The fundamental group of S^1.

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We will show that $\pi_1(S^1) \approx \mathbb{Z}$. We will do this by constructing a homotopy-preserving isomorphism from paths in S^1 to paths of $\mathbb{R}$ which start at the origin and end at an integer, and then construct an isomorphism from the path classes of these paths of $\mathbb{R}$ into $\mathbb{Z}$.

Our representation of S^1 is the unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. From now on, the base point for the fundamental group is $x_0 = (1, 0)$ in S^1. Let $p : \mathbb{R} \to S^1$ be the quotient map defined by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. This can be imagined as a spiral in $\mathbb{R}^3$ with an axis perpendicular to the XY plane being flattened onto the unit circle in this plane. We note that $p(0) = x_0$.

Lemma 1. *Let $f : I \to S^1$ be a loop with base point x_0. There is a unique path $\tilde{f} : I \to \mathbb{R}$ such that $f = p\tilde{f}$ and $\tilde{f}(0) = 0$. We will call $\tilde{f}$ the lift of f.*

Proof. Let $\{U_1, U_2\}$ be an open cover of S^1 such that both U_1 and U_2 are homeomorphic to an open interval in $\mathbb{R}$. For concreteness, let $U_1 = \{(x, y) \in S^1 \mid y > -\frac{1}{2}\}$ and $U_2 = \{(x, y) \in S^1 \mid y < \frac{1}{2}$.

We will show that we can break I into $0 = t_0 < t_1 < \ldots < t_n = 1$ so that $f([t_{i-1}, t_i])$ is entirely in U_1 or entirely in U_2, for $1 \le i \le n$. Since f is compact, we see that $\{f^{-1}(U_1), f^{-1}(U_2)\}$ is an open cover of I, and so we can let ε be the Lebesgue number of this open cover. Choose $t_0, \ldots, t_n$ so that $t_{i-1} < t_i$ and $t_i - t_{i-1} < \varepsilon$. Thus, $[t_{i-1}, t_i]$ is in either $f^{-1}(U_1)$ or $f^{-1}(U_2)$, and therefore $f([t_{i-1}, t_i])$ is in either U_1 or U_2.

Each of the path components of $p^{-1}(U_j)$ is homeomorphic to U_j, for $j = 1, 2$. For, letting W be a path component of $p^{-1}(U_j)$, $p|_W$ onto U_j is a homeomorphism.

Here is a sketch of induction on i. What we do is construct $\tilde{f}$ by starting with $\tilde{f}(0) = 0$ and extending the domain of the function with each $[t_{i-1}, t_i]$. Say $B_i = f([t_{i-1}, t_i])$ is in U_j. Then $\tilde{f}(t_{i-1})$ is in some path component W_j of $p^{-1}(U_j)$. If $\varphi : W_j \to U_j$ is the homeomorphism $p|_{W_j}$ onto U_j, then φ gives a homeomorphism between B_i and a closed interval in W_j. This extends $\tilde{f}$ by $\tilde{f}|_{[t_{i-1}, t_i]}(s) = \varphi^{-1}(f(s))$, and by the pasting lemma this is continuous.

The uniqueness follows. For, if $\tilde{f}_1$ and $\tilde{f}_2$ are paths such that $p\tilde{f}_1 = p\tilde{f}_2 = f$, then, we can use the above induction on i to and the homeomomorphims to show $\tilde{f}_1(t_i) = \tilde{f}_2(t_i)$. □

The following lemma follows from the fact that $f(1) = p\tilde{f}(1)$, and $f(1) = x_0$.

Lemma 2. *If $\tilde{f}$ is the lift of a loop f with base point x_0, then $\tilde{f}(1) \in \mathbb{Z}$.*

The following lemma is left as an exercise. (The proof is given as the answer to the exercise).

