Filtering Problems with Cox Jump Processes

Financial Mathematics Reading Group Talk held by Thomas Bernhardt
Introduction
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Two Topics

Cox processes
- counting processes
- jump rate corresponds to some intensity
- intensity can be described without the jumps
- goals: characterization, intensity manipulation

Filtering problem
- unknown intensity and known jumps
- exemplary calculation of a conditional expectation
- intensity is an Ornstein-Uhlenbeck process
Cox processes
Cox processes

Definition: On the given probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\)

Let \(N\) counting process, \(X\) non-negative \(\mathcal{F}_0 \otimes \mathcal{B}([0, \infty[)-\text{mb.}\) \(N\) is a Cox process with intensity \(X\) if for \(t \geq r \geq 0\) and \(n \in \mathbb{N}_0\)

\[
\mathbb{P}\left[ \int_{[0,t]} X_s \, ds < \infty \quad \forall \ t \geq 0 \right] = 1
\]

\[
\mathbb{P}\left[ N_t - N_r = n \mid \mathcal{F}_r \right] = e^{-\int_{[r,t]} X_s \, ds} \cdot \frac{\left( \int_{[r,t]} X_s \, ds \right)^n}{n!} \quad \mathbb{P}\text{-fs.}
\]

- Intensity constantly one leads to a Poisson process with respect to the given filtration
- Conditional distribution of jump times can be computed:

\[
\mathbb{P}\left[ T_{n+1} > t \mid \mathcal{F}_r \right] = \mathbb{P}\left[ N_t \leq n \mid \mathcal{F}_r \right] = \sum_{k=0}^{n-N_r} \mathbb{P}\left[ N_t - N_r = k \mid \mathcal{F}_r \right]
\]
Theorem: Characterization of Cox processes

\( N \) finite counting process, \( X \) non-negative \( \mathcal{F}_0 \otimes \mathcal{B}([0, \infty[)\)-mb with \( \mathbb{P}[\int_{0,t} X_s \, ds < \infty, \forall t \geq 0] = 1 \). Then is equivalent that:

(i) \( N \) is a Cox process with intensity \( X \),
(ii) \( M = N - \int_0 X_s \, ds \) is a local martingale,
(iii) \( \varphi \geq 0 \) predictable: \( \mathbb{E}[\int_0,\infty[ \varphi_s \, dN_s] = \mathbb{E}[\int_0,\infty[ \varphi_s X_s \, ds] \).

Proof:

(i) \( \Rightarrow \) (ii) jump times for localization and cond. distribution,
(ii) \( \Rightarrow \) (iii) measures implied by \( N, \int_0 X_s \, ds \) coincide on a \( \sigma \)-finite \( \cap \)-stable generator,
(iii) \( \Rightarrow \) (i) stochastic exponential contains Laplace transform.
Cox processes

**Theorem: Change of measure**

\( N \) Cox with intensity \( X \); \( Y \) non-negative \( \mathcal{F}_0 \otimes \mathcal{B}([0, \infty]) \)-mb with \( \mathbb{P}[\int_0^t Y_s X_s \, ds < \infty \, \forall \, t \geq 0] = 1 \);

\[ W := \exp\left(-\int_0^t (Y_s - 1) X_s \, ds \right) \cdot \prod_{s \in [0, t]} (1 + (Y_s - 1) \Delta N_s). \]

Then

- \( W \) is a non-negative right-continuous martingale,
- \( W_T = dQ/dP \) it follows that \( \mathbb{P} = Q \) on \( \mathcal{F}_0 \) and \( N^T \) is again Cox under \( Q \) with intensity \( 1_{[0, T]} YX \).

**Proof: martingale property**

\( W \) is a supermartingale (non-negative stochastic exponential).
\( W \) does not vary in expectation (using conditional distribution).
Cox processes

**Theorem: Change of measure** \( N \) Cox, intensity \( X; Y \) as \( X \\
\mathcal{W} := \exp(-\int_0^T (Y_s - 1)X_s \, ds) \cdot \prod_{s \in ]0,T]}(1 + (Y_s - 1)\Delta N_s). \\
Then \\
\bullet \ N^T \) Cox process with intensity \( \mathbb{1}_{[0,T]} YX \) under \( \mathbb{Q}. \)

**Proof: Cox property** \( \varphi \geq 0 \) predictable

\[
\mathbb{E}_\mathbb{Q}[\int_{0}^{T} \varphi_s \, dN_s] = \mathbb{E}[\mathcal{W}_T \int_{0}^{T} \varphi_s \, dN_s] = \mathbb{E}[\int_{0}^{T} \varphi_s \mathcal{W}_s \, dN_s] \\
= \mathbb{E}[\int_{0}^{T} \varphi_s \mathcal{W}_s - Y_s \, dN_s], \\
\mathbb{E}_\mathbb{Q}[\int_{0}^{T} \varphi_s Y_s X_s \, ds] = \mathbb{E}_\mathbb{Q}[\mathcal{W}_T \int_{0}^{T} \varphi_s Y_s X_s \, ds] \\
= \mathbb{E}[\int_{0}^{T} \varphi_s \mathcal{W}_s - Y_s X_s \, ds].
\]

