This is joint work with Tong Liu.

Notation 1. The following notation will be in effect throughout:

- k is a perfect field of characteristic $p > 0$, and $W := W(k)$;
- $K/W[\frac{1}{p}]$ is a finite extension, totally ramified of degree e, and π_0 is a uniformizer;
- $\overline{K}$ is a fixed algebraic closure, $G_K := \mathrm{Gal}(\overline{K}/K)$, and $\pi_n \in \overline{K}$ are elements satisfying $\pi_n^p = \pi_{n-1}$;
- $K_n := K(\pi_n)$, $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$, E is the minimal polynomial of π_0 over W (it's Eisenstein);
- $\mathfrak{S} := W[[u]]$, with φ the unique continuous (for the (p,u)-adic topology) map acting on W as the unique lift σ of the p-power map on k and sending u to u^p.

Definition 2. A (height-r filtered) *Breuil–Kisin (BK) module* is a triple $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ in which

- M is a finite free $\mathfrak{S}$-module;
- $\mathrm{Fil}^r M \subseteq M$ is a filtration by submodules, $E^r M \subseteq \mathrm{Fil}^r M$, $M/\mathrm{Fil}^r M$ p-torsion-free;
- $\varphi_{M,r} : \mathrm{Fil}^r M \to M$ is a φ-semilinear map whose image generates M over $\mathfrak{S}$.

Define $\varphi_M : M \to M$ by $\varphi_M(x) = \varphi_{M,r}(E^r x)$, and Let $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ be the category of these things, with filtration compatible and φ-equivariant morphisms of $\mathfrak{S}$-modules.

Theorem 3 (Kisin)**.** *Let V be a crystalline G_K-representation with Hodge-Tate weights in $\{0,\dots,r\}$. Then for any G_K-stable $\mathbb{Z}_p$-lattice $T \subseteq V$, there is a unique Breuil-Kisin module $M(T) \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ such that*

$$T_{\mathfrak{S}}(M(T)) := \mathrm{Hom}_{\mathfrak{S},\mathrm{Fil},\varphi}(M(T), \mathbf{A}_{\mathrm{inf}}) \cong T^\vee|_{G_\infty}.$$

Here, "unique" means that if M and M' are two objects of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ with G_∞ isomorphisms $T_{\mathfrak{S}}(M) \simeq T^\vee|_{G_\infty} \simeq T_{\mathfrak{S}}(M')$, then there is a unique isomorphism $M \simeq M'$ in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ realizing this composite after applying $T_{\mathfrak{S}}$.

Now fix $\mathfrak{X}/\mathcal{O}_K$ a smooth proper formal scheme. Put

$$T^i := (H^i_{\mathrm{et}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)/\mathrm{torsion})^\vee;$$

it is a $\mathbb{Z}_p$-lattice in a crystalline G_K-representation with Hodge-Tate weights in $\{0,\dots,i\}$. **Can we describe $M(T^i)$ cohomologically**?

Previous work:

(1) 2012 PhD thesis of N. Bär: (for schemes) constructed a perfect complex $\mathcal{M}(\mathfrak{X})$ of sheaves of φ-modules over $\mathcal{O}$, the ring of rigid analytic functions on the open disc $|z| < 1$ over $W[1/p]$, on $\mathfrak{X}_k$ and a natural isomorphism of φ-modules over $\mathcal{O}$ $H^i(\mathfrak{X}_k, \mathcal{M}(\mathfrak{X})) \cong M(T^i) \otimes_{\mathfrak{S}} \mathcal{O}$. Kisin has shown that base change along $\mathfrak{S} \to \mathcal{O}$ induces an equivalence of categories $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \otimes \mathbb{Q}_p \to \mathrm{Mod}_{\mathcal{O}}^{\varphi,r,0}$, where the superscript of 0 means *pure of slope zero*, so Bär's construction realizes $M(T^i)$ up to p-isogeny. However, reconstructing $M(T^i)$ from $M(T^i) \otimes_{\mathfrak{S}} \mathcal{O}$ (up to p-isogeny) is not so direct.

