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Galois representations & torsion classes

- joint work in progress w Allen, Calegari, Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne

§1. Locally symmetric spaces $\quad G/\mathbb{Q}$ conn real gp $\rightsquigarrow$ sym space $G(\mathbb{R})/K_\infty A_\infty^\circ$

$K \subset G(\mathbb{A}_f)$ compact open subgp $\rightsquigarrow X_K = {}_{G(\mathbb{Q})}\backslash[X \times G(\mathbb{A}_f)/K]$

V_λ irred alg rep'n of G over $\underbrace{E/\mathbb{Q}_p}_{\text{finite}}$ of highest wt λ

$\mathcal{O} \subset E$ ring of integers, let $\mathcal{V}_\lambda \subset V_\lambda$ be a K_p-stable $\mathcal{O}$-lattice ϖ unif.

$\rightsquigarrow$ local system $\mathcal{V}_\lambda$ on X_K.

$\mathbb{T} \curvearrowright H^*(X_K, \mathcal{V}_\lambda)$ (spherical) Hecke algebra

- $H^*(X_K, \mathcal{V}_\lambda)[1/p] \rightsquigarrow$ aut reps of G (Matsushima, Franke)

but $H^*(X_K, \mathcal{V}_\lambda)$ can contain torsion.

Assume $F = F_0 \cdot F^+$ s.t. p splits in F_0.
imquad, tot real

Let $G = \mathrm{Res}_{F/\mathbb{Q}} GL_n$, $\mathfrak{m} \subset \mathbb{T}$ max'l ideal in support of $H^*(X_K, \mathcal{V}_\lambda)$

The Calegari-Geraghty method is an extension of the Taylor-Wiles method (modularity lifting) to GL_n/F, where F can be a general number field.

Ingredients:

1). existence of Galois representations

- $\mathfrak{m} \rightsquigarrow \bar{\rho}_\mathfrak{m} : \mathrm{Gal}(\bar{F}/F) \to GL_n(\bar{\mathbb{F}}_p)$ s.t. Frobenius eigenvalues of $\bar{\rho}_\mathfrak{m}$ match the Satake parameters of $\mathfrak{m}$ at unramified places of F

Assume $\bar{\rho}_\mathfrak{m}$ abs irreducible $\rightsquigarrow \mathfrak{m}$ non-Eisenstein

$\rightsquigarrow$ $\rho_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(\mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}})$, where

$$\mathbb{T}(K,\lambda) := \mathrm{Im}(\mathbb{T} \to \mathrm{End}(H^*(X_K, \mathcal{V}_\lambda)))$$

s.t. char poly of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ at v unram place is determined in terms of Hecke operators in $\mathbb{T}(K,\lambda)_{\mathfrak{m}}$.

In our case (F = CM field) this is OK (Scholze, Boxer) up to $\mathbb{T}(K,\lambda)_{\mathfrak{m}}$ replaced by $\mathbb{T}(K,\lambda)_{\mathfrak{m}}/I$ I nilpotent ideal of bounded nilpotence degree.

2). local-global compatibility

let v be a prime of F above ℓ

Q: given $\mathfrak{m}$, what can we say about $\overset{(-)}{\rho_{\mathfrak{m}}}|\mathrm{Gal}(\bar{F}_v/F_v)$?

- if $p \neq \ell$ Helm
- if $p = \ell$ tricky because construction of $\overset{(-)}{\rho_{\mathfrak{m}}}$ increases the level at p, HT weights.

The statement of local-global compatibility needs to be formulated in general (Kisin's crystalline/pot semistable def rings - not constructed integrally)

3). vanishing of torsion

$\mathfrak{m}$ non-Eisenstein $\rightsquigarrow$ $H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ concentrated in a range of degrees of length ℓ_0 ($= \mathrm{rk}\, G(\mathbb{R}) - \mathrm{rk}\, K_\infty - \mathrm{rk}\, A_\infty$).

- completely open except for special cases

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§2. Local-global compatibility

Let F, p as above $\bar{\rho}_{\mathfrak{m}}$ abs irreducible.

Let v be a prime of F above p.

Assume $[F_v : \mathbb{Q}_p] \leq \frac{1}{2}[F:\mathbb{Q}]$. (Later edit: Also need $\mathrm{Im}\, \bar{\rho}_{\mathfrak{m}}$ large enough)

1). ordinary case:

loc. sym space: $\mathfrak{m}$ is ordinary at v if

i). $K_v \subset GL_n(\mathcal{O}_{F_v})$ is the Iwahori subgroup Iw_v.

ii). let U_v = double coset operator corr to $\left[\mathrm{Iw}_v \begin{pmatrix} 1 & & & 0 \\ & \varpi_v & & \\ & & \ddots & \\ 0 & & & \varpi_v^{n-1} \end{pmatrix} \mathrm{Iw}_v\right]$

then $\mathfrak{m}$ in support of $H^*(X_K, \mathcal{V}_\lambda)^{\mathrm{ord}_v}$ = largest direct summand of $H^*(X_K, \mathcal{V}_\lambda)$ on which U_v acts invertibly.