Lemma 3. *Let f, g be homotopic loops in S^1. Then $\tilde{f}(1) = \tilde{g}(1)$.*

Proof. Let $F : I \times I \to S^1$ be a homotopy from f to g (that is, $F(t,0) = f(t)$, $F(t,1) = g$, and $F(0,s) = F(1,s) = x_0$). For $s \in I$, let $h(s)$ be a path of the homotopy (where $h(s)(t) = F(t,s)$). Since F is continuous and $I \times I$ is compact, let δ be the Lebesgue number of the open cover $\{F^{-1}(U_1), F^{-1}(U_2)\}$, where U_1 and U_2 are as before. Let ℓ denote the operation which lifts a loop to a path in $\mathbb{R}$ (so $\ell f = \tilde{f}$).

Claim: if $s_1, s_2 \in I$ are such that $|s_1 - s_2| < \frac{\delta}{2}$, then $\ell(h(s_1))(1)$ and $\ell(h(s_2))(1)$ are in the same path component of $p^{-1}(U_1)$ or $p^{-1}(U_2)$. We see that $h(s_1)(t)$ and $h(s_2)(t)$ are always both in U_1 or in U_2 for all $t \in I$ since $|s_1 - s_2| < \delta$. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be so $t_i - t_{i-1} < \frac{\delta}{2}$. By definition, we see that $\ell(h(s_1))$ and $\ell(h(s_2))$ are in the same path component on $\{0\}$. We will proceed by induction on $0 < i \leq n$, assuming $\ell(h(s_1))(t)$ and $\ell(h(s_2))(t)$ are in the same path component for all $t \in [0, t_{i-1}]$. Since the diameter of $[t_{i-1}, t_i] \times [s_1, s_2]$ is less than δ, $F([t_{i-1}, t_i] \times [s_1, s_2])$ is contained entirely in U_j for some $j = 1, 2$. Thus, $\ell(h(s_1))([t_{i-1}, t_i])$ and $\ell(h(s_2))([t_{i-1}, t_i])$ are in the same path component of $p^{-1}(U_j)$. This completes the induction.

Using the same t_i as defined (for convenience), it follows that all $\ell(h(t_i))(1)$ are in the same path component of $p^{-1}(U_j)$, for some j. Since $\ell(h(t_0)) = \tilde{f}$ and $\ell(h(t_n)) = \tilde{g}$, it follows that $\tilde{f}(1)$ and $\tilde{g}(1)$ are in the same path component W of $p^{-1}(U_j)$, for some j. And, since W has a diameter less than one, we conclude $\tilde{f}(1)$ and $\tilde{g}(1)$ must be equal to the same integer. □

By this lemma, we see that f and g being homotopic implies $\tilde{f}$ and $\tilde{g}$ are homotopic by the homotopy $F(t,s) = (1-s)\tilde{f}(t) + s\tilde{g}(t)$.

Let G be the set of all path classes in $\mathbb{R}$ for paths which start at the origin and end at an integer. Because $\mathbb{R}$ is convex, each end point has exactly one path class. Since homotopic loops in S^1 have homotopic lifts, lifting induces an injection from $\pi_1(S^1)$ into G. The path $f(t) = (\cos 2\pi nt, \sin 2\pi nt)$ for $n \in \mathbb{Z}$ lifts to $\tilde{f}(t) = nt$, and $\tilde{f}(1) = n$, thus lifting is also a surjection onto G.

For α, β path classes in G, define path addition $\alpha + \beta$ to be $\alpha\tau_\alpha(\beta)$ where $\tau_\alpha(\beta)$ means translate β so it starts at the end point of class α, making the concatenation work. It is clear that this operation is well defined. We see that $fg = p(\tilde{f} + \tilde{g})$, where $\tilde{f}$ and $\tilde{g}$ are the lifted f and g, since $p(\tilde{f} + \tilde{g}) = p(\tilde{f}\tau_{\tilde{f}}(\tilde{g})) = p(\tilde{f})p(\tau_{\tilde{f}}(\tilde{g})) = fp(\tilde{g}) = fg$. Thus, this operation on G defines an isomorphism between G and $\pi_1(S^1)$ via lifting.

We will prove that G is isomorphic to $\mathbb{Z}$. We have already seen elements of G and $\mathbb{Z}$ are in one-to-one correspondence. If $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$, then $(\tilde{f} + \tilde{g})(1)$ is $n + m$. The operation of taking the endpoint is thus an isomorphism.

This proves our theorem that $\pi_1(S^1) \approx \mathbb{Z}$.