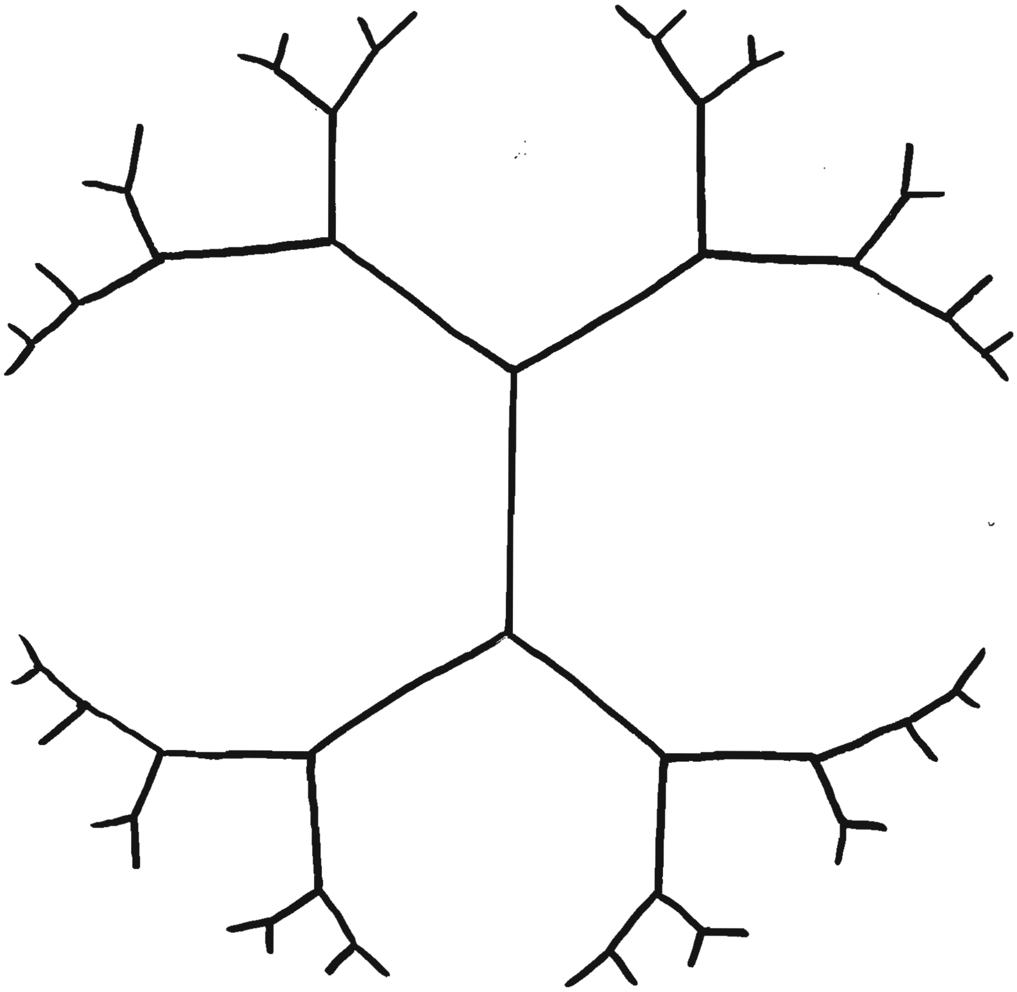
# A few motivating problems

1. A parlor trick:
   I tell you that $\frac{p}{q} = 0.058544\ldots$, where $p, q$ are integers and $q < 10^3$. This is enough information to identify $\frac{p}{q}$ uniquely, but how can you do it quickly and impress your friends?

2. Rational approximation:
   If I tell you that $x = 3.141592\ldots$, this time you might guess that $x$ is not rational. Nevertheless, you probably know that $x \approx \frac{22}{7}$, and you may even have learned that $x \approx \frac{355}{113}$. Where do approximations like these come from?

3. Questions about quadratic forms:
   When is there an integer solution to $Ax^2+Bxy+Cy^2=D$? If there are solutions, what is the family of solutions like? Finite? Infinite? Are there "symmetries"— changes of coordinates $(x,y) \mapsto (x', y')$ that take solutions to solutions?

# Our setting: The TOPOGRAPH



- It's a TREE
- Each vertex has degree 3
- It's planar, with open polygonal faces

