MATH 776
REVIEW OF HOMOLOGICAL ALGEBRA

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Let $\mathcal{A}$ be an abelian category. One can take $\mathcal{A}$ to be the category of left modules over a ring without losing much generality.

1. Chain complexes

A **chain complex** in $\mathcal{A}$ is a pair $(M_n, d_n)_{n\in\mathbf{Z}}$ where M_n is an object of $\mathcal{A}$ and $d_n \colon M_n \to M_{n-1}$ is a morphism such that $d_{n-1} \circ d_n = 0$ for all n. We write a chain complex as

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow \cdots$$

We typically just write d in place of d_n, and leave this implicit when writing a chain complex. A **morphism** of chain complexes $f \colon M_\bullet \to N_\bullet$ consists of giving for each $n \in \mathbf{Z}$ a morphism $f_n \colon M_n \to N_n$ in $\mathcal{A}$, such that the diagrams

$$\begin{array}{ccc} M_n & \xrightarrow{d} & M_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ N_n & \xrightarrow{d} & N_{n-1} \end{array}$$

commute for all n. In this way, we have a category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. It is again an abelian category, with kernels, cokernels, and images computed pointwise.

Let $M_\bullet$ be a chain complex. Since $d_n \circ d_{n+1} = 0$, it follows that $\operatorname{im}(d_{n+1}) \subset \ker(d_n)$. The **homology** of $M_\bullet$ is defined to be the quotient: specifically,

$$\mathrm{H}_n(M_\bullet) = \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})}.$$

The complex $M_\bullet$ is said to be **acyclic** if $\mathrm{H}_n(M_\bullet) = 0$ for all n. If $f \colon M_\bullet \to N_\bullet$ is a morphism of chain complexes then f naturally induces an isomorphism $\mathrm{H}_n(M_\bullet) \to \mathrm{H}_n(N_\bullet)$ for all n. The morphism f is said to be a **quasi-isomorphism** if these maps are all isomorphisms.

Remark 1.1. There is a variant notion called "cochain complex" that is sometimes used. The only difference is notation. In a cochain complex, the groups are indexed with a superscript (so M^0, M^1, etc.), and the differentials increase degree (so $d^0 \colon M^0 \to M^1$, etc.). All the concepts and theorems we prove can be translated to this language. □

2. Chain homotopies

Let $f \colon M_\bullet \to N_\bullet$ be a morphism of chain complexes. We say that f is **null homotopic** if there exist morphisms $s_n \colon M_n \to N_{n+1}$ such that

$$f_n = d_{n+1}s_n + s_{n-1}d_n.$$

Date: February 19, 2019.

The diagram is

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & M_{n+1} & \xrightarrow{d} & M_n & \xrightarrow{d} & M_{n-1} & \xrightarrow{d} & \cdots \\ & & & \swarrow^{s_n} & \downarrow f_n & \swarrow^{s_{n-1}} & & & \\ \cdots & \xrightarrow{d} & N_{n+1} & \xrightarrow{d} & N_n & \xrightarrow{d} & N_{n-1} & \xrightarrow{d} & \cdots \end{array}$$

Two morphisms $f, g\colon M_\bullet \to N_\bullet$ are said to be **chain homotopic** if $f - g$ is null homotopic. Two complexes $M_\bullet$ and $N_\bullet$ are said to be **homotopy equivalent** if there exist morphisms $f\colon M_\bullet \to N_\bullet$ and $g\colon N_\bullet \to M_\bullet$ such that fg and gf are each chain homotopic to the identity map.

The importance of this concept is due to the following observation:

Proposition 2.1. *Let $f, g\colon M_\bullet \to N_\bullet$ be chain homotopic maps of complexes. Then the maps $\mathrm{H}_n(M_\bullet) \to \mathrm{H}_n(N_\bullet)$ induced by f and g are equal. In particular, if f is null homotopic then it induces the zero map on homology.*

Proof. It suffices to treat the case where f is null homotopic. Let $y \in \mathrm{H}_n(M_\bullet)$. Let $x \in M_n$ be a lift of y with $dx = 0$. By definition, $f(y)$ is represented by $f(x) \in N_n$. Now, we have $f(x) = d_{n+1}(s_n(x)) + s_{n-1}(d_n(x)) = d_{n+1}(s_n(x))$ since $dx = 0$. But this shows that $f(x) \in \mathrm{im}(d_{n+1})$, and thus maps to 0 in $\mathrm{H}_n(N_\bullet)$. □

