GALOIS CORRESPONDANCE FOR BASEPOINTED COVERING SPACES

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1. QUICK SUMMARY

(1) Homotopy lifting property
(2) Lifting criterion
(3) Unique lifting property
(4) Existence of covering spaces

2. STATEMENT AND PROOF

First, we have to define a notion of equivalence between the covering spaces. We do it in the most obvious way. Let Y and Z be two covering spaces and $f : Y \to Z$ be a map between the covering spaces, then if f is a homeomorphism we say that f is an isomorphism, and Y and Z are called isomorphic when there exists an isomorphism between them. It is really easy to see that this defines an equivalence relation.

Also we note that the local conditions below mean that each point has arbitrarily small neighborhoods satisfying that property, as opposed to having one single neighborhood. The condition locally simply-connected can be replaced with a weaker condition called semilocally simply-connected, but this practically changes nothing, so we go with the more intuitive condition.

Theorem 1. *Let X be path-connected, locally path-connected, and locally simply-connected. Then there is a bijection between the set of base point preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, and the map is obtained by sending the covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ to the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.*

Remark 1. In this bijection the normal subgroups correspond to the Galois covers by what's shown in the previous talk.

Remark 2. Note that we are dealing with base-pointed maps here, there exists an alternative version which avoids fixing a basepoint. This will be done in the next talk.

Proof. The proof is actually almost done. We only need to show that two covering spaces induce the same image in $\pi_1(X, x_0)$ if and only if they are isomorphic.

The if part follows trivially by the functorial properties of the induced homomorphisms.

For the only if part, we will make use of the lifting properties. Let the covering spaces be (Y, y_0) and (Z, z_0), with the maps p and q into (X, x_0). By the lifting criterion, there exists a lift $\tilde{p}$ of the map p to the covering space (Z, z_0), and a $\tilde{q}$ which is defined in the same way. By definition, these maps satisfy $q\tilde{p} = p$ and

$p\tilde{q} = q$, and therefore $q\tilde{p}\tilde{q} = p\tilde{q} = q$. Now by the unique extension property, the only basepoint preserving lift of the map q to the covering space (Z, z_0) is the identity map, which implies that $\tilde{p}\tilde{q}$ is the identity mapping. Similarly, $\tilde{q}\tilde{p}$ is identity as well. This finishes the proof, since $\tilde{p}$ is then an isomorphism between the covering spaces (Y, y_0) and (Z, z_0). □

3. Exercises

(1) Show that there is no connected covering space of S^1 with a non-trivial finite fundamental group.

Proof. Choose an arbitrary basepoint x_0 in S^1. Assume that the connected covering space $p : (\tilde{X}, \tilde{x_0}) \to (S^1, x_0)$ has a non-trivial finite fundamental group. Then, the subgroup $p_*(\pi_1(S^1, x_0))$ is finite. Yet the only finite subgroup of the group $\mathbb{Z}$ is the trivial group. (1)

Then by the Galois correspondance $(\tilde{X}, \tilde{x_0})$ must be isomorphic to the universal cover, in the sense of covering space maps, in particular, they have to homeomorphic, and trivially homotopy equivalent, which means that their fundamental groups are isomorphic. This gives a contradiction, since $\pi_1(\tilde{X}, \tilde{x_0})$ was assumed to be non-trivial.

As an alternative, shorter proof we can use that p_* is injective, and everything follows immediately from (1).

□

(2) The space X satisfies the conditions of the theorem. Prove that if $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is the universal cover, and $q : (Y, y_0) \to (X, x_0)$ is a connected covering space, then there exists a map r such that $r : (\tilde{X}, \tilde{x_0}) \to (Y, y_0)$ is the universal cover of (Y, y_0).

Proof. By the lifting criterion there exists a lift $\tilde{p}$ of p to (Y, y_0). We can use this $\tilde{p}$ as the map r in the statement.

Take a point y in Y. There exists a neighborhood U of $q(y)$ such that the inverse image of U under both p and q are disjoint unions of open sets that are homeomorphically mapped to U. Take the one that contains y, and look at its inverse image under $\tilde{p}$, which is a disjoint union of open sets by using the fact that $q\tilde{p} = p$. This finishes the proof. □