Problem 1. The population of lemmings in a remote region of Norway can be modeled by means of the “logistic map.” According to this (somewhat simplistic) model, the current year’s lemming population $P$ (in thousands of lemmings) is the sole input determining next year’s lemming population, which is therefore a function of $P$; we call this function $\ell(P)$. Some values of $\ell(P)$ are given by the following table:

<table>
<thead>
<tr>
<th>$P$</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>350</th>
<th>400</th>
<th>450</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell(P)$</td>
<td>0</td>
<td>162</td>
<td>288</td>
<td>378</td>
<td>432</td>
<td>432</td>
<td>432</td>
<td>288</td>
<td>162</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(Notice that the higher $P$ is, the higher $\ell(P)$ is, up to a point. If $P$ is too large, then competition for resources drives the population down in the next generation.)

(a) Estimate $\ell(\ell(150))$. Show your work.  

We have $\ell(\ell(150)) = \ell(378)$. Interpolating a linear function between $(350, \ell(350))$ and $(400, \ell(400))$, we get an estimate of 

$$ \ell(378) \approx 378 + \frac{288 - 378}{400 - 350} \cdot (378 - 350) = 327.6. $$

(I also accepted 333, halfway between 288 and 378, as close enough.)

(b) Interpret your answer from part (a) in terms of lemming population. 

(Warning: Pay close attention to the nature of the input and output of $\ell$.)

If the current population of lemmings is 150,000, then the expected population in two years is 327,600.

(c) Estimate the derivative of $\ell(P)$ at $P = 150$. Show your work.

We can use the average rate of change of $\ell(P)$ on the interval [100, 200] as an estimate:

$$ \ell'(150) \approx \frac{432 - 288}{200 - 100} = 1.44 \text{ thousands of lemmings per thousand lemmings}. $$

(d) Interpret your answer from part (c) in terms of lemming population.

If the current population is close to 150,000 lemmings, an increase of 1,000 lemmings now will cause the population in one year to be about 1,440 higher than it would otherwise have been.

(e) Estimate the derivative of $\ell(\ell(P))$ at $P = 150$. Show your work.

By the chain rule, the derivative of $\ell(\ell(P))$ is $\ell'\ell'(P)$. In this case, that’s

$$ \ell'(378) \cdot \ell'(150) \approx \left( \frac{288 - 378}{400 - 350} \right) \cdot (1.44) = (-1.8)(1.44) = -2.592 \frac{\text{thou. lemmings}}{\text{thou. lemmings}}. $$

(f) Interpret your answer from part (e) in terms of lemming population.

If the current population is close to 150,000 lemmings, an increase of 1,000 lemmings now will cause the population in two years to be reduced by about 2,592, compared to what it would have been otherwise.
In case you need it for the next problem, here’s the quotient rule:
\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]

**Problem 2.** Let \( h(x) = (\ln x)/x \).

(a) Determine the intervals on which \( h(x) \) is (i) increasing, (ii) decreasing. Use the techniques of calculus; show your work, and do not round off the endpoints of your intervals. \([9 \text{ pts.}]\)

Applying the quotient rule, we get the derivative:
\[
h'(x) = \frac{(1/x)(x) - (\ln x)(1)}{(x)^2} = \frac{1 - \ln x}{x^2}.
\]

Notice that the denominator is positive for all \( x > 0 \) (we don’t care about \( x \leq 0 \), because \( h(x) \) is undefined for those \( x \)). So the sign of \( h'(x) \) is determined entirely by the numerator, \( 1 - \ln x \), which is positive if \( x < e \) and negative if \( x > e \). Accordingly, \( h(x) \) is increasing on \((0, e)\) and decreasing on \((e, \infty)\).

(b) Determine the intervals on which \( h(x) \) is (i) concave up, (ii) concave down. (Same provisos as above.) \([9 \text{ pts.}]\)

Applying the quotient rule again, we get
\[
h''(x) = \frac{(-1/x)(x^2) - (1 - \ln x)(2x)}{(x^2)^2} = \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-x - 2x + 2x \ln x}{x^4} = \frac{x^2(2 \ln x - 3)}{x^4} = \frac{2 \ln x - 3}{x^3}.
\]

Again the denominator is positive for all \( x > 0 \), so we only care about the sign of the numerator, which is negative if \( \ln x < 3/2 \) (i.e., if \( x < e^{3/2} \)) and positive if \( \ln x > 3/2 \) (i.e., if \( x > e^{3/2} \)). Thus \( h(x) \) is concave down on \((0, e^{3/2})\) and concave up on \((e^{3/2}, \infty)\).

**Problem 3.** Using your results from Problem 2, find the \( x \)- and \( y \)-coordinates of the highest point on the graph of \( y = x^{1/x} \) (over the domain \( x > 0 \)). Do not round off.

(Warning: Attempting to differentiate \( x^{1/x} \) is probably a bad idea. Look for another approach.) \([4 \text{ pts.}]\)

Observe that \( \ln(x^{1/x}) = \frac{1}{x} \cdot \ln x \), which is the function \( h(x) \) from Problem 2. Since \( \ln u \) is an increasing function of \( u \), we know that \( \ln(x^{1/x}) \) reaches its maximum at the same \( x \)-coordinate as \( h(x) \) does. This coordinate is \( x = e \), since that’s where \( h(x) \) changes from increasing to decreasing. Thus \( x^{1/x} \) reaches its highest point at \((e, e^{1/e})\).