(2) Bhatt–Morrow–Scholze: construct a perfect complex $\mathbb{R}\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X})$ of $\mathbf{A}_{\mathrm{inf}}$-modules with φ and, when $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W)$ is torsion-free, a natural isomorphism of φ-modules over $\mathbf{A}_{\mathrm{inf}}$:

$$H^i(\mathbb{R}\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X})) \cong M(T^i) \otimes_{\mathfrak{S}} \mathbf{A}_{\mathrm{inf}}.$$

This is a p-integral result, but it is not clear how to extract $M(T^i)$ from $M(T^i) \otimes_{\mathfrak{S}} \mathbf{A}_{\mathrm{inf}}$.

To explain our approach to this problem, we first fix some notation. For $n \geq 0$, define $\mathfrak{S}_n := W[[u_n]]$ with φ acting as σ on W and via $u_n \mapsto u_n^p$. Let $\theta_n : \mathfrak{S}_n \to \mathcal{O}_{K_n}$ via $u_n \mapsto \pi_n$. Let S_n be the p-adically completed PD-envelope of θ_n; concretely $\mathfrak{S}_n[\frac{E(u_n^{p^n})^m}{m!}]^{\wedge}_{m\geq 1}$. These rings fit together into:

$$\begin{tikzcd}
\mathfrak{S} = \mathfrak{S}_0 \arrow[r, hook, shift left, "u_0 \mapsto \varphi(u_1)"] \arrow[d, hook] & \mathfrak{S}_1 \arrow[l, bend left=10, "{\varphi,\cong}"] \arrow[r, hook, shift left, "u_1 \mapsto \varphi(u_2)"] \arrow[d, hook] & \mathfrak{S}_2 \arrow[l, bend left=10, "{\varphi,\cong}"] \arrow[r, hook] \arrow[d, hook] & \cdots \\
S := S_0 \arrow[r, hook, shift left] \arrow[d, "PD"] & S_1 \arrow[l, bend left=10, "\varphi"] \arrow[r, hook, shift left] \arrow[d, "PD"] & S_2 \arrow[l, bend left=10, "\varphi"] \arrow[r, hook] \arrow[d, "PD"] & \cdots \\
\mathcal{O}_K = \mathcal{O}_{K_0} \arrow[r, hook] & \mathcal{O}_{K_1} \arrow[r, hook] & \mathcal{O}_{K_2} \arrow[r, hook] & \cdots
\end{tikzcd}$$

Note that the Frobenius maps carry $\mathfrak{S}_n$ bijectively onto $\mathfrak{S}_{n-1}$, and carry S_n into S_{n-1} (but not onto). Our construction of the BK-module $M(T^i)$ from crystalline cohomology was motivated by the following:

Lemma 4. *As rings, $\mathfrak{S} \cong \varprojlim_{\varphi,n} S_n$ via $f(u) \mapsto \{f^{\sigma^{-n}}(u_n)\}$.*

This suggests that we should take some kind of inverse limit along Frobenius of the crystalline cohomology of $\mathfrak{X}$ over S_n (which is why we need divided powers in the first place) up the tower $\mathcal{O}_{K_n}$. Unfortunately, this does not literally work, as in general the result is too *small.* So we must enlarge the crystalline cohomology at each layer of this tower by allowing poles along certain special subschemes of S_n whose order is controlled by the filtration.

