The first three problems concern the justification of the power rule, \( \frac{d}{dx}(x^n) = nx^{n-1} \), in three cases: when \( n \) is a positive integer, a negative integer, or a rational number. Each case requires a different argument. (There is a fourth case which we won’t justify: the case of irrational powers, such as \( x^{\sqrt{2}} \) and \( x^\pi \). Justification of the power rule in that case requires some very advanced mathematics!)

**Problem \( \Delta \) (Pawn).** Examples 4 and 5 in §2.3 of your book show how to arrive at the derivatives of \( x^2 \) and \( x^3 \) directly from the limit definition. Start by reviewing those examples. Then, in the same spirit:

(a) Write a limit expression for the derivative of \( f(x) = x^4 \).

(b) Use algebra to simplify your expression and evaluate the limit, showing all the steps. The result should agree with the power rule.

(c) Suppose you were to repeat this problem for \( f(x) = x^5 \), \( x^6 \), etc. You could follow a similar process in each of these cases, but the details would vary. Explain, in as much detail as possible, the pattern that all these cases have in common. The fully-expanded limit expressions for \( \frac{d}{dx}(x^n) \) have more and more terms as \( n \) gets larger, yet, when fully simplified and evaluated, they always boil down to the familiar \( nx^{n-1} \): why?

**Problem \( \EuScript{Q} \) (Knight).** Now we consider the case of negative powers.

From Problem \( \Delta \), you know why \( \frac{d}{dx}(x^n) = nx^{n-1} \), given that \( n \) is a positive integer. Use this information together with the quotient rule to evaluate \( \frac{d}{dx}\left(\frac{1}{x^n}\right) \). Then, use algebra to confirm that your answer agrees with the power rule applied to \( \frac{d}{dx}(x^{-n}) \).

**Problem \( \clubsuit \) (Bishop).** Finally, we consider the case of fractional powers. Throughout this problem, you may assume that the power rule holds for integer powers (as you confirmed this in Problems \( \Delta \) and \( \EuScript{Q} \)).

Let \( f(x) = x^n \), \( g(x) = x^{m/n} \) (where \( m \) and \( n \) are integers and \( n \neq 0 \)). Let \( h(x) = f(g(x)) \).

(a) Write a simple formula for \( h(x) \).

(b) Based on your answer from part (a), what is the derivative of \( h(x) \)?

(c) Based on the chain rule, we know the derivative of \( h(x) \) is equal to \( f'(g(x)) \cdot g'(x) \). Set this expression equal to your answer from part (b), and solve for \( g'(x) \) in terms of \( x \). Simplify your answer as much as possible, using algebra to confirm that your answer agrees with the power rule applied to \( \frac{d}{dx}(x^{m/n}) \).
Problem R (Rook). When an object (such as a water balloon) is attached to an ideal spring, tugged, and released, its displacement is given as a function of time by

\[ y = A \cos \left( \sqrt{\frac{k}{m}} \cdot t \right), \]

where \( m \) is the mass of the object, \( k \) is the spring constant (representing the stiffness of the spring), and \( A \) is the amplitude of the oscillation.

(a) In terms of the constants \( m, k, \) and \( A \), find:
   (i) the first positive time \( t_1 \) at which the object is farthest from equilibrium,
   (ii) the first positive time \( t_2 \) when the object is moving at its maximum speed,
   (iii) the first positive time \( t_3 \) when the object is accelerating fastest (irrespective of sign), and
   (iv) the period of the oscillation, \( T \).

(b) If we think of \( T \) as a function of \( m \) (still holding \( k \) and \( A \) constant), what is \( \frac{dT}{dm} \)?

What is the practical meaning of its sign?

Problem Q (Queen). Consider the function \( f(x) = \ln(x^2 + a^2) \), where \( a \) is a positive constant.

(a) Find the critical point(s), if any. Classify as local maxima, minima, or neither, giving reasons for your answer (not based on a graph).

(b) Find the inflection point(s), if any. On what interval(s) is \( f(x) \) concave up? Concave down? Show your work.

(c) If the line \( y = -2 \) is tangent to the graph of \( y = f(x) \), what is the value of \( a \)?

Problem K (King). Skipping at a speed of \( v \) miles per hour burns approximately \( 100 + 29v^2 - \frac{1}{3}v^4 \) calories per hour, given that \( 1 \leq v \leq 6 \).

At what speeds would one burn the most and least calories per mile? (Careful: these are not necessarily the speeds at which one burns the most and least calories per hour!)

Give a complete justification for your answer; if you found the critical points of any function, indicate how you classified them as maxima and minima.