 B. Cais

Breuil-Kisin modules and crystalline coh

(jt w T Liu)

k = perf, char $p>0$, $W = W(k)$

$K / W[1/p]$ finite, tot ramified, deg e, unif π_0

Fix $\overline{K}$, $G_K = \mathrm{Gal}(\overline{K}/K)$, $\pi_n \in \overline{K}$ s.t. $\pi_n^p = \pi_{n-1}$ $n \geq 1$.

$K_n = K(\pi_n)$, $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$, E = min poly of π_0 / W. $= \dots + p$.

$\mathfrak{S} = W[\![u]\!] \circlearrowleft \varphi : u \mapsto u^p$

Def: A (ht r, filtered) Breuil-Kisin module (BK) is

$(M, \mathrm{Fil}^r M, \varphi_{M}, r)$

- M a fin free $\mathfrak{S}$-mod
- $\mathrm{Fil}^r M \subseteq M$ submodules, $E^r M \subseteq \mathrm{Fil}^r M$ s.t. $(M / \mathrm{Fil}^r M)$ is p torsion free
- $\varphi_{M,r} : \mathrm{Fil}^r M \longrightarrow M$ φ semilinear, img. generate $M / \mathfrak{S}$.

$\mathrm{Mod}_{\mathfrak{S}}^{\varphi, r}$ - category.

Def. $\varphi_M : M \to M$ by $\varphi_M(x) = \varphi_{M,r}(E^r x)$

Thm (Kisin): V a crystalline G_K-rep'n, HT wts in $\{0, \dots, r\}$,

for $T \subseteq V$ a G_K-stable $\mathbb{Z}_p$-lattice

$\exists!$ BK-module $M(T) \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi, r}$ s.t.:

$$\mathrm{Hom}_{\mathfrak{S}, \mathrm{Fil}, \varphi}(M(T), A_{\inf}) \simeq T^\vee |_{G_\infty}$$

Fix $\mathfrak{X} / \mathcal{O}_K$ smooth proper formal scheme.

$$T^i := \left(H^i_{et}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p) / \mathrm{tors} \right)^\vee$$ is a $\mathbb{Z}_p$-lattice in a crystalline G_K-rep'n w HT wts in $\{0, \dots, i\}$

B. Cais ②

Question Can we describe $M(T^i)$ cohomologically?

Previous work: 1). 2012 PhD thesis of N. Bär.

$\mathcal{M}(\mathfrak{X})$ perfect complex of sheaves of φ-modules over $\mathcal{O}$ = rigid analytic fns on $|z| < 1$ over $W[1/p]$ on $\mathfrak{X}_k$.

s.t. $H^i(\mathfrak{X}_k, \mathcal{M}(\mathfrak{X})) \simeq M(T^i) \otimes_{\mathfrak{S}} \mathcal{O}$

$$\Gamma\ \mathrm{Mod}_{\mathfrak{S}}^{\varphi, r} \otimes \mathbb{Q}_p \xrightarrow[\text{Kisin}]{\sim} \mathrm{Mod}_{\mathcal{O}}^{\varphi, r, 0} \quad \leftarrow \text{slope } 0$$

Note: so this work of Bär is recovering BK module up to p-isogeny.

2). BMS

$R\Gamma_{A_{inf}}(\mathfrak{X})$ = perfect complex of A_{inf}-modules w φ.

If $H^j_{crys}(\mathfrak{X}_k / W)$ is torsion-free for $j =$ ... i

then $H^i(R\Gamma_{A_{inf}}(\mathfrak{X}_k/W)) \simeq M(T^i) \otimes_{\mathfrak{S}} A_{inf}$

For $n \geq 0$, $\mathfrak{S}_n = W[[u_n]] \circlearrowleft \varphi : u_n \longmapsto u_n^p$.

$$\theta_n : \mathfrak{S}_n \longrightarrow \mathcal{O}_{K_n}$$
$$u_n \longmapsto \pi_n$$

S_n = p-adically completed PD envelope of θ_n

$$= \mathfrak{S}_n \left[\frac{E(u_n^{p^n})^m}{m!} \right]^{\wedge}_{m \geq 1}.$$

$$\begin{array}{ccccccc}
\mathfrak{S} = \mathfrak{S}_0 & \hookrightarrow & \mathfrak{S}_1 & \hookrightarrow & \mathfrak{S}_2 & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
S := S_0 & \hookrightarrow & S_1 & \hookrightarrow & S_2 & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_K = \mathcal{O}_{K_0} & \hookrightarrow & \mathcal{O}_{K_1} & \hookrightarrow & \mathcal{O}_{K_2} & \hookrightarrow &
\end{array}$$