To make this precise, fix i and put $\mathcal{M} := H^i_{\mathrm{crys}}(\mathfrak{X} \times \mathcal{O}_K/(p)/S)$; this is a φ-module over S with a filtration $\mathrm{Fil}^j\mathcal{M}$. Put $z_n := E(u_0)\varphi^{-1}(E(u_0))\cdots\varphi^{1-n}(E(u_0)) \in \mathfrak{S}_n$ for $n \geq 1$ and set $z_0 = 1$. These elements satisfy $\varphi(z_n) = \varphi(E)z_{n-1}$ for $n \geq 1$. Give $S_n[z_n^{-1}]$ the $\mathbb{Z}$-filtration by powers of z_n, *i.e.* $\mathrm{Fil}^j S_n[z_n^{-1}] := S_n \cdot z_n^j$. Define

$$\begin{aligned}
\underline{M}(\mathfrak{X}) &:= \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathcal{M} \otimes_S S_n[z_n^{-1}]) \\
&= \left\{ \{\xi_n\}_n \middle| \xi_n \in \sum_{j\geq 0} \frac{1}{z_n^j} \mathrm{Fil}^j(H^i_{\mathrm{crys}}(\mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_{K_n}/(p)/S_n) \text{ and } \varphi(\xi_n) = \xi_{n-1} \right\}
\end{aligned}$$

and set $\quad \mathrm{Fil}^j\underline{M}(\mathfrak{X}) := \left\{ \{\xi_n\}_n \middle| \xi_0 \in \mathrm{Fil}^j(\mathcal{M}) \right\}.$

Note that $\underline{M}(\mathfrak{X})$ is a filtered φ-module over $\varprojlim_{\varphi} S_n = \mathfrak{S}$ by Lemma 4.

Theorem 5 (Liu–C.)**.** *Assume $p > 2$, $i < p-1$, and $H^j_{\mathrm{crys}}(\mathfrak{X}_k/W)$ is torsion-free for $j = i, i+1$. Then there is a natural isomorphism*

$$\underline{M}(\mathfrak{X}) \cong M(T^i)$$

in $\mathrm{Mod}^{\varphi,i}_{\mathfrak{S}}$.

Remarks 6. Some remarks are in order:

(1) The definition of the filtration $\mathrm{Fil}^j\mathcal{M}$ is as follows. Set $D := H^i_{\mathrm{crys}}(\mathfrak{X}_k/W)$ and put $\mathcal{D} := D \otimes_W S[1/p]$, equipped with the monodromy operator $N_{\mathcal{D}} := \mathrm{id}_D \otimes N_S$, where

N_S is the derivation $-u\frac{d}{du}$. Denote by f_{π_0} the map

$$f_{\pi_0} : \mathcal{D} = D \otimes_W S[1/p] \longrightarrow D \otimes_W K \simeq H^i_{\mathrm{dR}}(\mathfrak{X}_K/K)$$

induced by the homomorphism $S[1/p] \to K$ mapping u to π_0 and the comparison isomophism between crystalline cohomology of the special fiber after extending scalars up to K and de Rham cohomology of the generic fiber. We then define $\mathrm{Fil}^0\mathcal{D} = \mathcal{D}$ and, inductively,

$$\mathrm{Fil}^j\mathcal{D} := \{x \in \mathcal{D} : N_{\mathcal{D}}(x) \in \mathrm{Fil}^{j-1}\mathcal{D} \text{ and } f_{\pi_0}(x) \in \mathrm{Fil}^j D_K\}$$

where $\mathrm{Fil}^j D_K = \mathrm{Fil}^j(D \otimes_W K)$ is the j-th piece of the Hodge filtration on de Rham cohomology. Now there is a canonical φ-equivariant isomorphism

$$\mathcal{M}[1/p] = H^i_{\mathrm{crys}}(\mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_K/(p)/S) \otimes_S S[1/p] \simeq H^i_{\mathrm{crys}}(\mathfrak{X}_k/W) \otimes_W S[1/p] =: \mathcal{D}$$

reducing to the identity modulo u, by which we obtain a filtration $\mathrm{Fil}^j(\mathcal{M}[1/p])$ on $\mathcal{M}[1/p]$ by "transport of structure." We then define $\mathrm{Fil}^j\mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^j(\mathcal{M}[1/p])$, with the intersection taking place inside $\mathcal{M}[1/p]$.