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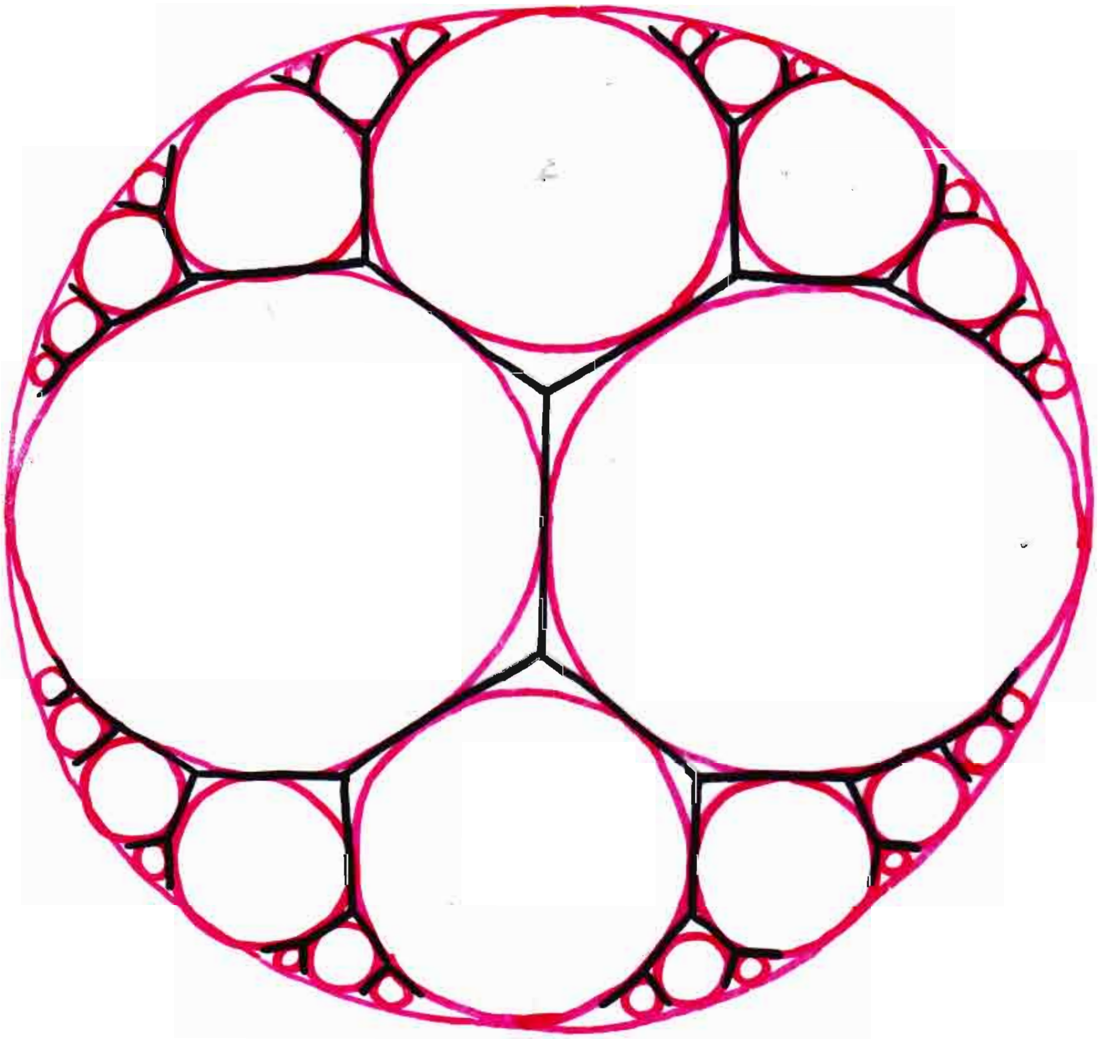
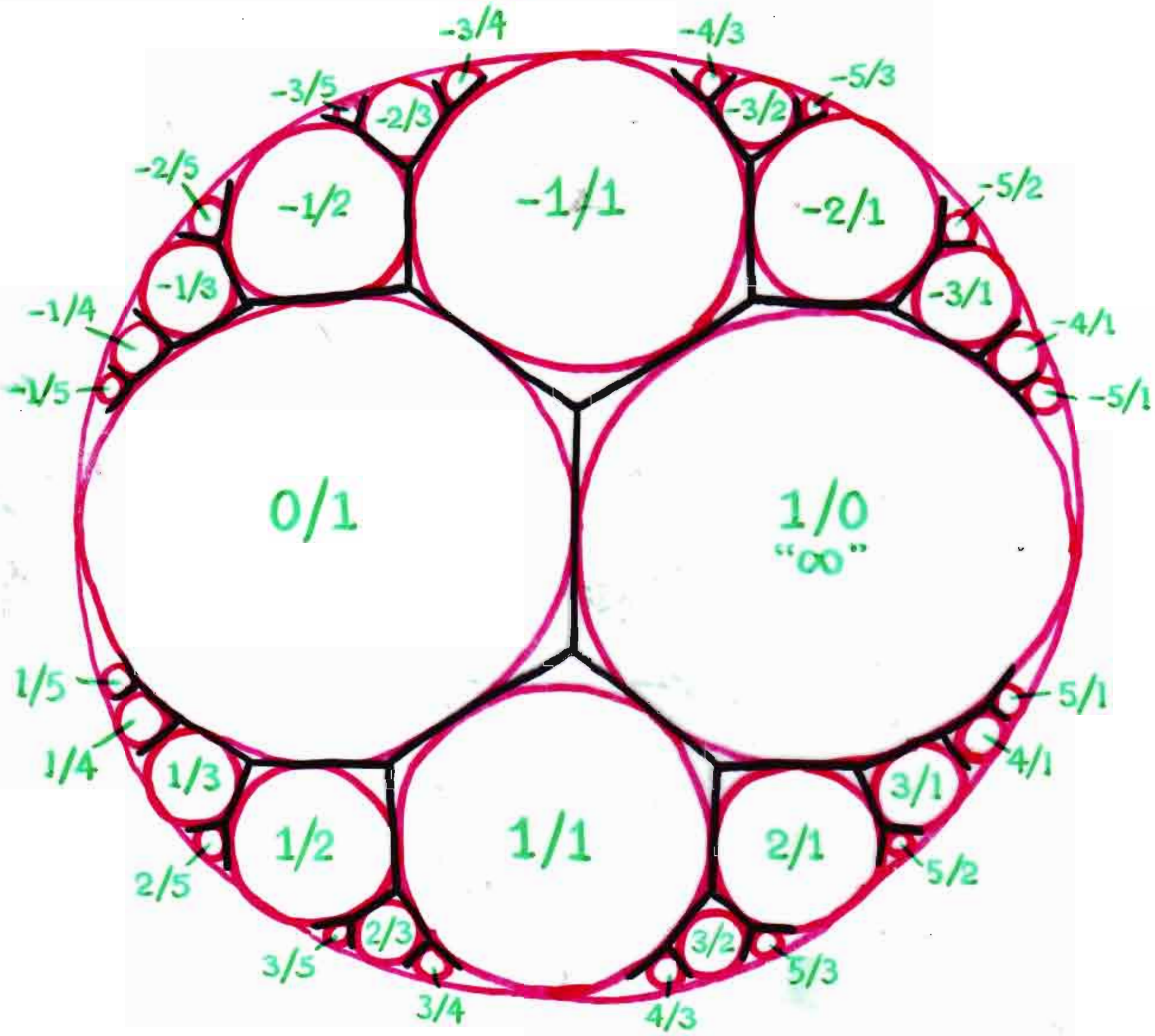


- It's a TREE

- Each vertex has degree 3

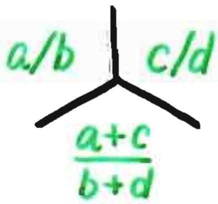
- It's hyperbolic planar, with open polygonal faces ... ...which have inscribed "Apollonian" circles.

