Note on Modular Manifolds for Elliptic Modules

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Abstract

This short note was intended to cover the Section 5 in Drinfeld's paper in the talk at October 20, 2017, which is still incomplete about details though. And the only reference is Drinfeld's paper [Dr]

Construction of modular schemes

We fix A be the ring of integers of a local field k of characteristic p. Let S be a scheme over S.

Definition 0.1. An **elliptic module** X of rank d over S is a pair (L, ϕ), where L is a line bundle with a commutative group scheme structure over S that is locally additive, and $\phi : A \to \mathrm{End}(L)$ is a ring homomorphism to the ring of endomorphism of the group scheme L over S, satisfying

- For any $a \in A$, the differential of $\phi(a)$ is multiplication by a.
- For any given $\mathrm{Spec}(K) \to S$, where S is a field, the induced homomorphism from $A \to K\{\tau\}$ is an elliptic module of rank d in the previous sense.

Note that this makes $X(S) = L(S)$ become an A-module.

Example 0.2. When $S = \mathrm{Spec}(B)$ and $L = \mathrm{Spec}(B[T])$ is the trivial bundle over S, the ring of endomorphism $\mathrm{End}(L)$ is canonically isomorphic to $B\{\tau\}$, so ϕ becomes a ring homomorphism from A to $B\{\tau\}$. And when $X = (L, \phi)$ is an elliptic module of rank d, the homomorphism is of the form

$$\phi(a) = \sum_{i=0}^{n} b_i \tau^i.$$

We note that the definition above will make b_i nilpotent for $i > d \log_p |a|$, and b_i invertible for $i = d \log_p |a|$.

Definition 0.3. We say an elliptic A-module of rank d is **standard** if for every $a \in A$, the endomorphism $\phi(a)$ is of the form

$$\sum_{i=0}^{d \log_p |a|} b_i \tau^i,\ b_i \in \mathrm{H}^0(S, L^{1-p^i}),$$

where $b_{d \log_p |a|}$ is by definition nowhere vanishing.

Now we introduce the level structure.

Definition 0.4. Let I be an ideal of A, and $X = (L, \phi)$ is an elliptic A-module of rank d. Then an **I-level structure** on X is a A-module morphism

$$\psi : (I^{-1}A/A)^d \longrightarrow X(S),$$

such that for each $m \in V(I)$, as a divisor $X_m := \ker(\phi(m)) \subseteq X$ coincides with the sum of divisors

$$\sum_{\alpha \in (m^{-1}A/A)^d} \psi(\alpha).$$

Remark 0.5. When the image of $S \to \mathrm{Spec}(A)$ does not intersect $V(I)$, the level structure of I is equivalent to an isomorphism

$$(I^{-1}A/A)^d \times S \cong X_I.$$

Roughly, this is analogues to the n-level structure of an elliptic curves over $\mathbb{F}_q$, where $\gcd(n, q) = 1$.

Theorem 0.6 (Existence of moduli scheme)**.** *Let $I \subseteq A$ be a nonzero ideal with $\#V(I) \geq 2$. Then the functor*

$$\mathrm{Sch}_A \longrightarrow \{elliptic\ A\ modules\ of\ rank\ d\ with\ I-level\ structure\}/iso$$

is represented by a scheme M_I^d, which is of finite type over A.

Proof.

Lemma 0.7 (Lemma 1)**.** *Let B be a ring of characteristic p with* $\mathrm{Spec}(B)$ *connected, $f_1, f_2 \in B\{\tau\}$, where $f_i = \sum_{j=0}^{d_i} a_{ij}\tau^j$ such that a_{id_i} is invertible in B. Let $h \in B\{\tau\}$ such that*

$$hf_1 = f_2h.$$

Then we have

- *If $d_1 \neq d_2$, then $h = 0$.*
- *If $d_1 = d_2$ and $h \neq 0$, then the leading coefficient of h is invertible.*

Lemma 0.8 (Lemma 2)**.** *Let B be a ring of characteristic p, $f = \sum_{i=0}^{n} a_i\tau^i$, $d > 0$, with a_d invertible and a_i nilpotent for $i > d$. Then there exists a unique element of the form*

$$h = 1 + \sum_{j=1}^{m} b_j\tau^j,\ b_j\ nilpotent$$

such that

$$\deg(hfh^{-1}) = d.$$

Proof. Strategy of the second lemmas: considering the ideal I generated by nilpotent coefficients, show by induction that we could take the conjugation to make the coefficient of index $> d$ into the ideal I^{2^N}. □

Now we could give a sketch of the proof. The proof is divided into two parts:

- The first part is a geometric argument, showing that when restricting the functor over some subcategories (that cover the whole category Sch_A), any elliptic A-module of rank d is trivialized. Since $V(I)$ has at least two primes, for any elliptic A-module of rank d X over S, there exists at least one $m \in V(I)$ such that m is not the characteristic of X; in other words, the image of $S \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(A)\backslash m \to \mathrm{Spec}(A)$. So we first restricts the functor above onto subcategories $Sch_{A,m}$, consisting of all of S such that $S \to \mathrm{Spec}(A)\backslash m \to \mathrm{Spec}(A)$. But after taking the restriction, any choice of a nonzero element in $(m^{-1}A/A)^d$ gives a trivialization of X (a section $\psi(a)$ which is nowhere vanishing on S). So each elliptic A-module of rank d over such a S is over a trivial bundle.
- Now by the Lemma 2, under the isomorphism, there exists a unique standard elliptic A-module of rank d over the trivial line bundle. Based on this, the moduli of standard elliptic module with level I structure can be constructed by the parametrization of coefficients of $\phi(a_i)$ and ψ, where a_i are finite generators of A over $\mathbb{F}_q$, thus exists and affine.

So at the end, we could glue them (by the uniqueness) to get the whole moduli scheme M_I^d.

□

Deformations of elliptic modules

In this section, we will use the result in last talk to show the geometric properties of M_I^d.

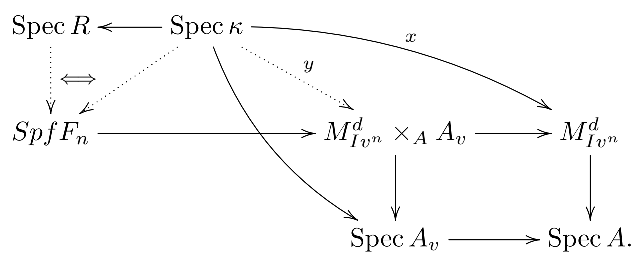
We let $v \in \mathrm{Spec}(A)$, $\mathcal{O} = A_v$, $\kappa = \overline{A_v/v}$, and denote by $\mathcal{C}$ by the category of complete local $\widehat{\mathcal{O}}^{nr}$-algebras whose residue field is κ. We pick X to be an elliptic A-module of rank d over κ with level structure v^n.

Now we consider the functor

$$R \in \mathcal{C} \longmapsto \{deformations\ of\ level\ v^n\ of\ X\ over\ R\}/iso.$$

Here the deformation is in the sense of elliptic A-module of rank d , not the one for formal groups.

Then this functor can be represented as follows: Let I be a nontrivial ideal in A with $v \notin V(I)$. We lift the v^n level structure on X to a level structure of level Iv^n, which corresponds to a point $x \in M_{Iv^n}^d(\kappa)$. Then we take the image y of x in $M_{Iv^n}^d \times_A A_v^{nr}$, and let F_n be the completion local ring at y. The ring F_n represents our functor. To show the representability, we have the picture as belows



On the other hand, given $R \to C$ with an elliptic A-module of rank d Y over R, we could consider the induced limit

$$\widehat{Y} = \varinjlim Y_{v^n},$$

which is a divisible A_v-module in the category of formal schemes. (We note that the divisibility can be showed by Hensel's Lemma.) Besides, a v^n-level structure on Y induces a level structure of level n on $\widetilde{Y}$, in the sense of divisible formal group (recall). We recall the ring E_n constructed in the last talk, which parametrizes (represents) the all possible deformation of divisible formal group $\widetilde{Y}$ over R. Then by induced structure, we could get a morphism

$$\mathrm{Spec}(F_n) \longrightarrow \mathrm{Spec}(E_n).$$

Here is the main result in this section

Theorem 0.9. *The homomorphism $E_n \to F_n$ given above is an isomorphism.*

Granting the theorem, the local property of M_I^d can be deduced from what we know about the ring E_n last time, and we get the following geometric properties about the modular scheme.

Corollary 0.10. *Assume I is a nontrivial ideal in A such that $\#V(I) \geq 2$. Then M_I^d is a smooth variety of dimension d over $\mathbb{F}_q$. The morphism $M_I^d \to \mathrm{Spec}(A)$ is smooth over $\mathrm{Spec}(A) \backslash V(I)$. And if $J \subseteq I$, the morphism $M_J^d \to M_I^d$ is finite and flat.*

Proof. The proof is divided into three parts:

- Since a level structure on $\widehat{Y} = \varinjlim Y_{v^n}$ will also endow a v^n-level structure on Y, we could reduce to $n = 0$.
- From the embedding $\mathbb{F}_p[x] \to A$, any elliptic A-module of rank d has a natural structure of elliptic $\mathbb{F}_p[x]$-modules, so we could get a natural map from $E_0' \to E_0$ ($F_0' \to F_0$). And after the discussion about the action of A_x^* on those rings, we could reduce from general A to $A = \mathbb{F}_p[x]$.

