Practice Exercises: Differentiation of Elementary Functions

Activity 1 (High road, low road). Many problems can be solved in more than one way; that applies to taking derivatives, too. For example, if \( f(x) = \frac{x^3+1}{x} \), then you can use the quotient rule together with the power rule to find \( f'(x) \):

\[
f'(x) = \frac{(3x^2) \cdot (x) - (x^3 + 1) \cdot (1)}{(x)^2},
\]

or you can convert \( 1/x \) to \( x^{-1} \) and use the product rule (again with the power rule):

\[
f(x) = (x^3 + 1)(x^{-1});
\]

\[
f'(x) = (3x^2)(x^{-1}) + (x^3 + 1)(-1 \cdot x^{-2}),
\]

or you can avoid the product and quotient rules by distributing first:

\[
f(x) = \frac{x^3 + 1}{x}
\]

\[
= x^2 + x^{-1};
\]

\[
f'(x) = 2x + (-1 \cdot x^{-2}).
\]

Are these answers equivalent? Yes, but some simplification is required (try it!).

Each of the following exercises asks you to compute the derivative of a function in two or more ways. The answers are self-checking: if you come up with the same derivative twice (after simplification), you probably did everything correctly; if not, look for mistakes.\(^1\)

These exercises are meant to serve two purposes: first, they’ll give you practice in applying the rules of differentiation; second, they’ll build your experience in recognizing which methods of attack are most efficient. As you practice breaking down complicated

\(^{1}\)But keep in mind that simplification isn’t always easy, and two formulas which look different may in fact be equivalent! One way to check is to graph both.
functions in multiple ways, pay attention to which choices turn out to be the most convenient!

1. Compute \( \frac{d}{dx} \left( \frac{x e^x}{2x+1} \right) \) in two ways:
   - by using the quotient rule first, with \( f(x) = xe^x \) and \( g(x) = 2x + 1 \);
   - by using the product rule first, with \( f(x) = x \) and \( g(x) = \frac{e^x}{2x+1} \).

(Whichever rule you use first, you’ll still have to use the other rule to compute \( f'(x) \) or \( g'(x) \).)

Compare your answers. Do they jibe?

2. Compute \( \frac{d}{dx} \left[ \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 \right] \) in two ways:
   - by using the chain rule, with outer function \( f(y) = y^2 \);
   - by expanding first, then differentiating term by term.

3. Compute \( \frac{d}{dx} \left[ (e^x)^3 + (e^x)^2 + e^x + 1 \right] \) in two ways:
   - by using laws of exponents to simplify first, then using the rule \( \frac{d}{dx}(e^{kx}) = ke^{kx} \);
   - by using the chain rule with \( g(x) = e^x \), \( f(y) = y^3 + y^2 + y + 1 \).

4. Compute \( \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right) \) in three ways:
   - by using the quotient rule directly;
   - by rewriting \( \frac{\cos x}{\sin x} \) as \( \frac{1}{\tan x} \) and using the quotient rule;
   - by rewriting \( \frac{\cos x}{\sin x} \) as \( (\tan x)^{-1} \) and using the chain rule with the power rule.

5. Compute \( \frac{d}{dx} (\cos^2 x - \sin^2 x) \) in three ways:
   - by rewriting \( \cos^2 x - \sin^2 x \) as \( (\cos x)(\cos x) - (\sin x)(\sin x) \) and using the product rule;
• by using the chain rule twice, with outer function \( f(y) = y^2 \) each time;

• by using a trigonometric identity to simplify the original function first.

6. Compute the derivative of \( f(x) = \sqrt{1-x^2} \) in two ways:

• by using the chain rule;

• by differentiating both sides of the equation \( x^2 + f(x)^2 = 1 \) and solving the resulting equation for \( f'(x) \).

7. Pick a strategy for computing \( \frac{d}{dx} \left[ e^{5 \ln(1/t^2)} \right] \), and do it. Check your answer by another method.

8. Pick a strategy for computing \( \frac{d}{dx} \left[ \sqrt{x \sqrt{x \sqrt{x}}} \right] \), and do it. If you’re unsure of your answer, check it the long way (i.e., using the chain rule and product rule multiple times).

9. Compute the derivative of \( f(x) = \arctan(x) + \arctan(1/x) \), and simplify fully. Try graphing \( y = f'(x) \). Are you surprised? Reconsider whether you have simplified as much as possible. Then try to explain this result by considering what you know about the tangent and arctangent functions.

Activity 2 (Looking for patterns). It’s often possible to save work by noticing patterns. For example, as we discussed in class, the 1000\(^{th} \) derivative of \( \sin x \) is simply \( \sin x \): this is true because the fourth derivative of \( \sin x \) in \( \sin x \), and 1000 is a multiple of 4. (How would you find the 2011\(^{th} \) derivative of \( \sin x \)?)

Another example: We saw in class that the \( n \)^{th} derivative of \( x^n \) is \( n! \), and the \((n+1)\)^{th} derivative of \( x^n \) is zero (as are all higher derivatives). There is no need to actually calculate all the derivatives, once you understand the pattern.
The following exercises only look tedious (or impossible). Keep your eye out for useful patterns!

1. What are the first, second, third, fourth, and fifth derivatives of \( \ln x \)? Find the pattern. Can you write down a formula for the \( n \)th derivative of \( \ln x \), in terms of \( x \) and \( n \)?

2. Repeat the above exercise for \( xe^x \).

3. What are the derivatives of \( e^{e^x} \)? \( e^{e^x} \)? \( e^{e^{e^x}} \)? Find the pattern. (Note: The order of exponentiation in a tower of powers is from the top down, so \( e^{e^x} \) is equivalent to \( e^{(e^x)} \), not \( (e^e)^x \).)

4. (a) Figure out the derivative of \( f(x)g(x)h(x) \) by using the product rule twice.

(b) Figure out the derivative of \( f(x)g(x)h(x)j(x) \). (In the spirit of Activity 1, there are several ways you can do this! You may find it helpful to use your answer from part (a).)

(c) Generalize: if \( F(x) = f_1(x)f_2(x)f_3(x)\cdots f_n(x) \), what is the derivative of \( F(x) \) in terms of \( f_1, f_2, f_3, \ldots, f_n \)? Can you justify your answer?

5. (a) Figure out the derivative of \( f(g(h(x))) \) by using the chain rule twice.

(b) Figure out the derivative of \( f(g(h(j(x)))) \).

(c) Generalize: what is the chain rule for a function composed from arbitrarily many functions?