# Our setting: The TOPOGRAPH

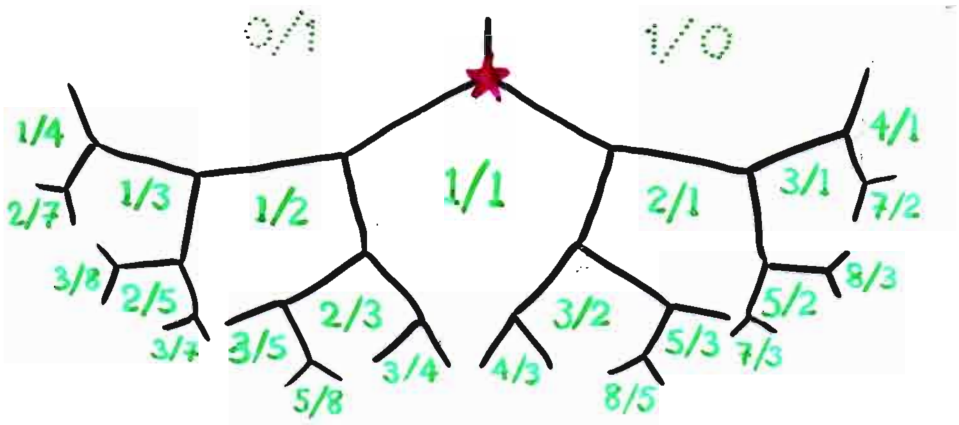



- It's a TREE
- Each vertex has degree 3
- It's hyperbolic planar, with open polygonal faces … …which have inscribed "Apollonian" circles.

We write a rational number on each face.
Around each vertex, one of the three numerators is the sum of the other two;
likewise with denominators.

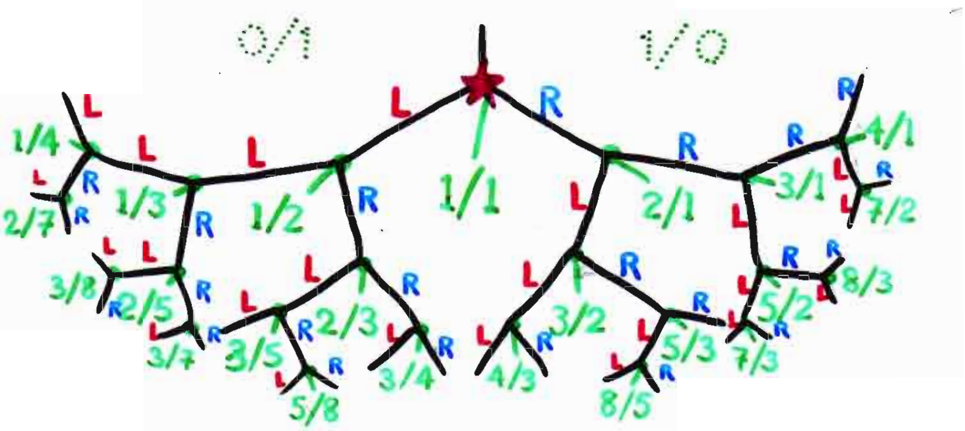


# The positive part.



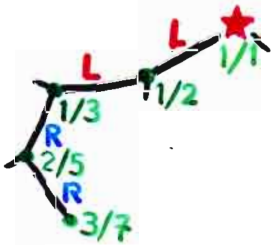

A binary tree rooted at ★

# The positive part.




A binary tree rooted at ★

Giving directions: $\frac{3}{7}$ is ★ L L R R

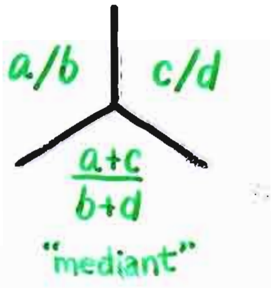



Driving without a map:
Start at $\frac{1}{1}$. If you're at vertex $\frac{p}{q}$, and $\frac{p}{q} > \frac{3}{7}$, go LEFT. But if $\frac{p}{q} < \frac{3}{7}$, go RIGHT.

## Sorting property of the tree:

Once you go LEFT from $\frac{p}{q}$, you only reach numbers less than $\frac{p}{q}$. (Going RIGHT, the opposite is true.)

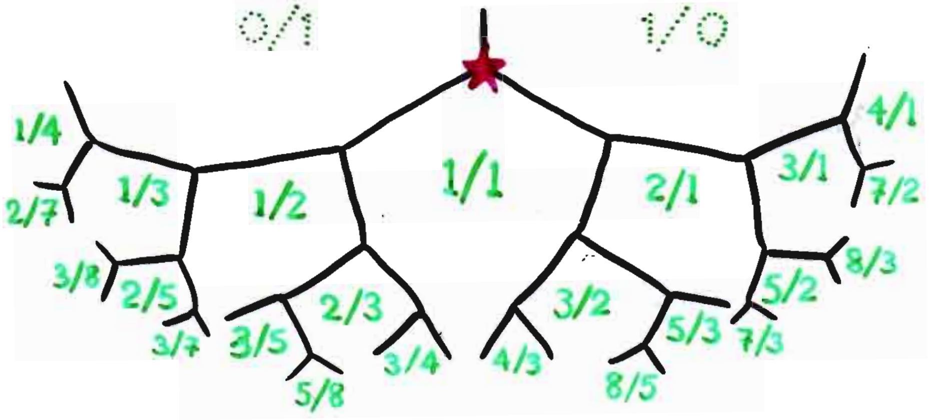
Idea of the proof:




$\boxed{\text{If } \frac{a}{b} < \frac{c}{d}, \text{ then } \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.}$ + induction

Proof. $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc \Rightarrow ab + ad < ab + bc$
$\Rightarrow a(b+d) < b(a+c) \Rightarrow \frac{a}{b} < \frac{a+c}{b+d}$. Similarly $\frac{a+c}{b+d} < \frac{c}{d}$.

# The positive part.




A binary tree rooted at ★

- **Proposition:** Every rational number that labels a face of the tree occurs in <u>reduced</u> form (i.e., $\frac{p}{q}$ where $\gcd(p,q)=1$).

