Exercises from February 9th

18.904 Spring 2011

These are exercises from JJ's and Kyle's talks. Solutions are on the following page.

Exercise 1 (JJ)**.** Let D be the open unit disc and let $f : D \to D$ be a continuous map. Does the Brouwer Fixed Point Theorem hold in this context? That is, must $f(D)$ have a fixed point?

Exercise 2 (Kyle)**.** Let f, g be homotopic loops in S^1. Show that $\tilde{f}(1) = \tilde{g}(1)$, where $\tilde{f}$ and $\tilde{g}$ denote the lifts to maps $\mathbf{R} \to S^1$ taking 0 to the basepoint of S^1.

Solution to Exercise 1: The theorem does not hold in this case, because we cannot find a retract from the open disc to the 1-sphere, a step that is key to our proof of the theorem.

More specifically, consider the continuous map that shrinks the distance between a point in D and the border of D by half:

$$f(x,y) = \left(\frac{1}{2}(\sqrt{1-y^2}+x), y\right).$$

The map f does not fix any points in the open disc.

Solution to Exercise 2: Let $F : I \times I \to S^1$ be a homotopy from f to g (that is, $F(t,0) = f(t)$, $F(t,1) = g$, and $F(0,s) = F(1,s) = x_0$). For $s \in I$, let $h(s)$ be a path of the homotopy (where $h(s)(t) = F(t,s)$). Since F is continuous and $I \times I$ is compact, let δ be the Lebesgue number of the open cover $\{F^{-1}(U_1), F^{-1}(U_2)\}$, where U_1 and U_2 are as before. Let ℓ denote the operation which lifts a loop to a path in $\mathbf{R}$ (so $\ell f = \tilde{f}$).

Claim: if $s_1, s_2 \in I$ are such that $|s_1 - s_2| < \frac{\delta}{2}$, then $\ell(h(s_1))(1)$ and $\ell(h(s_2))(1)$ are in the same path component of $p^{-1}(U_1)$ or $p^{-1}(U_2)$. We see that $h(s_1)(t)$ and $h(s_2)(t)$ are always both in U_1 or in U_2 for all $t \in I$ since $|s_1 - s_2| < \delta$. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be so $t_i - t_{i-1} < \frac{\delta}{2}$. By definition, we see that $\ell(h(s_1))$ and $\ell(h(s_2))$ are in the same path component on $\{0\}$. We will proceed by induction on $0 < i \leq n$, assuming $\ell(h(s_1))(t)$ and $\ell(h(s_2))(t)$ are in the same path component for all $t \in [0, t_{i-1}]$. Since the diameter of $[t_{i-1}, t_i] \times [s_1, s_2]$ is less than δ, $F([t_{i-1}, t_i] \times [s_1, s_2])$ is contained entirely in U_j for some $j = 1, 2$. Thus, $\ell(h(s_1))([t_{i-1}, t_i])$ and $\ell(h(s_2))([t_{i-1}, t_i])$ are in the same path component of $p^{-1}(U_j)$. This completes the induction.

Using the same t_i as defined (for convenience), it follows that all $\ell(h(t_i))(1)$ are in the same path component of $p^{-1}(U_j)$, for some j. Since $\ell(h(t_0)) = \tilde{f}$ and $\ell(h(t_n)) = \tilde{g}$, it follows that $\tilde{f}(1)$ and $\tilde{g}(1)$ are in the same path component W of $p^{-1}(U_j)$, for some j. And, since W has a diameter less than one, we conclude $\tilde{f}(1)$ and $\tilde{g}(1)$ must be equal to the same integer.