- Prove the case for $n = 0$, $A = \mathbb{F}_p[x]$, $v = (x)$. Here by what we know before about E_0, and the modular interpretation of F_0, those two rings are isomorphic to $\mathbb{F}_p[x, \alpha_1, \ldots, \alpha_{d-1}]$. We then show that the induced map on tangent spaces is injective, thus the morphism we have should be an isomorphism.

□

The action by adeles

In this section, we introduce the action of $\mathrm{GL}(d, \mathbb{A}_f)$ on the modular scheme.

We let $\mathbb{A}$ be the ring of adeles of k, and $\mathring{A}_f$ be the ring of finite adeles. And let $\widehat{A} = \varprojlim A/I$ be its completion.

Since $M_J^d \to M_I^d$ is finite flat between qcqs schemes, we could define the inverse limit as the big modular scheme.

$$M^d = \varprojlim M_I^d.$$

We now use the modular interpretation to define the action of $\mathrm{GL}(d, \mathbb{A}_f)/k^*$ on M^d.

Let S be a A-scheme, X be a elliptic A-module of rank d over S together with a morphism $\psi : (k/A)^d \to X(S)$, such that for each $I \subset A$, the restriction on $(I^{-1}/A)^d$ is a structure of level I. Let $g \in \mathrm{GL}(d, \mathbb{A}_f)$ be a matrix with coefficient in $\widehat{A}$, which is an endomorphism of $(k/A)^d$ with finite kernel P. We denote by H to be the sum of divisors

$$\sum_{\alpha \in P} \psi(\alpha).$$

Then H corresponds to a A-finite submodule of X, and X/H is also an elliptic A-module of rank d, since ψ commutes with A-actions. This actually endows a unique morphism $\psi_1 : (k/A)^d \to X/H(S)$ such that the following diagram commutes

$$\begin{array}{ccc} (k/A)^d & \xrightarrow{\psi} & X(S) \\ \downarrow g & & \downarrow \\ (k/A)^d & \xrightarrow{\psi_1} & X/H(S). \end{array}$$

Here in order to show that the morphism $\psi_1|_{I^{-1}/A}$ is a level structure, we will need a small result in the last talk. And after that, by checking the local coordinates, we see g gives a left action on M^d. But note that since the scalar matrix k^* acts trivially (this is because $X/X_a \cong X$ for $a \in A$, and ψ commutes with A-actions, so $\psi_1 = \psi$), we could extend the action to $\mathrm{GL}(d, \mathbb{A}_f)/k^*$. So follow this we get a morphism $g : M^d \to M^d$.

This action actually makes the modular scheme of given level become the quotient of the big modular scheme. Let $I \subset A$ be a nontrivial ideal with $\#V(I) \geq 2$, and let $U_I = \ker(\mathrm{GL}(d, \widetilde{A}) \to GL(d, A/I))$. Then for any $J \subseteq I$, the morphism

$$M_J^d \times_A (\mathrm{Spec}(A) \backslash V(J)) \longrightarrow M_I^d \times_A (\mathrm{Spec}(A) \backslash V(J))$$

is a finite etale Galois covering (bundle) with structure group U_I/U_J. So since M_I^d is normal, by reducing to the affine, the whole M_I^d equals to $U_I \backslash M_J^d$, so we get

$$M_I^d = U_I \backslash M^d.$$

Congruence relations

At the end of the note, we give the congruence relation of modular schemes.

Let $v \in \mathrm{Spec}(A)$, and let $M^d_{(v)}$ be the fiber of M^d over v. Then to each point of $M^d_{(v)}$ it corresponds to a elliptic A-module of rank d , which also corresponds to a formal A_v-module, since $\phi(v^n)(T)$ is approaching to 0 as $n \rightsquigarrow \infty$. Here is our main theorem:

Theorem 0.11. *Let $W \subseteq M^d_{(v)}$ be the subset of points that correspond to formal A_v-modules of height 1. Then we get*

1. *The subset W is an open, affine and everywhere dense subset in $M^d_{(v)}$ that is $\mathrm{GL}(d, \mathbb{A}_f)$-invariant.*
2. *Let $B \subset \mathrm{GL}(d, k_v)$ be a group of matrices (a_{ij}) with $a_{i1} = 0$ for $i > 1$. Let B' be the preimage of B in $\mathrm{GL}(d, \mathbb{A}_f)$. Then the $\mathrm{GL}(d, \mathbb{A}_f)$-scheme W is the induced scheme by some (any) B'-scheme W^0 that has the following properties*
   - *(a) The matrices $(a_{ij}) \in B \subset B'$ for which $|a_{11}|_v = 1$ and the lower right corner $(i, j > 1)$ coincides with the identity acts trivially on W^0_{red}.*
   - *(b) The matrix*

$$\begin{pmatrix} \pi_v & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

     acts on W^0_{red} as the geometric Frobenius of A_v/π_v.

References

[Dr] V.G. Drinfel'd. Elliptic Modules, Mathematics of the USSR-Sbornik (1974), 23 (4): 561.