- **Proof.** We'll actually show that if $\frac{a}{b}, \frac{c}{d}$ are adjacent faces, then $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$.

  The determinant, defined as $ad-bc$.

  - Why will this suffice for our original proposition?

We use induction on the *depth* of the edge separating $\frac{a}{b}, \frac{c}{d}$.

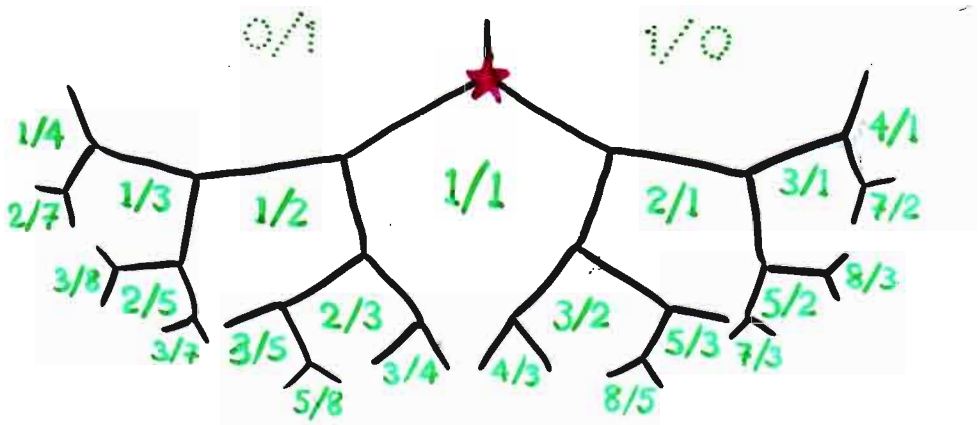
Base case: $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$

Inductive step: If faces $\frac{a}{b}, \frac{c}{d}$ are adjacent, then with no loss of generality, $\frac{a}{b}$ is also adjacent to a *parent* of $\frac{c}{d}$ (namely $\frac{c-a}{d-b}$). By the inductive hypothesis, we may assume $\begin{vmatrix} a & c-a \\ b & d-b \end{vmatrix} = \pm 1.$

Thus $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad-bc = ad-ab+ab-bc = a(d-b)-b(c-a)$

$= \begin{vmatrix} a & c-a \\ b & d-b \end{vmatrix} = \pm 1.$ $\square$

# The positive part.




A binary tree rooted at ★

- **Proposition:** EVERY (positive) rational number occurs in the tree somewhere (... and in only one place).

- **Step 1 of proof:** with $a,b,c,d>0$

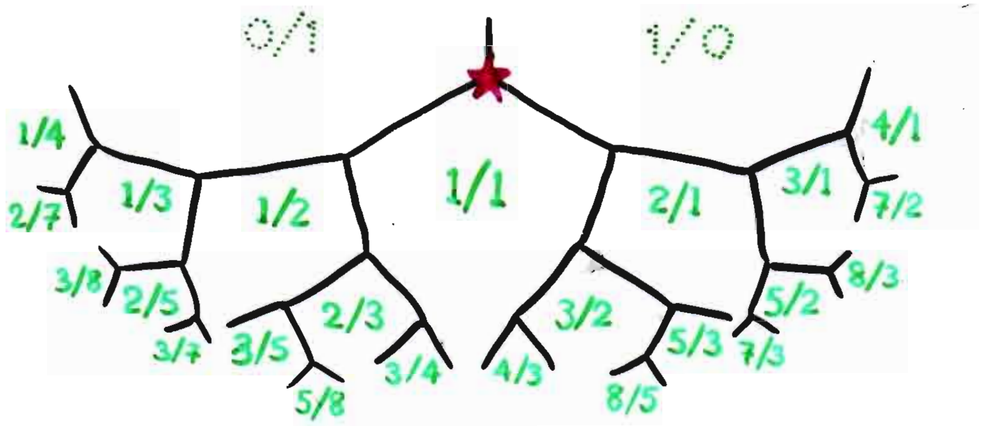
  If $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$, then $\frac{a}{b}$ and $\frac{c}{d}$ both occur in the tree, and they are adjacent. (This is like a converse to our last Prop.)

  The proof of this is similar to the last proof we saw. Suppose there's a "least" counterexample — say, with $a+b+c+d$ as small as possible. With no loss of generality, $a \le c$. We can also assume $b<d$ (why?). Now $\begin{vmatrix} a & c-a \\ b & d-b \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$, so by our assumption, $\frac{a}{b}$ and $\frac{c-a}{d-b}$ must occur in the tree, adjacent to each other. But then their "child" is $\frac{c}{d}$.

- **Step 2 of proof:** For any reduced fraction $\frac{c}{d}$, there exists some $\frac{a}{b}$ so that $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$, i.e. $ad-bc = \pm 1$.

  In fact, we can solve $ad-bc=1$ <u>and</u> $ad-bc=-1$, and even force $0 \le a < c$, $0 \le b < d$ (obtaining the two parents of $\frac{c}{d}$). How ?

# The positive part.




A binary tree rooted at ★

## Driving without a map, revisited: *ARE WE THERE YET?*

Define a <u>turn</u> as a change from going left to going right, or v.v. So the directions to $\frac{3}{7}$ (★LLRR) include just two turns, ★ & $\frac{1}{3}$ (counting the initial departure from ★ as a turn).

Directions to $\frac{5}{8}$ (★LRLR) include four turns (★, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$).

At every turn, the denominator is > twice what it was two turns prior (why?).