The **homotopy category** of $\mathbf{Ch}(\mathcal{A})$, denoted $\mathbf{K}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are where $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(M, N)$ is the set of equivalence classes of morphisms of complexes under chain homotopy. Thus two complexes are homotopy equivalent if and only if they are isomorphic in $\mathbf{K}(\mathcal{A})$. The above proposition shows that homology yields a well-defined functor $\mathrm{H}_n\colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}$.

3. Long exact sequences

Suppose that

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence in $\mathbf{Ch}(\mathcal{A})$. Let c be an element of C_n with $dc = 0$. Lift c arbitrarily to an element $b \in B_n$. Since db maps to $dc = 0$, it follows that $a = db \in A_{n-1}$. We have $da = d^2 b = 0$.

Proposition 3.1. *There is a well-defined morphism $\partial\colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_{n-1}(A_\bullet)$ given by $c \mapsto a$.*

Proof. Suppose that b' is a second lift of c. Then $b' = b + \epsilon$ for some $\epsilon \in A_n$. Thus $a' = db' = db + d\epsilon = a + d\epsilon$ and so a' and a differ by $d\epsilon$, and thus represent the same class in $\mathrm{H}_{n-1}(A_\bullet)$. Thus the construction is independent of the choice of lift b. We therefore have a well-defined map $\widetilde{\partial}\colon \ker(d_n\colon C_n \to C_{n-1}) \to \mathrm{H}_{n-1}(A_\bullet)$.

Now suppose that $c = d(c')$ for some $c' \in C_{n+1}$. Let $b' \in B_{n+1}$ be a lift of c', so that $b = db'$ is a lift of c. Then $a = db = d^2 b' = 0$. Thus $\widetilde{\partial}$ kills $\mathrm{im}(d_{n-1}\colon C_{n+1} \to C_n)$, and therefore induces a map ∂ as claimed. □

The morphism ∂ in the above lemma is called the **connecting homomorphism**. Its importance is due to the following result:

Proposition 3.2. *The sequence*

$$\cdots \to \mathrm{H}_n(A_\bullet) \to \mathrm{H}_n(B_\bullet) \to \mathrm{H}_n(A_\bullet) \xrightarrow{\partial} \mathrm{H}_{n-1}(A_\bullet) \to \mathrm{H}_{n-1}(B_\bullet) \to \mathrm{H}_{n-1}(A_\bullet) \to \cdots$$

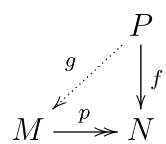
is everywhere exact.

Proof. Left as an exercise. □

The sequence in the above proposition is called the **long exact sequence** associated to the original short exact sequence of chain complexes. It is functorial in the short exact sequence, in the obvious sense.

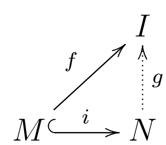
4. Projectives and injectives

An object P of $\mathcal{A}$ is **projective** if in any diagram



where p is a given surjection and f is a given morphism, one can find g making the diagram commute. Equivalently, the functor $\mathrm{Hom}(P, -)$ is exact. The category $\mathcal{A}$ is said to have **enough projectives** if every object is a quotient of a projective.

The dual notion to "projective" is "injective." Precisely, an object I is called **injective** if in any diagram



where i is a given injection and f is a given morphism, one can find g making the diagram commute. Equivalently, the functor $\mathrm{Hom}(-, I)$ is exact. The category $\mathcal{A}$ is said to have **enough injectives** if every object injects into an injective.

