acute{e}t}(\ , \mathbb{Q}_\ell)$ as $GL_2(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$-reps.
- description of $H^1_{pro\acute{e}t}(\ , \mathbb{Q}_\ell)$. Gunayol, Faltings - Fargues

X rigid analytic curve$/C$ (smooth, connected)

<u>Thm</u>: (i) If X is proper, have:

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\text{proét}}(X, \mathbb{Q}_p(1)) \arrow[r] \arrow[d, "d\log"] & (B^+_{st} \otimes H^1_{HK}(X))^{N=0, \varphi=p} \arrow[r] \arrow[d, "\theta\otimes 1_{HK}"'] & H^1(X, \mathcal{O}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega^1(X) \arrow[r] & H^1_{dR}(X) \arrow[r] & H^1(X, \mathcal{O}) \arrow[r] & 0 \end{tikzcd}$$

(ii) X Stein $\Rightarrow (H^1(X, \mathcal{O}) = 0)$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(X)/C \arrow[r] \arrow[d, equal] & H^1_{\text{proét}} \arrow[r] \arrow[d, "d\log"] & (B^+_{st} \otimes H^1_{HK})^{N=0, \varphi=p} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}(X)/C \arrow[r] & \Omega^1(X) \arrow[r] & H^1_{dR}(X) \arrow[r] & 0 \end{tikzcd}$$

(iii) X overconvergent affinoid same diagram (HK coh defined by Grosse-Klönne)

(iv) ? X affinoid

ad hoc definition

$$0 \to \mathcal{O}(X)/C \longrightarrow H^1_{\text{proét}} \longrightarrow (B^+_{st} \otimes H^1_{HK})^{N=0, \varphi=p} \oplus \left[\mathbb{Q}_p \widehat{\otimes} \mathcal{O}^{an}(X) / \mathbb{Q}_p \otimes \mathcal{O}^{an}(X)\right] \longrightarrow 0.$$

<u>Remarks</u>:
- in cases (i), (iii), (iv), H^1_{HK} is a finite-dim'l $\breve{\mathbb{Q}}_p$-Banach space.
  $\Rightarrow (B^+_{st} \otimes H^1_{HK})^{N=0, \varphi=p}$ is a fin Dimensional Banach Space.
- in (ii), H^1_{HK} is a countable product of fin dim'l $\breve{\mathbb{Q}}_p$-Banach spaces.
- $H^1_{\text{proét}}$ is always separated.
- several proofs (relative Fontaine rings, syntomic cohomology, symbols & p-adic integration)

X affinoid: $X \hookrightarrow \overline{X}$ proper, $\overline{X} \setminus X = \bigsqcup_{\text{finite}} D_i$

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Hyodo-Kato cohomology:

X rigid analytic curve $\rightsquigarrow$ Berkovich curve

$\Gamma^{an}(X)$ skeleton $= \{x,$ x doesn't have a nbhd which is an open disk$\}$

$\uparrow$ connected, $\overset{\text{metrizable}}{\text{locally}}$ finite graph; all pts are types 2,3

Vertices $\Sigma^{an}(X)$ nodes genus $\geqslant 1$

genus 0 and valence $\geqslant 3$

genus 0 boundary of the curve.

$\Gamma^{an}(X) \smallsetminus \Sigma^{an}(X) =$ disjoint union of open segments

or a $\underset{\text{Tate curve}}{\text{circle}}$ or a $\underset{\mathbb{G}_m}{\text{line}}$ or a $\underset{\text{punctured open ball}}{\text{half-line}}$

$\Gamma^{an}(X)$ can be empty if

$X = \mathbb{P}^1, \mathbb{A}^1,$ open ball.

$\Sigma^{an}(X)$ empty if one of the cases above

$\underset{\text{open}}{\text{or}}$ annulus or Tate curve

Assume $\Sigma^{an}(X) \neq \emptyset$

edge a $\longleftrightarrow$ annulus $Y_a = \{\alpha < v_p(z) < \beta\}$

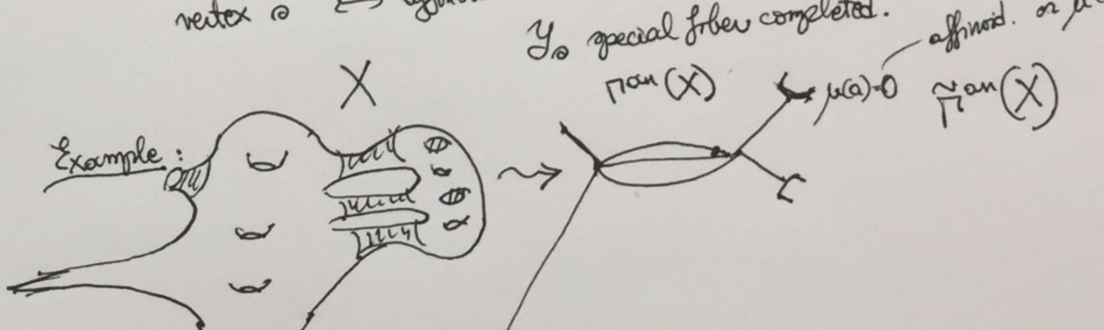
$\mu(a) \hookrightarrow \beta - \alpha$

vertex s $\longleftrightarrow$ affinoid Z_s w good reduction

Y_s special fiber completed.

affinoid. or $\mu(a) = 0^+$ if overconvergent.

Example:



$$H^1_{HK}(X) = H^1(\tilde{\Gamma}^{an}(X), \breve{\mathbb{Q}}_p) \oplus H^1_c(\tilde{\Gamma}^{an}(X), \breve{\mathbb{Q}}_p)^*[1]$$

$$\oplus \prod_{s\in\Sigma^{an}(X)} H^1_{rig}(Y_s/\mathcal{O}_{\breve{\mathbb{Q}}_p})^{\textcircled{1}} \rightsquigarrow \text{take the part of slope 1 if } s \text{ on boundary}$$

φ : natural Frob on 1st term, pFrob on second term,

$(\phi(a))_a \longrightarrow (\mu(a)\phi(a))$

N : $H^1_c(\ )^* \longrightarrow H^1(\tilde{\Gamma}_{..})$

$\parallel$ Ker Δ $\qquad$ coker $\partial : K^{\Sigma} \longrightarrow K^{A}$

$\Delta : K^A \to K^{\Sigma}$ $\qquad$ $\partial\phi(u) = \phi(s_2(u)) - \phi(s_1(u))$

- Same formula for ℓadic cohomology.
- Coleman integration : $\mathcal{L} \in C \rightsquigarrow H^1_{HK} \hookrightarrow H^1_{dR}$