($\varphi: \mathfrak{S}_1 \to \mathfrak{S}_0$, $\varphi: \mathfrak{S}_2 \xrightarrow{\simeq} \mathfrak{S}_1$; $u_0 \mapsto \varphi(u_1)$; $\varphi: S_1 \to S_0$, $\varphi: S_2 \to S_1$)

Lemma $\mathfrak{S} \xrightarrow{\sim} \varprojlim_{\varphi, n} S_n$ (isom of rings)

$f(u) \longmapsto \{ f^{\sigma^{-n}}(u_n) \}$ (σ: Frobenius on W)

[Aside: you also have $\mathfrak{S} \xrightarrow{\sim} \varprojlim_{\varphi, n} W(\mathcal{O}_{K_n})$

$u \longmapsto \{[\pi_n]\}$

$\{ z_n \in W(\mathcal{O}_{K_n}) \mid \varphi(z_n) = z_{n+1} \}$]

Fix i. $\mathcal{M} = H^i_{crys}(\mathfrak{X} \times \mathcal{O}_K/(p) / S)$

φ-module$/S$ w $\mathrm{Fil}^j \mathcal{M}$.

Set $z_n = E(u_0)\,\varphi^{-1}(E(u_0)) \cdot \ldots \cdot \varphi^{1-n}(E(u_0)) \in \mathfrak{S}_n$, $n \geq 1$.

so $\varphi(z_n) = \varphi(E)\, z_{n-1}$.

Give $S_n[z_n^{-1}]$ the z-filtration by powers of z_n.

Def: $\underline{M}(\mathfrak{X}) := \varprojlim_{\varphi, n} \mathrm{Fil}^0(\mathcal{M} \otimes_S S_n[z_n^{-1}])$

$= \{ \{\xi_n\}_{n \geq 0} \mid \xi_n \in \sum_{j \geq 0} \frac{1}{z_n^j} \mathrm{Fil}^j(H^i_{crys}(\mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_K/(p) / S_n))$ s.t. $\varphi(\xi_n) = \xi_{n-1} \}$ ← Module over $\varprojlim_{\varphi, n} S_n = \mathfrak{S}$.

$\mathrm{Fil}^j \underline{M}(\mathfrak{X}) = \{ \{\xi_n\}_n \mid \xi_0 \in \mathrm{Fil}^j(\mathcal{M}) \}$

Thm (Liu-C). Assume $p > 2$, $i < p-1$ and $H^j_{crys}(\mathfrak{X}_k / W)$ is torsion-free for $j = i, i+1$.

Then $\underline{M}(\mathfrak{X}) \simeq M(T^i)$ in $\mathrm{Mod}^{\varphi, i}_{\mathfrak{S}}$.

Remarks: 1. $\mathcal{M}[1/p] \simeq \underbrace{H^i_{crys}(\mathfrak{X}_k / W)}_{D} \otimes_W S[1/p]$

$\mathcal{M}[1/p] = \mathcal{D}$

$D_K = D \otimes_W K \simeq H^i_{dR}(\mathfrak{X}_K / K)$ ← Hodge filtration.

Define $\mathrm{Fil}^j \mathcal{M} = \mathcal{M} \cap \mathrm{Fil}^j(\mathcal{M}[1/p]) \subseteq \mathcal{M}[1/p]$

$\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$, $\mathrm{Fil}^j \mathcal{D} = \{ x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{j-1} \mathcal{D},\ f_{\pi_0}(x) \in \mathrm{Fil}^j D_K \}$

where: $N_{\mathcal{D}} = id \otimes N_{S}$ on $\mathcal{D} = D \otimes_W S[1/p]$,

$N_S = -u \frac{\partial}{\partial u}$

$f_{\pi_0} : \mathcal{D} = D \otimes_W S[1/p] \longrightarrow D \otimes_W K = H^i_{dR}(\mathfrak{X}_K / K)$

$u \longrightarrow \pi_0$.

Hope: $\mathrm{Fil}^i \mathcal{M} = \varprojlim_m H^i_{crys}\left(\left(\mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_K(p^m) / S/p^m \right)_{crys}, \mathcal{J}_m^{[i]} \right)$ ↘ $\mathcal{M}$ (don't know it's a sub)

expect: true after inverting p.

Pf: (of Thm) 1). $\mathrm{Mod}_{\mathfrak{S}}^{\varphi, r} \xrightarrow{\sim} \mathrm{Mod}_{S}^{\varphi, r}$ (when $r < p-1$), q-inv: $\mathcal{M} \longmapsto \varprojlim_{\varphi, r} \mathrm{Fil}^0(\mathcal{M} \otimes_S S_n[1/z_n])$

2). $\mathrm{Hom}_{S, \mathrm{Fil}, \varphi}(\mathcal{M}, A_{cris}) \simeq T^i$

follows under our hypotheses from BMS

+ commutative algebra.