How much to put in a tontine

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The UK pension freedom since 2015

building a personal pension pot, and at retirement...

**buy annuity**
- savings for **guaranteed income**, $\mathbb{E}[\text{give}] = \mathbb{E}[\text{take}]$
- **mortality pooling** (law of large numbers)

**go into drawdown**
- savings spent over time
- **investments** (fluctuating)
- bequest
The UK pension freedom since 2015

building

cost

buy annuity
- saving
- income
- mortality pooling (law of large numbers)
go into drawdown
- savings spent over time
- investments (fluctuating)
- bequest

credit value

investment
annuity

age
The UK pension freedom since 2015

building a personal pension pot, and at retirement...

**buy annuity**
- savings for guaranteed income, $\mathbb{E}[\text{give}] = \mathbb{E}[\text{take}]$
- mortality pooling (law of large numbers)

$\Downarrow$

mortality credits at high ages, unpopular choice

**go into drawdown**
- savings spent over time
- investments (fluctuating)
- bequest

$\Downarrow$

investment returns at low ages, risk of outliving
Tontines

$Tontine = \text{mortality credits} + \text{investment return}$

- surrender savings to a group of people, to get mortality credits
- no guarantees, to be able to invest

add bequest
- allow to choose $\alpha$, how much to surrender, to have a bequest (comes with reduction in mortality credits)
in the background mortality credits boost wealth and bequest

(a) Before re-balancing.

(b) After re-balancing.
Tontines

Tontine = mortality credits + investment return
- surrender savings to a group of people, to get mortality credits
- no guarantees, to be able to invest

add bequest
- allow to choose $\alpha$, how much to surrender, to have a bequest
  (comes with reduction in mortality credits)

mathematical description
- mortality credits = additional $\alpha$-weighted stream of income
- in a Black-Scholes market and force of mortality $\lambda$...

\[
\frac{dX_t}{X_t} = r(1 - \pi_t)dt + \mu \pi_t dt + \sigma \pi_t dW_t - c_t dt + \alpha \lambda_t dt
\]
Numerical results

optimization problem including lifespan $\tau$, bequest motive $b$, and constant relative risk aversion $1 - \gamma$

- $\sup_{\alpha,c,\pi} \mathbb{E} \left[ \int_0^\tau U(s, cX_s) \, ds + b B(\tau, (1 - \alpha)X_\tau) \right]$
- $U(s, x) = B(s, x) = e^{-\rho s} x^{\gamma} / \gamma$
- $\mathbb{P}[\tau > x] = \exp \left( - \int_0^x \lambda_s \, ds \right)$
solution for optimal $\alpha$, given bequest motive $b$ and risk aversion $1 - \gamma$

- risk seeking, low $1 - \gamma$
  - down and up
  - changes from 0% to 100%
Numerical results

Force of mortality

Age (years)

0.0 0.5 1.0 1.5 2.0 2.5 3.0

Forces of mortality at Age

0% 20% 40% 60% 80% 100%

Solution for optimal $\alpha$, given bequest motive $b$ and risk aversion $1 - \gamma$.

Risk seeking, low $1 - \gamma$• down and up• changes from 0% to 100%

Risk averse, high $1 - \gamma$• around 80%• stable even for changes in $\mu$, $\sigma$, $r$ and slight changes with $\rho$, $\lambda$.
Numerical results

Force of mortality

Consumption rate = 0.09 and $\alpha = 0.8$

Bequest account value at Age

Constant relative risk aversion $1 - \gamma$

Risk seeking, low $1 - \gamma$

• down and up
• changes from 0% to 100%

Consumption rate = 0.09 and $\alpha = 0.8$

Risk averse, high $1 - \gamma$

• around 80%
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Numerical results

**Force of mortality**

Consumption rate = 0.09 and $\alpha = 0.8$

**Bequest account value at Age**

Consumption rate = 0.09 and $\alpha = 0.8$

**constant relative risk aversion $1 - \gamma$**

in the tontine

0% 20% 40% 60% 80% 100%

$b=1$

$b=2$

$b=3$

$b=6$

$b=7$

risk averse, high $1 - \gamma$

• around 80%

• stable even for changes in $\mu$, $\sigma$, $r$ and slight changes with $\rho$, $\lambda$
Numerical results

solution for optimal $\alpha$, given bequest motive $b$ and risk aversion $1 - \gamma$

risk seeking, low $1 - \gamma$
- down and up
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Numerical Results

direct comparison to drawdown (same consumption, no fluctuation)

`Tontine with bequest' gives higher bequest after age 87

Drawdown account hits zero by age 88
given $\tau$ independent from $\mathcal{F}_t$, is it true that...

$$\sup_{\alpha, c, \pi} \mathbb{E} \left[ \int_0^\tau U(s, c_s X_s) \, ds \right] = \sup_{\alpha, c', \pi'} \mathbb{E} \left[ \int_0^\infty U(s, c'_s X_s) \mathbb{P}[\tau > s] \, ds \right]?$$

- on the left side, $c, \pi$ adapted to $\mathcal{F}_t \vee \{\tau > s | t > s\}$
- on the right side, $c', \pi'$ adapted to $\mathcal{F}_t$

for $c$ there is $\mathcal{F}_t$-adapted $c'$ such that $c_{t \wedge \tau} = c'_{t \wedge \tau}$, but $c'$ might not be locally integrable! For example...

- $\mathcal{F}_t = \{\Omega, \emptyset\}$, $\tau \sim \mathcal{U}(0, 1)$, $c_t = (1 - \tau \wedge t)^{-1}$
  $$\Rightarrow c'_t = (1 - t)^{-1}$$
do we know that the optimal controls are deterministic before solving the HJB?

\[ V(t, x) = \sup_{\alpha, c, \pi} \mathbb{E} \left[ \int_0^\tau e^{-\rho s} X_s^{\gamma} / \gamma \, ds \bigg| X_t = x \right] \]

- \( V(t, x) \) and \( V(t, y) \) only differ by a constant
  \( \Rightarrow \) any \( x \) at \( t \) gives same optimal controls
  \( \Rightarrow \) (heuristic) optimal controls are independent of \( X \)
  \( \Rightarrow \) optimal controls are deterministic
Mathematical features 3

does the transversality condition holds true for all $X$...

$$\lim_{t \to \infty} V(t, X_t) = \lim_{t \to \infty} e^{-\rho t} P[\tau > t] E[\log X_t] = 0 \ ?$$

we have

- $V(0, x) = E \left[ \int_0^\infty e^{-\rho s} P[\tau > s] \log(c_s X_s) \, ds \right] > -\infty$

$\Rightarrow E \left[ \int_0^\infty e^{-\rho s} P[\tau > s] \log(X_s) \, ds \right] \in \mathbb{R}$

$\Rightarrow \lim_{n \to \infty} V(t_n, X_{t_n}) = \lim_{n \to \infty} e^{-\rho t_n} P[\tau > t_n] E[\log X_{t_n}] = 0$

but bad choice of $X$ tends to $\infty$ on another sequence
Future research

- how many members so that law of large numbers holds true?
- study Mathematical features 2 in more detail

Thank you for your attention.
Do you have any questions or feedback?