An Introduction to Categories.

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Today we're going to talk about categories. We're not going to do anything deep. Rather, this is meant to introduce category theoretic words. Our topics are the following.

- What is a category? (and examples).
- What is a functor? (and examples).
- What are equivalent categories?

1 What is a category?

A **category** $\mathcal{C}$ is:

- a collection $\text{Ob}(\mathcal{C})$ of *objects*,
- a collection $\text{Mor}(X, Y)$ of *morphisms* for each pair $X, Y \in \text{Ob}(\mathcal{C})$, including an identity morphism $1 = 1_X \in \text{Mor}(X, X)$ for each $X \in \text{Ob}(\mathcal{C})$,
- A composition of morphisms function $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$ for each triple $X, Y, Z \in \text{Ob}(\mathcal{C})$ satisfying $f \circ 1 = f$, $1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.

The identity morphism is unique, for if $1, 1' \in \text{Mor}(X, X)$, then $1 = 1 \circ 1' = 1'$.

1.1 Examples

By "small set," we mean all subsets of some universe (otherwise we might run into Russel's paradox).

The category **Set**, where objects are small sets, and morphisms are maps between them. Axioms follow from properties of functions.

More relevant is the category **Top**, where objects are all topological spaces and where morphisms are continuous maps between them. Axioms follow since continuous maps compose into a continuous maps. Note that restricting the collection of morphisms to homeomorphisms and the result is still a category. The same goes for restricting the objects to, say, manifolds.

The category **Top**. is of pointed topological spaces, and morphisms are base-point-preserving continuous maps.

Individual groups are categories in the following way: if G is a group, then a category is one object X, and $\mathrm{Mor}(X, X) = G$.

But there is also **Grp**, whose objects are small groups, and whose morphisms are all group homomorphisms between them.

Less relevant is the category **BanAnaMan** of Banach analytic manifolds, which I put here only because of its name.

Recall that a morphism $\varphi : X \to Y$ of coverings $p_X : X \to B$ and $p_Y : Y \to B$ is a continuous map such that $p_X = p_Y \varphi$. These are the morphisms for a category whose objects are covering spaces of B.

2 What is a functor?

A **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$ and to each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ a morphism $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$, such that $F(1) = 1$ and $F(f \circ g) = F(f) \circ F(g)$.

(Note: this is like a directed graph homomorphism).

2.1 Examples

If Y is an object in $\mathcal{D}$, then mapping all objects in $\mathcal{C}$ to Y and all morphisms of $\mathcal{C}$ to 1_Y is a functor. Or, there is the identity functor which does what it sounds like.

Let $\mathscr{P}X$ denote the power set of a set X, and $\mathscr{P}f : \mathscr{P}X \to \mathscr{P}Y$ be the map which sends each $S \subset X$ to its image $f(S)$. Then $\mathscr{P} : \mathbf{Set} \to \mathbf{Set}$ is a functor.

We saw that $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ maps a topological space to its set of path components. And, continuous maps $f : X \to Y$ induce a map between path components $\pi_0(f)$ such that $\pi_0(fg) = \pi_0(f)\pi_0(g)$.

Similarly, $\pi_1 : \mathbf{Top}. \to \mathbf{Grp}$ maps a pointed topological space to its fundamental group. We've seen that continuous maps f have an induced group homomomorphism f_* such that $(fg)_* = f_* g_*$, and the identity map induces the identity homomorphism.

3 Category equivalence

Let $\mathcal{C}, \mathcal{D}$ be two categories. An **equivalence of categories** consists of of functor $F : \mathcal{C} \to \mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are the identity functors on $\mathcal{D}$ and $\mathcal{C}$, respectively.

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- **full** if for any two objects $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Mor}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Mor}_{\mathcal{D}}(F(X_1), F(X_2))$ is surjective;

- **faithful** if for any two objects $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Mor}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Mor}_{\mathcal{D}}(F(X_1), F(X_2))$ is injective;

- **essentially surjective** if each object $Y \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to an object in the image of F (that is, there is $X \in \mathrm{Ob}(\mathcal{D})$, and morphisms $f : Y \to F(X)$ and $g : F(X) \to Y)$.

One can show a functor yields an equivalence of categories iff it is full, faithful, and essentially surjective. (That is, if F satisfies these, then there exists a G so that FG and GF are identity functors).