Thus the # of turns on the way to $\frac{p}{q}$ is at most

$$2 \log_2 q \approx 2.885 \ln q,$$

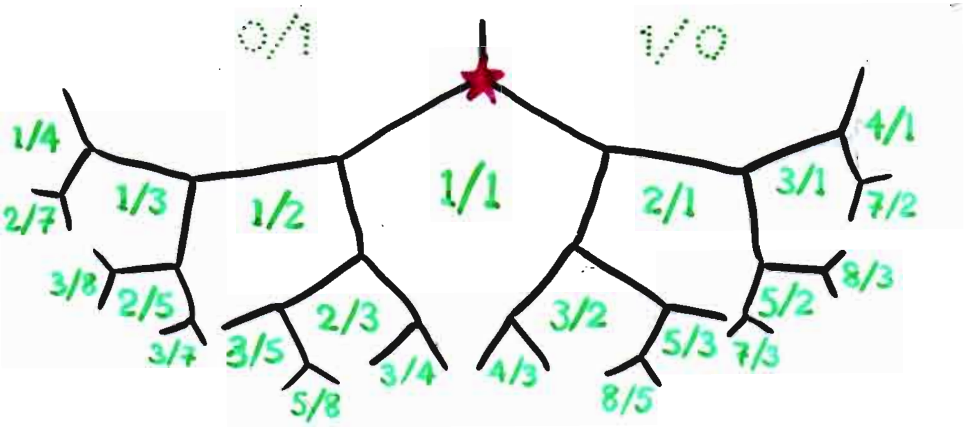
but in fact, an asymptotic estimate for the worst paths (★LRLRLR...) is

$$\log_{\frac{1+\sqrt{5}}{2}} q \approx 2.078 \ln q$$

turns.

This is also the worst running time for computing $\gcd(p,q)$ using the Euclidean algorithm.

# The positive part.

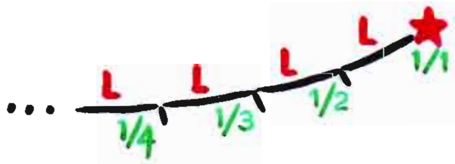



A binary tree rooted at ★

Is it fair to count turns rather than L's and R's?

Yes... because we can "see the next turn."

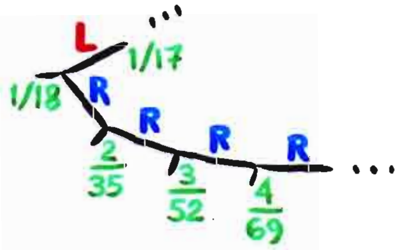
Let's search for .058544... .




Will we ever turn **RIGHT**?

Only when we reach some $\frac{1}{n} < .058544$.

It's easy to check that this occurs at $n=18$.

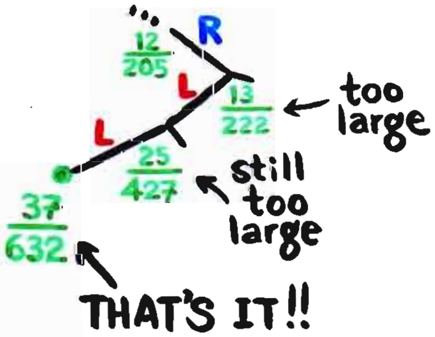



Now we're on the road $\left\{\frac{n}{1+17n} : n=1,2,3,\ldots\right\}$.

We'll only turn when $\frac{n}{1+17n} > .058544$.

Again, we can easily solve this inequality to see that the smallest such (integer) $n$ is 13.

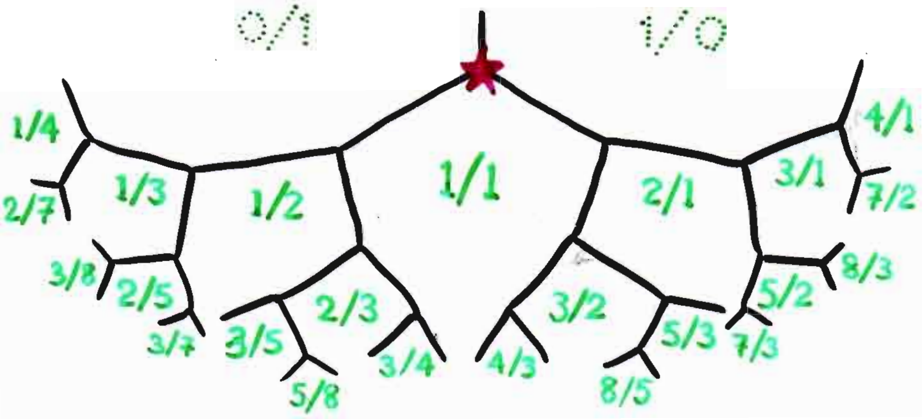
So we'll turn after 12 R's, at $\frac{13}{222}$.




This is how the trick works.

$$\frac{37}{632} = .058544\ldots$$

# The positive part.




A binary tree rooted at ★

## Some questions for YOU to ponder:

- To make sure $\frac{p}{q}$ can be uniquely identified, you must put a cap on $q$ and then demand a certain number of decimal digits. What is the relationship between these two requirements?

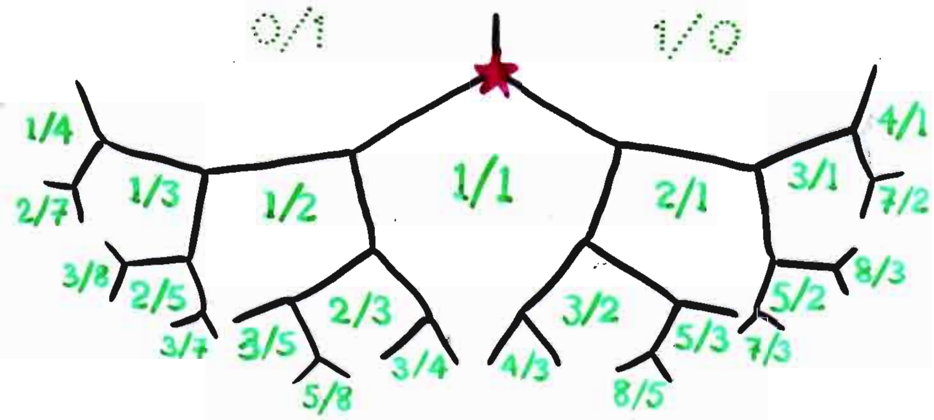
- Our directions to $\frac{37}{632}$ turned out to be

  ★LLLLLLLLLLLLLLLLLRRRRRRRRRRRRLL.

  Good thing the "running time" of the parlor trick depends on the number of turns, not L's and R's! But are such long strings of L's or R's unusual? After all, each position in the tree is akin to a run of coin tosses ...