Example 4.1. Suppose $\mathcal{A}$ is the category of R-modules. Then any free R-module is projective. If R is a Dedekind domain, then any ideal of R is projective; this yields examples of projective modules that are not free. If $R = \mathbf{Z}$ then a module is injective if and only if it is divisible; thus $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are examples of injective $\mathbf{Z}$-modules. For any R, the category $\mathcal{A}$ has enough projectives and enough injectives. □

5. Projective resolutions

Let M be an object of $\mathcal{A}$. A **projective resolution** of M is an exact complex

$$\cdots \to P_2 \to P_1 \to P_0 \stackrel{\epsilon}{\to} M \to 0 \to 0 \to \cdots$$

where each P_i is projective. One typically regards $P_\bullet$ as a complex, which is 0 in negative degrees, and refers to the above complex with M tacked on as the augmented complex. One can also view a projective resolution as a quasi-isomorphism of complexes $\epsilon \colon P_\bullet \to M$, where M is regarded as a complex concentrated in degree 0:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Existence of projective resolutions is straightforward:

Proposition 5.1. *Suppose $\mathcal{A}$ has enough projectives. Then every object of $\mathcal{A}$ has a projective resolution.*

Proof. Since $\mathcal{A}$ has enough projectives, we can find a surjection $\epsilon \colon P_0 \to M$ with P_0 projective. Suppose now we have constructed a partial projective resolution

$$P_n \xrightarrow{d_n} \cdots \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0.$$

That is, each P_i is projective, and the sequence is exact away from P_n. We can then extend one more step by choosing a surjection $P_{n+1} \to \ker(d_n)$ with P_{n+1} projective. This is possible since there are enough projectives. □

Projective resolutions are obviously not unique in general. For example, if $M = 0$ and P is an projective then

$$\cdots \to 0 \to P \to P \to M \to 0$$

is a projective resolution of M. However, they are unique up to homotopy. We deduce this from the following more general result.

Proposition 5.2. *Let $\epsilon \colon P_\bullet \to M$ be a projective resolution, let $\delta \colon Q_\bullet \to N$ be any exact augmented complex, and let $f \colon M \to N$ be a morphism. Then there exists a morphism of complexes $g \colon P_\bullet \to Q_\bullet$ lifting f. Moreover, if g' is a second lift then g and g' are chain homotopic.*

Proof. We first construct g_0. Consider the diagram

$$\begin{array}{ccc} P_0 & \xrightarrow{\epsilon} & M \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle f} \\ Q_0 & \xrightarrow{\delta} & N \end{array}$$

Since δ is surjective and P_0 is projective, the lifting property of projectives allows us to find g_0. Suppose now we have constructed $g_0, \ldots, g_n$ and we want to construct g_{n+1}. Consider the diagram

$$\begin{array}{ccccc} P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} \\ \downarrow{\scriptstyle g_{n+1}} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} \\ Q_{n+1} & \longrightarrow & Q_n & \longrightarrow & Q_{n-1} \end{array}$$

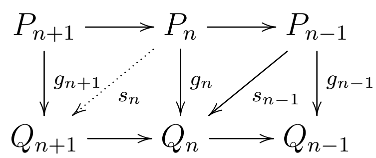
(When $n = 0$ the right column should consist of M and N.) Let $K = \ker(d \colon Q_n \to Q_{n-1})$. Since the bottom row is exact, the differential gives a surjection $Q_{n+1} \to K$. Of course, the composition $g_n d$ maps P_{n+1} into K. Thus, by the lifting property of projectives, we can find $g_{n+1} \colon P_{n+1} \to Q_{n+1}$.

We now prove the uniqueness claim. If g and g' are two lifts of f then $g - g'$ is a lift of 0. It thus suffices to show that if $f = 0$ then g is null-homotopic. We thus construct maps $s_n \colon P_n \to Q_{n+1}$ having the requisite properties. To construct s_0, consider the diagram

$$\begin{array}{ccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & M \\ \downarrow{\scriptstyle g_1} & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle 0} \\ Q_1 & \longrightarrow & Q_0 & \longrightarrow & N \end{array}$$

Since the right square commutes, g_0 maps P_0 into $\ker(Q_0 \to N)$. Since the bottom row is exact, Q_1 surjects onto this kernel. Thus, by the mapping property for projectives, we can find a map $s_0 \colon P_0 \to Q_1$ such that $g_0 = ds_0$. Note that $P_n = 0$ for $n < 0$, and so $s_n = 0$ for $n < 0$. We thus have $g_0 = ds_0 + s_{-1}d$, as requied.