Using characterization gives the claim.
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Example with an OU intensity

- model \( dX = dJ - \lambda X dt \) where \( J \) is compound Poisson process and \( M \) its Poisson process
- aim the conditional Laplace transform with additional information about \( M \)

\[
E \left[ \exp \left( \alpha X_R + \beta X_S + \gamma \int_{]R,S]} X_s \, ds \right) \mid \mathcal{F}_R^N \lor \mathcal{F}_R^M \right]
\]

Equivalent to calculate (solving the SDE, using independency)

\[
E \left[ \exp \left( \int_{]R,S]} B_s \, dJ_s \right) \right] \cdot E \left[ \exp \left( (\alpha + B_R)X_R \right) \mid \mathcal{F}_R^N \lor \mathcal{F}_R^M \right]
\]

where \( B_s = \gamma/\lambda + (\beta - \gamma/\lambda) \cdot \exp (\lambda(s - S)) \).
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The left term $\mathbb{E}\left[ \exp \left( \int_{[R,S]} B_s \, dJ_s \right) \right]

$M$ be a Poisson process and $\xi$ independent size of a jump of $J$.

$\mathbb{E}\left[ \exp \left( \int_{[R,T]} B_s \, dJ_s \right) \bigg| \mathcal{F}_\infty \right] = \prod_{s \in [R,S]} \left( 1 + \left( \mathbb{E}\left[ \exp(B_s \xi) \right] - 1 \right) \Delta M_s \right)

\Rightarrow \mathbb{E}\left[ \exp \left( \int_{[R,T]} B_s \, dJ_s \right) \right] \overset{!}{=} \exp \left( a \int_{[R,S]} \mathbb{E}[\exp(B_s \xi)] - 1 \, ds \right)

cos following process is a martingale (measure change theorem)

$t \quad \mapsto \quad \frac{\prod_{s \in [R,t]} \left( 1 + \left( \mathbb{E}[\exp(B_s \xi)] - 1 \right) \Delta M_s \right)}{\exp(a \int_{[R,t]} \mathbb{E}[\exp(B_s \xi)] - 1 \, ds)}.$
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The right term \( \mathbb{E}\left[ \exp \left( (\alpha + B_R) X_R \right) | \mathcal{F}_R^N \vee \mathcal{F}_R^M \right] \)

**General approach:** \( X \) positive, choose \( Y = 1/X \)

Measure \( Q \) with \( N \) independent of \( X \) and untouched distribution of \( X \).

Look at \( H \mathcal{F}_0 \)-mb. Bayes, independency, measurability:

\[
\mathbb{E}\left[ H | \mathcal{F}_R^N \vee \mathcal{F}_R^M \right] = \frac{\mathbb{E}_Q[H \frac{dP}{dQ} | \mathcal{F}_R^N \vee \mathcal{F}_R^M]}{\mathbb{E}_Q[\frac{dP}{dQ} | \mathcal{F}_R^N \vee \mathcal{F}_R^M]} = \frac{\mathbb{E}[(H \frac{dP}{dQ})^{m,n}]}{\mathbb{E}[(dP/dQ)^{m,n}]} \bigg|_{N=n} = M=m.
\]
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The formula (for $\Gamma(b, \rho)$ distributed jump sizes)

\[
\mathbb{E} \left[ \exp \left( \alpha X_R + \beta X_S + \gamma \int_{[R,S]} X_s \, ds \right) \bigg| \mathcal{F}_R^N \lor \mathcal{F}_R^M \right] = \exp \left( a \int_{[R,S]} \frac{b^p}{(b - B_s)^p} - 1 \, ds \right) \cdot \frac{Z_R^{\alpha + B_R}}{Z_R^0}
\]

where

\[
Z_R^\alpha = \int_{M_R} \exp \left( \alpha X_{R, j}^M \right) \exp \left( \int_{[0,R]} 1 - X_{s,j}^M \, ds \right) \prod_{s \in [0,R]} \left( 1 + (X_{s,j}^M - 1)\Delta N_s \right) \, d\Gamma_j
\]

and

\[
X_{s,j}^M = e^{-\lambda s} x_0 + e^{-\lambda s} \sum_{n=1}^{M_s} e^{\lambda \theta n} \cdot j_n
\]

as well as $M_R$ as subscript of the integrals the number of integrations over the positive real line means with respect to the multidimensional variable $j$. 