- **BONUS POINTS:**
  Optimize this parlor trick for performance.

# The positive part.



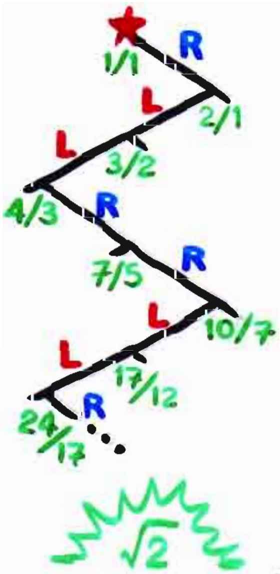

A binary tree rooted at ★

## Searching for irrational numbers.

The path ★RLRLRL... is interesting: it gives $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots$ — all numbers of the form $\frac{Fib_{n+1}}{Fib_n}$. Their limit is the golden mean, $\Phi = \frac{1+\sqrt{5}}{2} = 1.618\dots$. It's irrational — in fact, we've proved this! (How?)

Every infinite path into the tree must have an irrational limit, and every positive irrational number is the limit of some path.

## Searching for $\sqrt{2}$:



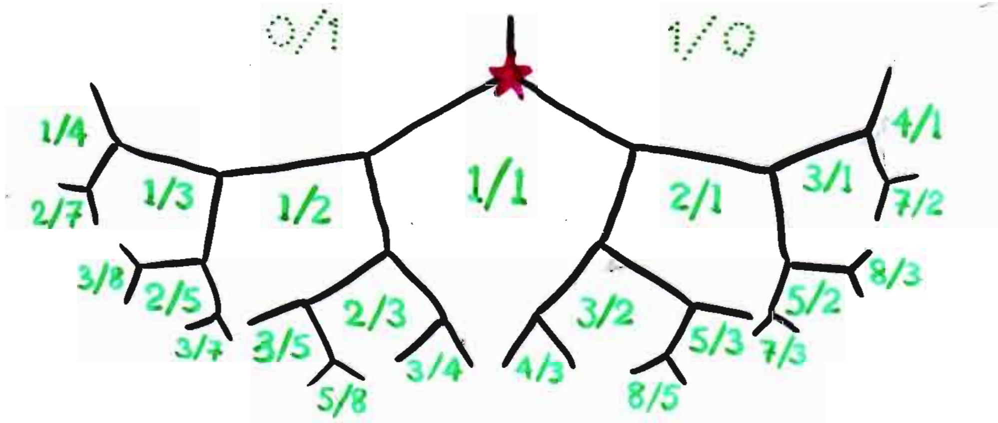

= 1.0000
2.0000
1.5000
1.3333
1.4000
1.4285
1.4166
1.4117
⋮
(= 1.4142...)

These are in some sense the best rational approximants of $\sqrt{2}$.

The path seems to be periodic:

★R LL RR LL RR ....

# The positive part.




A binary tree rooted at ★

"Are we there yet?"

"No."

"Are we there yet?"

"No."

"Are we there yet?"

"No."

"Are we there yet?"

"No."

"Are we there yet?"

"No."

"Are we there yet?"

"No."

"Are we there yet?"

"Almost..."

"Are we there yet?"

"No."

# Introduction to Quadratic Forms

- Pell's equation.

  We know $x^2-2y^2=0$ has no integer solutions.
  What about $x^2-2y^2=\pm 1$? For large $x$ and $y$, if $x^2-2y^2=\pm 1$, then $\frac{x}{y}\approx\sqrt{2}$. Another way to rationally approximate $\sqrt{2}$! But how do we find such $x,y$?

- The general question.

  How can we see whether $\underbrace{Ax^2+Bxy+Cy^2}_{\text{"binary quadratic form"}} = D$ has integer solutions?
  If it does, what is the solution set like?
  How do we *find* the solutions?

# Basic properties of quadratic forms.

Let $Q(x,y) := Ax^2+Bxy+Cy^2$, where $A,B,C \in \mathbb{Z}$. Then

(i) $Q(-x,-y) = Q(x,y)$

(ii) $Q(kx,ky) = k^2 Q(x,y)$

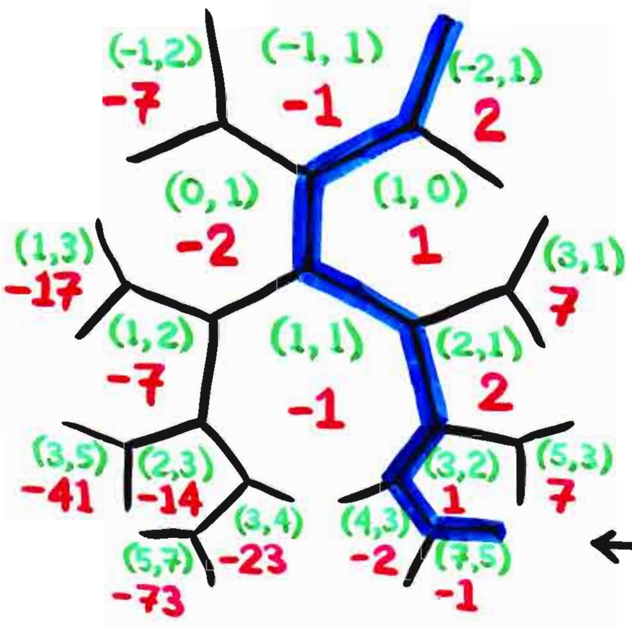
(iii) $Q(x-x', y-y')$, $\quad Q(x,y)+Q(x',y')$, $\quad Q(x+x', y+y')$
form a three-term arithmetic progression.

Thanks to (i) and (ii), our TOPOGRAPH is perfectly suited to "graphing" the values of $Q$ (up to square factors).

We no longer think of the cells as rational numbers— just as pairs $(x,y)$ which are relatively prime, with no distinction between $(x,y)$ and $(-x,-y)$.

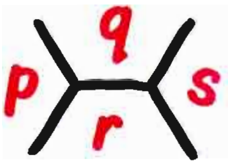
On each face of the graph, we write $Q(x,y)$.

Example: $x^2-2y^2$




← What's that blue path ??

