Instructions:

- Do not open this exam until you are told to begin.
- This exam has 8 pages including this cover. There are 7 problems.
- Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
- Please read the instructions for each problem carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
- Show an appropriate amount of work for each problem so that the graders can see how you obtained your answer. Include units in your answers where appropriate.
- You may use your calculator.
- Please turn off all cell phones and pagers and remove all headphones.

For graders’ use (do not write here):

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**Problem 1.** Suppose the population of Islandia has a doubling time of 64 years, and is on course to reach 4 million at the beginning of the year 2050. Assume the population is an exponential function of time.

(a) Find a formula for $P(t)$, the population (in millions) $t$ years after the beginning of the year 2000. Give your answer both in the form $P_0 a^t$ and in the form $P_0 e^{rt}$.

(b) In one sentence, explain how you know that $P$ is an invertible function.

(c) Find a formula for $P^{-1}(t)$.

(d) What is the average rate of change of $P(t)$ from 2000 to 2050, with units?

(e) During precisely what year (between 2000 and 2050) will the instantaneous rate of population growth be closest to the average rate of growth between 2000 and 2050?

(f) Compute $P^{-1}(5)$ and $(P^{-1})'(5)$. What do these numbers mean in terms of population?
Problem 2. As you may have noticed, there are a lot of problems in your textbook that involve yams. As it happens, most of the M-STEM instructors and coordinators love yams! In fact, we had a yam-eating contest recently. Let $A(t)$ be the number of yams that I (Austin Shapiro) ate during the first $t$ minutes of the contest, and let $D(t)$ be the number of yams Darryl Koch (M-STEM Coordinator) ate during the first $t$ minutes. Suppose that I told you I was doing computations during the yam-eating contest (what else would I be doing?) and computed the following:

- Austin abruptly quit eating after 19.5 minutes
- Austin ate 12.5 yams before quitting
- $A'(t) = 2.5$ for $t$ just before 19.5
- Darryl abruptly quit eating after 20 minutes
- Darryl ate 11.5 yams before quitting
- $D'(t) = 8$ for $t$ just before 20

After the competition was over, Austin and Darryl had the following conversation:

* Austin: “I won! I’m the champion of the world!”
* Darryl: “You got lucky! We both know that if we had both eaten until $t = 20.5$ minutes, then I would have won!”

Using all of the information provided, give a mathematical argument to support or cast doubt on Darryl’s claim. [8 pts.]
Problem 3. Suppose $f(x)$ is continuous for all $x$, and $f(0) = 0$. Suppose this is a graph of the derivative, $f'(x)$:

(a) Draw possible graphs of $f(x)$ and $f''(x)$ (on separate axes). Be sure to show clearly what happens at $x = 0$, $x = a$, and $x = b$.  

(b) Which of these is a plausible formula for $f'(x)$ (the function whose graph was given to you)? Circle your answer; no explanation is needed.
Problem 4. Decide whether each of the following statements is true or false. (If it is not always true, select “false.”)

If the statement is true, explain why. If the statement is false, rewrite it with a single word or number changed so that the altered statement is true.

(a) “If $f(x)$ is concave down for all $x$, then $f'(2)$ is greater than the average rate of change of $f(x)$ on $[1, 4]$.”

(b) “If a car’s value is depreciating as an exponential function of time, then the car loses more value during the third year than during the fourth year.”

(c) “The period, amplitude, and first positive $x$-intercept of the function $y = 0.5 \sin (0.5(x + 0.5))$ are all different in value.”

(d) “Let $T(x)$ be the annual property tax (in dollars) on a house with an assessed value of $x$ thousand dollars, and let $P(t)$ be the assessed value of Darryl’s house (in thousands of dollars) $t$ years after 2000. If $T'(11) = 15$, $P(11) = 180$, and $P'(11) = 5$, then the property tax on Darryl’s house is currently rising at a rate of $75$ per year.”
Problem 5. Let $g(x) = xe^{-3x}$.

(a) On what intervals is $g(x)$ increasing and decreasing? [6 pts.]

(b) On what intervals is $g(x)$ concave up and concave down? [6 pts.]
Problem 6.

(a) For a general function $f(x)$ and a point $a$ in its domain, write down the limit definition of $f'(a)$. [2 pts.]

(b) For the specific function $f(x) = x^{\tan x}$, write down a limit expression for $f'($π/3$)$. Except for trivial simplifications, don’t try to evaluate the limit! [4 pts.]

(c) Estimate the limit from part (b) by plugging in small values of $h$. (Use your calculator for this. Make sure it’s in radian mode.) [2 pts.]

(d) Can you differentiate $x^{\tan x}$ by applying the chain rule with $\tan x$ as the inner function? If so, do it. If not, explain why not. [2 pts.]

(e) Let $h(x) = \ln(x^{\tan x})$. Simplify $h(x)$ using laws of logarithms, then differentiate $h(x)$ by using the product rule. (Reminder: The derivative of $\tan x$ is $1/\cos^2 x$.) [4 pts.]

(f) By the chain rule, $\frac{d}{dx}[\ln f(x)] = f'(x)/f(x)$.

With this in mind, use your work from part (e) to evaluate the derivative of $x^{\tan x}$ at $x = \pi/3$. (This is tricky—if you aren’t getting it, skip it and come back at the end.) [2 pts.]
Problem 7. A radar scanner consists of a transmitter and an antenna. It works by sending out pulses of radio waves which bounce off any object in their path. The waves which hit the target object scatter in all directions, and a small proportion of their energy is returned to the antenna of the radar scanner.

The strength of the return signal varies roughly in inverse proportion to the fourth power of the distance between the scanner and the target. This means that it can be modeled by a function $k/x^4$, where $x$ is the distance and $k$ is a constant specific to that radar transmitter, representing its power. (The stronger the transmitter, the higher the value of $k$.)

Suppose there are two radar scanners spaced 1000 feet apart along a straight road. The first scanner has three times the transmitting power of the second scanner. A car is traveling between the two scanners; let $x$ be its distance from the first scanner, where $0 < x < 1000$.

(a) Write down a function of $x$ which represents the total energy returned by the car to both scanners. You may use a single unknown constant in your formula.

(b) Using the techniques of calculus, find the position $x$ (between 0 and 1000) at which the car is least detectable—i.e., the point at which your function from part (a) reaches its minimum value. Show how you know it’s a minimum (as opposed to, say, a maximum).