Exercises from February 28th

18.904 Spring 2011

These are exercises from Dannys's and Umuts's talks. Solutions are on the following page.

Exercise 1 (Danny)**.** Prove that the map $\phi : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_0)$ (with $\tilde{X}$ the universal cover) defined by $[\gamma] \mapsto g[\gamma]$, where g is in $\pi_1(X, x_0)$ and acts on the universal cover, is continuous.

Exercise 2 (Danny)**.** Find all connected pointed coverings (up to isomorphism) of the circle.

Exercise 3 (Umut)**.** Show that there is no connected covering space of S^1 with a non-trivial finite fundamental group.

Exercise 4 (Umut)**.** The space X satisfies the conditions of the theorem. Prove that if $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is the universal cover, and $q : (Y, y_0) \to (X, x_0)$ is a connected covering space, then there exists a map r such that $r : (\tilde{X}, \tilde{x_0}) \to (Y, y_0)$ is the universal cover of (Y, y_0).

Solution to Exercise 1: We claim that $\phi^{-1}(U_{[\gamma]}) = U_{g^{-1}[\gamma]}$, where $U_{[\gamma]}$ is a basis element of $\tilde{X}$. This is enough to show continuity. We first note that any element of π_1 acting on $\tilde{X}$ preserves endpoints.

Let $(g^{-1}[\gamma])[\eta]$ be any element of $U_{g^{-1}[\gamma]}$ (so η is a path that stays inside U and has $\eta(0) = g^{-1}\gamma(1) = \gamma(1)$). Clearly $(g^{-1}[\gamma])[\eta] = g^{-1}[\gamma\eta]$. Therefore $g((g^{-1}[\gamma])[\eta]) = g(g^{-1}[\gamma\eta]) = (gg^{-1})[\gamma\eta] = [\gamma][\eta]$, which is in $U_{[\gamma]}$. This proves one inclusion.

Now suppose that $[\gamma'] \in \phi^{-1}(U_{[\gamma]})$. Then $g[\gamma'] \in U_{[\gamma]}$, so $g[\gamma'] = [\gamma\eta]$ for some η defined as before. Therefore $[\gamma'] = (g^{-1}g)[\gamma'] = g^{-1}(g[\gamma']) = g^{-1}[\gamma\eta]$, so $[\gamma'] \in U_{g^{-1}[\gamma]}$, which proves the other inclusion.

Solution to Exercise 2: It suffices to exhibit one such covering for each subgroup of $\mathbb{Z}$.

The trivial subgroup corresponds to the universal cover, as the induced map from the trivial group (the fundamental group of the universal cover) must have trivial image.

The only other subgroups are of the form $n\mathbb{Z}$ for some positive integer$n$. Cover S^1 by itself with the map p_n for which $z \mapsto z^n$. This is obviously a covering. Consider the induced map on the fundamental group. Then a loop which makes m windings of around S^1 (i.e. corresponds to a^n where a is the generator of $\mathbb{Z}$) will be mapped to one which makes nm windings around S^1, so the image of the induced map is $n\mathbb{Z}$. Hence we have the cover we need.

Solution to Exercise 3: Choose an arbitrary basepoint x_0 in S^1. Assume that the connected covering space $p : (\tilde{X}, \tilde{x_0}) \to (S^1, x_0)$ has a non-trivial finite fundamental group. Then, the subgroup $p_*(\pi_1(S^1, x_0))$ is finite. Yet the only finite subgroup of the group $\mathbb{Z}$ is the trivial group. (1)

Then by the Galois correspondance $(\tilde{X}, \tilde{x_0})$ must be isomorphic to the universal cover, in the sense of covering space maps, in particular, they have to homeomorphic, and trivially homotopy equivalent, which means that their fundamental groups are isomorphic. This gives a contradiction, since $\pi_1(\tilde{X}, \tilde{x_0})$ was assumed to be non-trivial.

As an alternative, shorter proof we can use that p_* is injective, and everything follows immediately from (1).

Solution to Exercise 4: By the lifting criterion there exists a lift $\tilde{p}$ of p to (Y, y_0). We can use this $\tilde{p}$ as the map r in the statement.

Take a point y in Y. There exists a neighborhood U of $q(y)$ such that the inverse image of U under both p and q are disjoint unions of open sets that are homeomorphically mapped to U. Take the one that contains y, and look at its inverse image under $\tilde{p}$, which is a disjoint union of open sets by using the fact that $q\tilde{p} = p$. This finishes the proof.