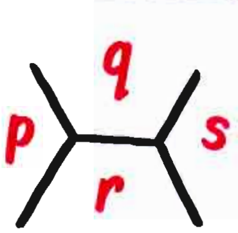
Using property (iii), it was easy to fill in the red values.




$p$, $q+r$, $s$ are always in arith. progression...

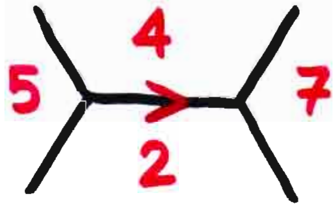
... so three adjacent cells are all we need to get started.

# The climbing lemma.

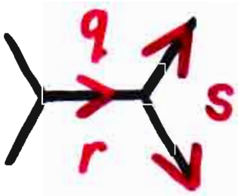
Given 

, either $p < q+r < s$, or $p > q+r > s$
(or possibly $p = q+r = s$).

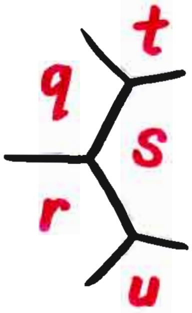
We draw an arrow to indicate the direction of <u>increase</u>.

e.g.: 


The lemma says that if $q, r > 0$ and the arrow points to $s$, then the other arrows at the $q$-$r$-$s$ junction point away:




The proof is simple and uses the progression property:




$$\begin{aligned} t = 2(q+s)-r &= 2q+2s-r \\ &> 2q+s+(q+r)-r \\ &= 3q+s \\ &> q+s. \end{aligned}$$

(Likewise $u > r+s$.)

Inductively applying the lemma, we see that continuing along any path away from the $q$-$r$-$s$ node, the arrows stay with us, taking us to ever-higher values of $Q(x,y)$.

(A similar result holds if $q, r < 0$ and we draw arrows in the direction of decrease. Thus absolute values also "climb" as we head into the negative region.)

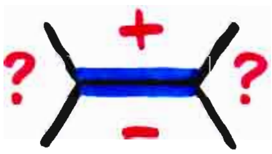
# Proposition.

If $Q(x,y)$ takes on both positive & negative values, but never 0, then the positive & negative regions of the topograph are separated by an infinite *river* — a simple path that never forks.

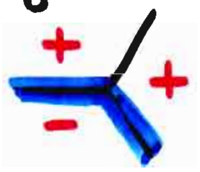
Also, for integer-valued quadratic forms, the values along the river repeat periodically (as do the bends of the river, in terms of left & right).

## Proof:

Let "the river" just mean the collection of edges that separate positive from negative values. We know there is at least one such edge:




But whatever the sign of each ?, the river winds its way...





There's no way for it to end.
The climbing lemma guarantees that there's only one river.
Periodicity is more technical to prove. You must believe me (or verify!) that $d = qr - \frac{1}{4}(s-q-r)^2$ is the same at all nodes

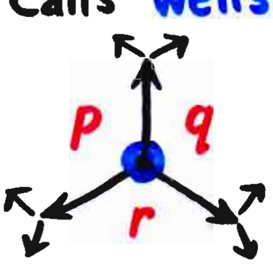

; $d$ is an invariant of the quadratic form (called the *determinant*). Now if $q, r$ lie across the river, then $d \leq qr < 0$. There are only finitely many configurations $(q,r,s)$ for which $qr - \frac{1}{4}(s-q-r)^2 = d$ could be true. Once one recurs, the river repeats (why?). □

# Other types of quadratic forms.

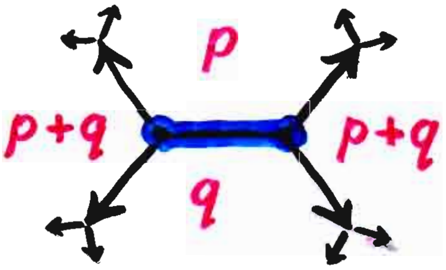
We just considered what are known as +− forms, meaning that their values are both pos. & neg. but never zero.

There are five other (non-trivial) types, each with its own topography:

A + form ("positive definite") has one or two lowest points, which Conway calls **wells**:



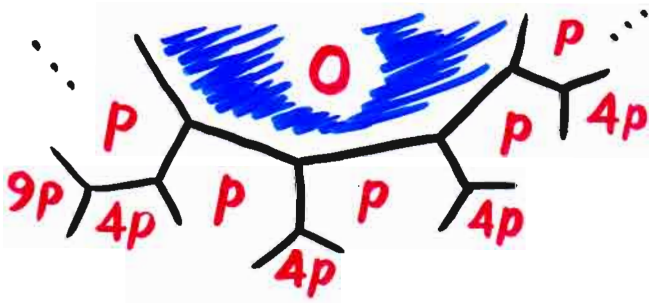

or



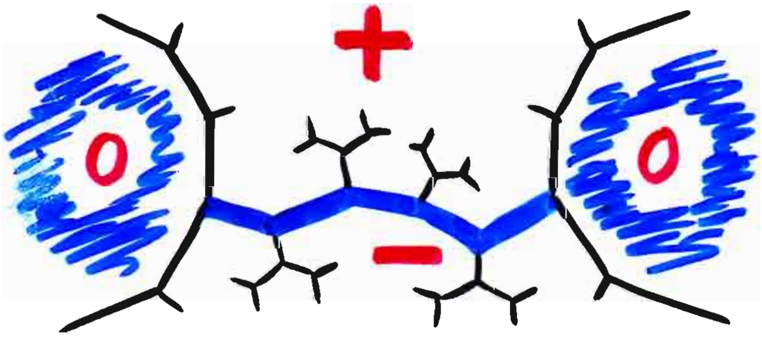

$(p<q+r,\ q<r+p,\ r<p+q)$

All arrows point away from these wells. A − form is similar.