Suppose now that we have constructed $s_0, \ldots, s_{n-1}$ satisfying the appropriate identities. Consider the diagram



Consider $h = g_n - s_{n-1}d$. We have

$$dh = dg_n - ds_{n-1}d = g_{n-1}d - ds_{n-1}d = (g_{n-1} - ds_{n-1})d = (s_{n-2}d)d = 0.$$

Thus h maps into $K = \ker(d \colon Q_n \to Q_{n-1})$. Since Q_{n+1} surjects onto K, the mapping property allows us to lift h to a map $s_n \colon P_n \to Q_{n+1}$. Since $h = ds_{n+1}$, we have $g_n = ds_n + s_{n-1}d$, as required. □

Corollary 5.3. *Let $\epsilon \colon P_\bullet \to M$ and $\delta \colon Q_\bullet \to M$ be two projective resolutions of M. Then $P_\bullet$ and $Q_\bullet$ are homotopy equivalent.*

Proof. The identity map $M \to M$ lifts to morphisms of complexes $f \colon P_\bullet \to Q_\bullet$ and $g \colon Q_\bullet \to P_\bullet$. Since fg and $\mathrm{id}_{P_\bullet}$ are both lifts of the identity on M, they are chain homotopic. Since gf and $\mathrm{id}_{Q_\bullet}$ are chain homotopic. □

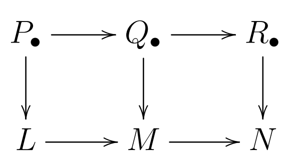
Corollary 5.4. *Assume $\mathcal{A}$ has enough projectives. There exists a well-defined functor $\mathcal{A} \to \mathbf{K}(\mathcal{A})$ sending an object of $\mathcal{A}$ to its projective resolution.*

We need one more result about projective resolutions:

Proposition 5.5 (Horseshoe lemma)**.** *Consider an exact sequence in $\mathcal{A}$:*

$$0 \to L \to M \to N \to 0.$$

Let $\epsilon \colon P_\bullet \to L$ and $\varphi \colon R_\bullet \to N$ be projective resolutions. Then there exists a projective resolution $\delta \colon Q_\bullet \to M$ such that $Q_n = P_n \oplus R_n$ and the differential $Q_n \to Q_{n-1}$ has the form $d_n(x, y) = (d_n(x) + g_n(y), d_n(y))$ for some $g_n \in R_n \to P_{n-1}$. In particular, we have a commutative diagram



where each row is an exact sequence.

Proof. We have already defined the groups Q_n, the only problem is to define the differentials and the augmentation. We begin with the latter. Since $M \to N$ is surjective, the augmentation $\varphi \colon R_0 \to N$ lifts through it; let $\delta' \colon R_0 \to M$ be a lift. Then we define $\delta \colon Q_0 \to M$ by $\delta(x, y) = \epsilon(x) + \delta'(y)$. One readily verifies that it is surjective.

We now construct the differential $d\colon Q_{n+1} \to Q_n$. Consider the diagram

$$\begin{array}{ccccc}
P_{n+1} & \longrightarrow & Q_{n+1} & \longrightarrow & R_{n+1} \\
\downarrow & & \vdots & & \downarrow \\
P_n & \longrightarrow & Q_n & \longrightarrow & R_n \\
\downarrow & & \downarrow & & \downarrow \\
P_{n-1} & \longrightarrow & Q_{n-1} & \longrightarrow & R_{n-1}
\end{array}$$

(When $n = 0$, the bottom row should be replaced with the given short exact sequence.) Let $y \in R_{n+1}$. Then

$$0 = d^2(0, dy) = d(g_n(dy), 0) = (dg_n(dy), 0).$$

Thus $g_n \circ d$ maps into $\ker(d\colon P_{n-1} \to P_n)$. Since this is surjected onto from P_n, the mapping property yields a lift $g_{n+1}\colon R_{n+1} \to P_n$; thus $dg_{n+1}(y) = g_n(dy)$. We use this g_{n+1} to define the differential $Q_{n+1} \to Q_n$ we leave the remainder of the proof as an exercise. $\square$

Remark 5.6. Everything in this section has an injective analog. Injective resolutions are usually written using cochain complexes. Thus an injective resolution of M is an exact complex

$$0 \to M \to I^0 \to I^1 \to \cdots$$

If $\mathcal{A}$ has enough injectives then every object has an injective resolution, and they are unique up to homotopy. $\square$

6. Derived functors

We now assume that $\mathcal{A}$ has enough projectives. Left $\mathcal{B}$ be a second abelian category and let $F\colon \mathcal{A} \to \mathcal{B}$ be a right-exact functor. Recall that this means that F is additive (i.e., commutes with direct sums) and that whenever

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$, the sequence

$$F(M_1) \to F(M_2) \to F(M_3) \to 0$$

is exact in $\mathcal{B}$. Note that this is not a short exact sequence: the first map is not required to be injective.