Galois side: $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}} \simeq \begin{pmatrix} \psi_1 & * & \cdots & * \\ & \psi_2 & * & \cdots * \\ & & \ddots & \\ & & & \psi_n \end{pmatrix}$ st. HT wts of ψ_i are determined by $\{\lambda_{\tau,i}\}$ $\tau: F_v \hookrightarrow \overline{\mathbb{Q}_p}$ & $\psi_i(\mathrm{Art}_{F_v}(\varpi_v)) \leftrightarrow$ certain Hecke eigenvalues in $\mathfrak{m}$.

Thm 1: If $\mathfrak{m}$ is ordinary at both v & v^c, then:

1). $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is ordinary & satisfies desired compatibilities.

2).

$$\begin{array}{ccccc} R^{\square,\,\mathrm{ord}}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}} \text{ (local def ring)} & \longrightarrow & R_{\bar{\rho}_{\mathfrak{m}}} \text{ (global def ring)} & \longrightarrow & \mathbb{T}(K,\lambda)^{\wedge}_{\mathfrak{m}}/I \end{array}$$

factors through

$R^{\square,\mathrm{ord}}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}}$ ordinary local def ring (up to possibly increasing I)

(not the naive definition ...)

2) Fontaine-Laffaille case

loc sym space: $\mathfrak{m}$ is Fontaine-Laffaille at v if in support of $H^*(X_K, \mathcal{V}_\lambda)$,

where $\begin{cases} \cdot\ K_v \text{ is hyperspecial} \\ \cdot\ \text{HT wts determined by } (\lambda_\tau)_{\tau: F_v \hookrightarrow \overline{\mathbb{Q}_p}} \end{cases}$

are in the range $[0, p-2]$

Galois side: $\bar\rho_{\mathfrak{m}}|_{G_{F_v}}$ is Fontaine-Laffaille if in essential image of

the functor: $\mathcal{MF}_{\mathbb{F}_\ell} \longrightarrow \{\text{cts } G_{F_v}\text{ reps on fin dim'l } \mathbb{F}_\ell\text{-vector spaces}\}$

Thm 2: If $\mathfrak{m}$ is FL at v, v^c, then: (Later edit: we actually need the corr. $\tilde\lambda$ to be in FL range $\rightsquigarrow$ slightly stronger condition on λ)

1). $\bar\rho_{\mathfrak{m}}|_{G_{F_v}}$ is FL w. prescribed HT wts

2). $\rho_{\mathfrak{m}}|_{G_{F_v}} \longrightarrow \mathbb{T}(K,\lambda)_{\mathfrak{m}}/I$ is FL (up to increasing I)

§3. Strategy of proof: modify the construction of Galois representations.

Let $\tilde G$ = quasisplit unit gp $/\mathbb{Q}$ preserving non-degenerate alternating Hermitian form on F^{2n} (signature (n,n) at every $\tau: F^+ \hookrightarrow \mathbb{R}$)

$\tilde K \subset \tilde G(\mathbb{A}_f)$ compact open $\rightarrow \tilde X_{\tilde K}$ Shimura variety

$\tilde\lambda$ weight for $\tilde G$ (we'll have to relate this to λ)

$\tilde{\mathfrak{m}}$ max'l ideal in support of $H^*(\tilde X_{\tilde K}, \mathcal{V}_{\tilde\lambda})$

Assume: 1). $\bar\rho_{\tilde{\mathfrak{m}}} = \bar\rho_1 \oplus \bar\rho_2$, ρ_i abs irred, n-dim'l.

2). $\exists$ auxiliary prime $\ell \neq p$ s.t. genericity cond at ℓ holds for $\bar\rho_{\tilde{\mathfrak{m}}}$ (OK if you have large image)

Thm (C-Scholze, in progress): 1). $H^i(\tilde X_{\tilde K}, \mathcal{V}_{\tilde\lambda}/p)_{\tilde{\mathfrak{m}}} \neq 0$

$\Rightarrow i \geq d = \dim_{\mathbb{C}} \tilde X_{\tilde K}$

2). $H^i_c(\tilde X_{\tilde K}, \mathcal{V}_{\tilde\lambda}/p)_{\tilde{\mathfrak{m}}} \neq 0 \rightarrow i \leq d = \dim_{\mathbb{C}} \tilde X_{\tilde K}$.

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Cor.: we have

$$H^d(X_{\tilde K}, \mathcal{V}_{\tilde\lambda})_{\tilde m}[1/p] \xleftarrow{(i)} H^d(\mathcal{X}_{\tilde K}, \mathcal{V}_{\tilde\lambda})_{\tilde m} \xrightarrow{(ii)} H^d(\partial \tilde X^{BS}_{\tilde K}, \mathcal{V}_{\tilde\lambda})_{\tilde m}$$

know LGC here (Varma) — deduce LGC here

Left with: relate $H^i(X_K, \mathcal{V}_\lambda)_m$ to $H^d(\partial \tilde X^{BS}_{\tilde K}, \mathcal{V}_{\tilde\lambda})_{\tilde m}$
$i = 0, \dots, d-1$

uses integral version of Kostant's formula (Polo-Tilouine) $\rightsquigarrow$ FL case
+ ordinary analogue of Kostant's formula $\rightarrow$ ordinary case

Applications*: . Sato-Tate for elliptic curves E / CM fields F ($\in$ non-CM)
· Ramanujan-Petersson conj. for π cuspidal coh rep'n of $GL_2(\mathbb{A}_F)$ of weight 0.

*: + important ingredients from automorphy lifting thms that are not discussed here.