A +0 (or −0) form has a **lake** with a constant-value border:




A +−0 form has **two lakes** connected by a finite **river** or a **weir**:

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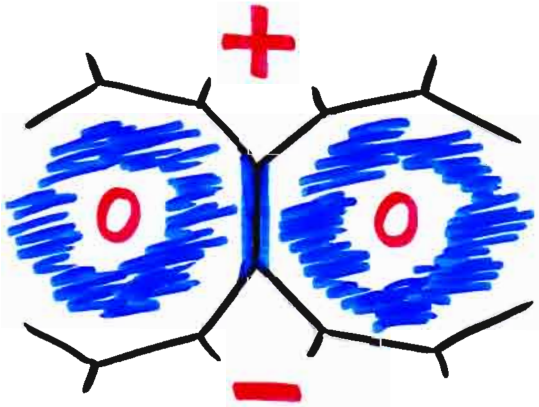


Now we know how to "see" all values of any quadratic form. Thanks to the climbing lemma, we know that if we map away from the "water" far enough to reach an altitude of $D$, and it doesn't appear, then

square-free!

$$Ax^2+Bxy+Cy^2=D$$ has no solution.

But!

We've also resolved...

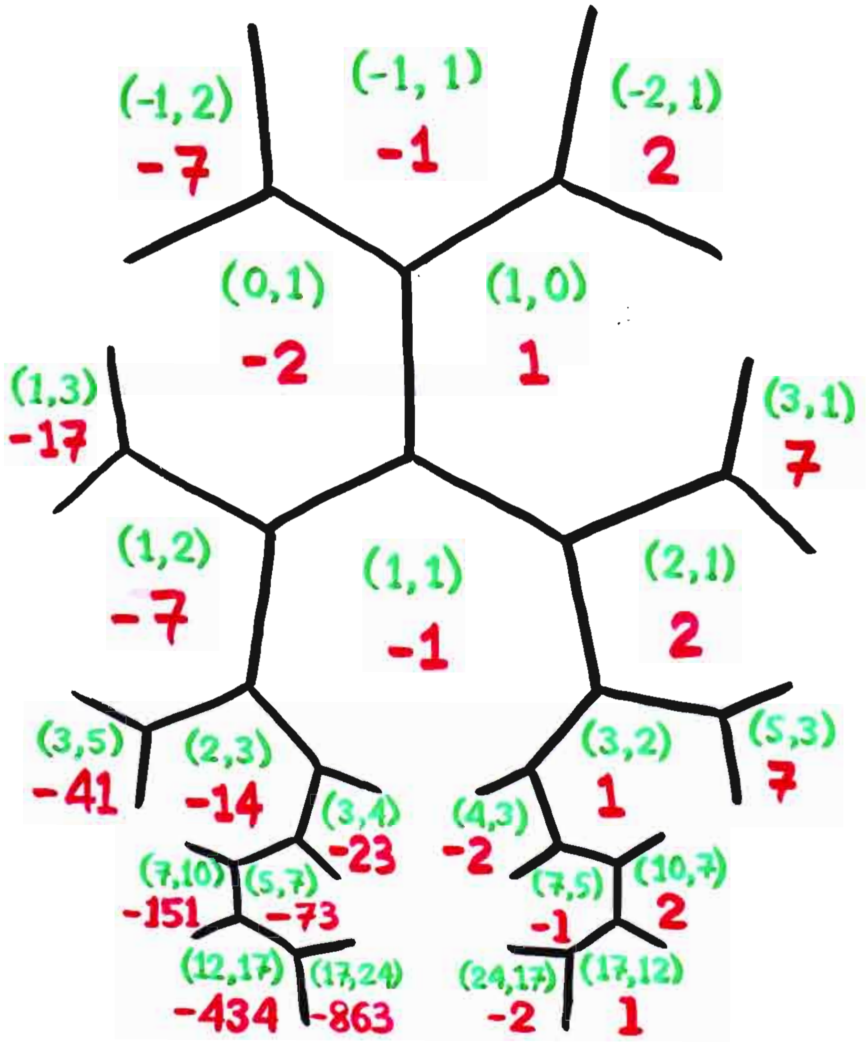
- The isometry problem.

  An isometry is a change of coordinates $(x,y)\mapsto(x',y')$ so that $Q(x,y)=Q(x',y')$. How can we "see" isometries on the topograph?

- The equivalence problem.

  When are two quadratic forms the same up to a change of coordinates, i.e., $Q_1(x,y)=Q_2(x',y')$?

  This can also be checked visually by comparing topographs...




Topographic plot of $x^2-2y^2$

The positive values* are $1, 2, 7, \ldots$.

The negative values* are $-1, -2, -7, -14, -17, -23, \ldots$.

No other values* can be realized by $x^2-2y^2$ for $x, y \in \mathbb{Z}$!

*up to square factors

Possible isometries of this form?

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 17 & 24 \\ 12 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Verify:

$$(x')^2-2(y')^2 = (3x+4y)^2-2(2x+3y)^2 = 9x^2+24xy+16y^2 - (8x^2+24xy+9y^2) = x^2-2y^2.$$

What's the <u>isometry group</u>?

# Other directions to go from here...

(homework!)

Existence of good rational approx. to an irrational number $\xi$ ($\frac{p}{q}$ is good if $\left|\frac{p}{q}-\xi\right|<\frac{1}{2q^2}$)

Those curious Apollonian circles (hyperbolic embedding of the topograph) — see Ford circles, Farey series, Soddy's "kiss precise"...

HERE ★

Relations between the tree of rational numbers, the Euclidean algorithm, and continued fractions

Quadratic forms in more than two variables

Conway's "Theorem of 15": A pos. def. matrix-integral quadratic form in any number of variables that assumes all values from 1 to 15 must take ALL pos. integers as values.

**CONTACT ME:**
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**REFERENCE:**
John H. Conway, *The Sensual (Quadratic) Form.*

"Ambiguous number" problem:

A number $\xi$ is called <u>ambiguous</u> if there are rational numbers $\frac{p}{q}<\xi<\frac{p'}{q'}$ such that $\xi-\frac{p}{q}=\frac{p'}{q'}-\xi$, and there is no $\frac{p''}{q''}$ with $q''<\max\{q,q'\}$ such that $\left|\xi-\frac{p''}{q''}\right|<\xi-\frac{p}{q}$. How many ambiguous rational numbers have value between 0 and $\frac{1}{100}$, and denom. $\leq 10^8$?