Example 6.1. Let R be a commutative ring and let $\mathcal{A} = \mathcal{B} = \mathrm{Mod}_R$. Let N be an R-module. Then the functor $F\colon \mathcal{A} \to \mathcal{B}$ given by $F(M) = M \otimes_R N$ is right-exact. $\square$

Definition 6.2. Let $i \geq 0$ be an integer. The ith **left derived functor** of F, denoted $\mathrm{L}_i F$, is the functor $\mathcal{A} \to \mathcal{B}$ defined by $(\mathrm{L}_i F)(M) = \mathrm{H}_i(F(P_\bullet))$, where $P_\bullet \to M$ is any projective resolution. We put $\mathrm{L}_i F = 0$ for $i < 0$. $\square$

The way the definition is formulated, it is perhaps not clear that $\mathrm{L}_i F$ is well-defined. To make this clear, we can rephrase as follows. Let $\Pi\colon \mathcal{A} \to \mathbf{K}(\mathcal{A})$ be the functor assigning to an object its projective resolution (Corollary 5.4). Then $\mathrm{L}_i F$ is the composition

$$\mathcal{A} \xrightarrow{\Pi} \mathbf{K}(\mathcal{A}) \xrightarrow{F} \mathbf{K}(\mathcal{B}) \xrightarrow{\mathrm{H}_i} \mathbf{B}$$

The only point on which we have not remarked yet is that F induces a well-defined functor $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$. But this is clear: the definition of homotopy simply passes through a functor.

Proposition 6.3. *We have* $\mathrm{L}_0 F = F$.

Proof. Let $P_\bullet \to M$ be a projective resolution of M. The sequence

$$P_1 \to P_0 \to M \to 0$$

is exact. Applying F, the sequence remains exact:

$$F(P_1) \to F(P_0) \to F(M) \to 0.$$

By definition, $(\mathrm{L}_0 F)(M)$ is the cokernel of $F(P_1) \to F(P_0)$. The above shows that this is canonically identified with $F(M)$. □

The most important property of the left derived functor is the following:

Proposition 6.4. *Consider a short exact sequence in* $\mathcal{A}$*:*

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then there is an associated long exact sequence in $\mathcal{B}$*:*

$$\cdots \to (\mathrm{L}_i F)(M_1) \to (\mathrm{L}_i F)(M_2) \to (\mathrm{L}_i F)(M_3) \to (\mathrm{L}_{i-1} F)(M_1) \to \cdots$$

Moreover, this long exact sequence is functorial in the original short exact sequence.

Proof. Let $P_\bullet \to M_1$ and $P''_\bullet \to M_3$ be projective resolutions. Let $P'_\bullet \to M_2$ be the projective resolution produced by the horseshoe lemma. Recall that

$$0 \to P_\bullet \to P'_\bullet \to P''_\bullet \to 0$$

is a short exact sequence of complexes, and at each index is split. Since F is additive, the sequence

$$0 \to F(P_\bullet) \to F(P'_\bullet) \to F(P''_\bullet) \to 0$$

remains exact. The result now follows from Proposition 3.2. □

Remark 6.5. Suppose $T_i \colon \mathcal{A} \to \mathcal{B}$ are functors satisfying the following conditions:

(a) $T_i = 0$ for $i < 0$.
(b) $T_0 = F$.
(c) $T_i(P) = 0$ for $i > 0$ and P projective.
(d) To every short exact sequence in $\mathcal{A}$ there is functorially associated a long exact sequence in the T's.