(2) It would be better to have a more direct, cohomological description of this filtration. For ease of notation, put $\mathfrak{X}_{(m)} := \mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_K/(p^m)$ and $S_{(m)} := S/(p^m)$ We hope that, in fact, one has:

$$\mathrm{Fil}^j\mathcal{M} = \varprojlim_m H^i((\mathfrak{X}_{(m)}/S_{(m)})_{\mathrm{crys}}, \mathfrak{J}_m^{[j]}),$$

where $\mathfrak{J}_m$ is the sheaf of PD-ideals on the big crystalline site $(\mathfrak{X}_{(m)}/S_{(m)})_{\mathrm{crys}}$ whose value on an object $(U \hookrightarrow T, \delta)$ is $\ker(\mathcal{O}_T \to \mathcal{O}_U)$, and we expect this to be true after inverting p. This latter statement would follow if one knew that the comparison map

$$\mathcal{M}[1/p] \simeq D \otimes_W S[1/p] \xrightarrow{f_{\pi_0}} D \otimes_W K \simeq H^i_{\mathrm{dR}}(\mathfrak{X}_K/K)$$

carried $\varprojlim_m H^i((\mathfrak{X}_{(m)}/S_{(m)})_{\mathrm{crys}}, \mathfrak{J}_m^{[j]})[1/p]$ isomorphically onto $\mathrm{Fil}^j H^i_{\mathrm{dR}}(\mathfrak{X}_K/K)$.

(3) The use of the rings S_n in our construction may be somewhat "artificial." It would perhaps be more natural to use the rings $W(\mathcal{O}_{K_n})$ in their place, as the canonical projection (to the 0-*th* Witt coordinate) $W(\mathcal{O}_{K_n}) \to \mathcal{O}_{K_n}$ is a divided power thickening, so we could instead make the same construction by considering crystalline cohomology over $W(\mathcal{O}_{K_n})$. A key point is that the map $\mathfrak{S} \to \varprojlim_{\varphi,n} W(\mathcal{O}_{K_n})$ sending u to $\{[\pi_n]\}_{n\geq 0}$ is an isomorphism of rings, and we expect that this alternate construction would again yield the Breuil–Kisin module.

(4) It is worth pointing out that the definition of $\underline{M}(\mathfrak{X})$ (using either the rings S_n or $W(\mathcal{O}_{K_n})$) makes sense, and produces a filtered φ-module over $\mathfrak{S}$, in general (*i.e.* without any restriction on i). However, without the connection to Breuil modules that is possible when $i < p-1$, we do not know if this resulting $\mathfrak{S}$-module coincides with the Breuil–Kisin module (indeed, we do not even know that it is an object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,i}$). It would be interesting to try to understand this construction in the general case.

The proof of Theorem 5 has two main ingredients:

(1) The functor $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_{S}^{\varphi,r}$ is an equivalence for $r < p-1$, $p > 2$, with the quasi-inverse $\mathcal{M} \mapsto \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathcal{M} \otimes_S S_n[z_n^{-1}])$. This one proves using "pure" semilinear algebra via some delicate calculations with the rings S_n and their Frobenius maps.

(2) As the Breuil–Kisin module $M(T^i)$ is built from T^i, to complete the proof of Theorem 5 it suffices to prove that $\mathcal{M}$ *is* the Breuil module associated to T^i, or what is the same thing, that one has an isomorphism

$$\mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, \mathbf{A}_{\mathrm{crys}}) \cong T^i.$$

To do this, we use some basic commutative algebra and the work of Bhatt–Morrow–Scholze to reduce to proving that

$$\mathcal{M} \otimes_S A_{\mathrm{crys}} \simeq H^i(\mathbb{R}\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X})) \otimes_{\mathbf{A}_{\mathrm{inf}}} A_{\mathrm{crys}},$$

which follows from their work using the hypothesis that $H^{i+1}(\mathfrak{X}_k/W)$ is torsion-free.