Then $T_i \cong \mathrm{L}_i F$. The proof of this is left as an exercise. □

Remark 6.6. There is a dual version of everything here. Suppose $G \colon \mathcal{A} \to \mathcal{B}$ is a left-exact functor and $\mathcal{A}$ has enough injectives. Then one has right derived functors $\mathrm{R}^i G \colon \mathcal{A} \to \mathcal{B}$. The definition is as follows: $(\mathrm{R}^i G)(M) = \mathrm{H}^i(G(I^\bullet))$, where $M \to I^\bullet$ is an injective resolution of M. □

7. Morphisms of derived functors

Let $F, G\colon \mathcal{A} \to \mathcal{B}$ be right-exact functors of abelian categories where $\mathcal{A}$ has enough injectives. Thus we have derived functors $\mathrm{R}^\bullet F$ and $\mathrm{R}^\bullet G$. A **morphism** of derived functors $\varphi^\bullet\colon \mathrm{R}^\bullet F \to \mathrm{R}^\bullet G$ consists of a natural transformation $\varphi^i\colon \mathrm{R}^i F \to \mathrm{R}^i G$ for each i such that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence in $\mathcal{A}$ then we obtain a morphism of long exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & (\mathrm{R}^i F)(M_1) & \longrightarrow & (\mathrm{R}^i F)(M_2) & \longrightarrow & (\mathrm{R}^i F)(M_3) & \longrightarrow & (\mathrm{R}^{i+1} F)(M_1) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \varphi^i} & & \downarrow{\scriptstyle \varphi^i} & & \downarrow{\scriptstyle \varphi^i} & & \downarrow{\scriptstyle \varphi^{i+1}} & \\ \cdots \longrightarrow & (\mathrm{R}^i G)(M_1) & \longrightarrow & (\mathrm{R}^i G)(M_2) & \longrightarrow & (\mathrm{R}^i G)(M_3) & \longrightarrow & (\mathrm{R}^{i+1} G)(M_1) & \longrightarrow \cdots \end{array}$$

The cheif fact we need is:

Proposition 7.1. *A morphism of derived functors is determined by its 0th member. That is, if $\varphi^\bullet$ and $\psi^\bullet$ are morphisms of derived functors $\mathrm{R}^\bullet F \to \mathrm{R}^\bullet G$ such that $\varphi^0 = \psi^0$ then $\varphi^i = \psi^i$ for all i.*

Proof. It suffices to assume $\varphi^0 = 0$ and show $\varphi^i = 0$ for $i > 0$. We proceed inductively, so suppose that we have shown $\varphi^i = 0$. Let M be a given object of $\mathcal{A}$, and choose a short exact sequence

$$0 \to M \to I \to N \to 0$$

with I injective. Since $(\mathrm{R}^{i+1} F)(I) = 0$, and similarly for G, we obtain a diagram

$$\begin{array}{ccccc} (\mathrm{R}^i F)(N) & \longrightarrow & (\mathrm{R}^{i+1} F)(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi^{i+1}} & & \\ (\mathrm{R}^i G)(N) & \longrightarrow & (\mathrm{R}^{i+1} G)(M) & \longrightarrow & 0 \end{array}$$

Thus $\varphi^{i+1} = 0$. □

8. Ext

The most important example of a derived functor is Ext, which is the derived functor of Hom. To be precise, for objects M and N of $\mathcal{A}$, we have left-exact functors

$$\Phi_M\colon \mathcal{A} \to \mathbf{Ab}, \quad X \mapsto \mathrm{Hom}(M, X) \qquad\qquad \Psi_N\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}, \quad Y \mapsto \mathrm{Hom}(Y, N)$$

If $\mathcal{A}$ has enough injectives, we can form the derived functor $\mathrm{R}^\bullet \Phi_M$. If $\mathcal{A}$ has enough projectives, then $\mathcal{A}^{\mathrm{op}}$ has enough injectives, and we can form the derived functor $\mathrm{R}^\bullet \Psi_N$. The important fact is that when both are defined they agree, in the following sense:

Proposition 8.1. *Suppose $\mathcal{A}$ has enough projectives and enough injectives. Then $(\mathrm{R}^i \Phi_M)(N) = (\mathrm{R}^i \Psi_N)(M)$ for all M and N.*

Proof. Fix M. We will show that $N \mapsto (\mathrm{R}^i \Psi_N)(M)$ is the ith derived functor of Φ_M. Let $P_\bullet \to M$ be a projective resolution. We note that $(\mathrm{R}^i \Psi_N)(M) = \mathrm{H}_i(\mathrm{Hom}(P_\bullet, N))$, by definition. We check the conditions of Remark 6.5:

- We have $(\mathrm{R}^i \Psi_N)(M) = 0$ for $i < 0$ by definition.

- We have $(\mathrm{R}^0\Psi_N)(M) = \Psi_N(M) = \Phi_M(N)$.
- Let I be an injective. Since I is injective, the functor $\mathrm{Hom}(-, I)$ is exact, and so $\mathrm{Hom}(P_\bullet, I)$ is exact away from degree 0. Hence $(\mathrm{R}^i\Psi_I)(M) = 0$ for $i > 0$.
- Consider a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

Applying $\mathrm{Hom}(P_i, -)$ yields an exact sequence, since P_i is projective. Thus, applying $\mathrm{Hom}(P_\bullet, -)$, we obtain an exact sequence of complexes

$$0 \to \mathrm{Hom}(P_\bullet, N_1) \to \mathrm{Hom}(P_\bullet, N_2) \to \mathrm{Hom}(P_\bullet, N_3) \to 0.$$

Taking homology, we thus get a long exact sequence in the $(\mathrm{R}^\bullet\Psi_{N_i}(M)$, as required.

The proposition now follows from Remark 6.5. □

Definition 8.2. Assume $\mathcal{A}$ has enough projective or enough injectives. We then define Ext^i to be the ith right derived functor of Hom. □

To be completely clear, we spell out exactly how to compute Ext. Let M and N be given. Suppose that $P_\bullet \to M$ is a projective resolution of M. Then $\mathrm{Ext}^i(M, N)$ is the homology of the sequence

$$\mathrm{Hom}(P_{i-1}, N) \to \mathrm{Hom}(P_i, N) \to \mathrm{Hom}(P_{i+1}, N).$$

Similarly, suppose that $N \to I^\bullet$ is an injective resolution of N. Then $\mathrm{Ext}^i(M, N)$ is the homology of the sequence

$$\mathrm{Hom}(M, I^{i-1}) \to \mathrm{Hom}(M, I^i) \to \mathrm{Hom}(M, I^{i+1}).$$

The proposition ensures that the two computations give the same answer, when they are both defined.

We now compute a few simple examples.

Proposition 8.3. *Let $\mathcal{A} = \mathbf{Ab}$. Then*

$$\mathrm{Ext}^i(\mathbf{Z}/n\mathbf{Z}, M) = \begin{cases} M[n] & i = 0 \\ M/nM & i = 1 \\ 0 & i > 1 \end{cases}$$

Proof. We have the following projective resolution of $\mathbf{Z}/n\mathbf{Z}$:

$$\cdots \to 0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0.$$

Applying $\mathrm{Hom}(-, M)$ to $P_\bullet$, we obtain the complex

$$\mathrm{Hom}(\mathbf{Z}, M) \xrightarrow{n} \mathrm{Hom}(\mathbf{Z}, M) \to 0 \to \cdots.$$

Of course, $\mathrm{Hom}(\mathbf{Z}, M) = M$. The result thus follows. □

9. Tor

Let R be a ring (not necessarily commutative). Let ${}_R\mathrm{Mod}$ and Mod_R denote the category of left and right R-modules. Given a right R-module M and a left R-module N, we have right-exact functors

$$\Phi_M \colon {}_R\mathrm{Mod} \to \mathbf{Ab}, \quad X \mapsto M \otimes_R X \qquad\qquad \Psi_N \colon \mathrm{Mod}_R \to \mathbf{Ab}, \quad Y \mapsto Y \otimes_R N$$

Since module categories always have enough projectives (simply use free modules), we can form the left-derived functors of Φ_M and Ψ_N. As with Ext, the two derived functors agree:

Proposition 9.1. *We have* $(\mathrm{L}_i\Phi_M)(N) = (\mathrm{L}_i\Psi_N)(M)$ *for all* M *and* N.

Definition 9.2. We define Tor_i to be the ith left-derived functor of either Φ_M or Ψ_N. □

Thus to compute $\mathrm{Tor}_i(M, N)$, one picks a projective resolution of M, applies $- \otimes_R N$, and computes homology; or one picks a projective resolution of N, applies $M \otimes_R -$, and computed homology.

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