K-theory seminar
Lecture 4 • Andrew Snowden • February 6, 2015

1. K-theory of schemes

1.1. Definitions. Let $X$ be a scheme. We define $\mathcal{P}(X)$ to be the category of locally free sheaves on $X$ of finite rank, regarded as an exact category. We define $K_i(X)$ to be $K_i(\mathcal{P}(X))$.

Suppose that $X$ is noetherian. Then the category $\mathcal{M}(X)$ of coherent sheaves on $X$ is abelian, and we define $G_i(X) = K_i(\mathcal{M}(X))$. The inclusion $\mathcal{P}(X) \subset \mathcal{M}(X)$ induces a map $K_i(X) \to G_i(X)$. If $X$ is regular, then this map is an isomorphism (since every coherent sheaf then admits a finite length resolution by locally free sheaves).

Let $M$ be a contravariant functor on the category of noetherian schemes with flat morphisms. If every coherent sheaf on $X$ is flat and $Y$ is a contravariant functor from schemes to abelian groups, then $M$ embeds into a sheaf in $\mathcal{M}(X)$ and $\mathcal{P}(X)$.

1.2. Pull-back maps. Let $f: X \to Y$ be a map of schemes. Then there is an exact functor $f^*: \mathcal{P}(Y) \to \mathcal{P}(X)$, which induces a homomorphism $f^*: K_i(Y) \to K_i(X)$. In this way, $K_i$ is a contravariant functor from schemes to abelian groups.

Now suppose that $f$ is flat and $X$ and $Y$ are noetherian. Then $f^*: \mathcal{M}(Y) \to \mathcal{M}(X)$ is an exact functor, and so there is an induced map $f^*: G_i(Y) \to G_i(X)$. In this way, $G_i$ is a contravariant functor on the category of noetherian schemes with flat morphisms.

More generally, let $\mathcal{M}(Y, f^*) \subset \mathcal{M}(Y)$ be the category of sheaves $M$ such that $L_i f^*(M) = 0$ for $i > 0$. Then $f^*$ induces an exact functor $\mathcal{M}(Y, f^*) \to \mathcal{M}(X)$, and thus homomorphisms $K_i(\mathcal{M}(Y, f^*)) \to G_i(X)$. If $f$ has finite Tor dimension (i.e., $L_i f^* = 0$ for $i \gg 0$, which holds if $Y$ is regular) and every coherent sheaf on $Y$ is a quotient of a vector bundle (e.g., $Y$ is projective over an affine scheme) then every coherent sheaf on $Y$ admits a finite resolution by sheaves in $\mathcal{M}(Y, f^*)$, and so $K_i(\mathcal{M}(Y, f^*)) = G_i(Y)$. In this case, $f^*$ induces a map $G_i(Y) \to G_i(X)$. For $i = 0$, this is an Euler characteristic construction: for a coherent sheaf $M$ on $Y$ we have

$$f^*([M]) = \sum_{i \geq 0} (-1)^i[L_i f^*(M)],$$

where $[-]$ denotes the class in $G_0$. The sum is finite due to the finiteness of Tor dimension.

1.3. Push-forward maps. Let $f: X \to Y$ be a finite morphism of noetherian schemes. Then $f_*$ induces an exact functor $\mathcal{M}(X) \to \mathcal{M}(Y)$, and we get homomorphisms $f_*: G_i(X) \to G_i(Y)$.

More generally, suppose that $f$ is proper. Then $R^if_* = 0$ for $i \gg 0$, and each $R^if_*$ takes coherent sheaves to coherent sheaves. Let $\mathcal{M}(X, f_*) \subset \mathcal{M}(X)$ be the category of sheaves $M$ on which $R^if_*(M) = 0$ for $i > 0$. Then $f_*$ induces homomorphisms $K_i(\mathcal{M}(X, f_*)) \to K_i(Y)$. If every coherent sheaf on $X$ embeds into a sheaf in $\mathcal{M}(X, f_*)$ (which is automatic if $X$
admits an ample line bundle) then $K_i(\mathcal{M}(X, f_\ast)) = K_i(\mathcal{M}(X)) = G_i(X)$, and we get a homomorphism $f_\ast : G_i(X) \to G_i(Y)$. For $i = 0$, this is an Euler characteristic:

$$f_\ast(M) = \sum_{i \geq 0} (-1)^i [R^i f_\ast(M)].$$

**Proposition 1.** Let $X$ and $Y$ be noetherian schemes admitting ample line bundles, and let $f : X \to Y$ be a proper map of finite Tor dimension. Then $f_\ast$ induces a map $K_i(X) \to K_i(Y)$, and the diagram

$$
\begin{array}{ccc}
K_i(X) & \longrightarrow & G_i(Y) \\
\downarrow f_\ast & & \downarrow f_\ast \\
K_i(Y) & \longrightarrow & G_i(Y)
\end{array}
$$

commutes.

**Proof.** Let $\mathcal{P}(X, f_\ast)$ be the category of vector bundles $\mathcal{E}$ on $X$ such that $R^i f_\ast(\mathcal{E}) = 0$ for all $i > 0$. Using the ample line bundle on $X$, every vector bundle on $X$ embeds into one in $\mathcal{P}(X, f_\ast)$ such that the quotient is also a vector bundle. Thus $K_i(\mathcal{P}(X, f_\ast)) = K_i(\mathcal{P}(X)) = K_i(X)$.

Let $\mathcal{H}(Y) \subset \mathcal{M}(Y)$ be the category of sheaves of finite Tor dimension. We claim that for $\mathcal{E} \in \mathcal{P}(X, f_\ast)$, we have $f_\ast(\mathcal{E}) \in \mathcal{H}(Y)$. This is local on $Y$, so we can assume $Y$ is affine. Let $\{U_i\}$ be a finite open cover of $X$. Then the Čech complex

$$0 \to H^0(X, \mathcal{E}) \to \bigoplus_i H^0(U_i, \mathcal{E}) \to \bigoplus_{ij} H^0(U_{ij}, \mathcal{E}) \to \cdots$$

is exact (since $\mathcal{E} \in \mathcal{P}(X, f_\ast)$) and of finite length, and its terms have finite Tor dimension (since $f$ does). This shows that $H^0(X, \mathcal{E}) = f_\ast(\mathcal{E})$ has finite Tor dimension.

Since $Y$ admits an ample line bundle, every coherent sheaf on $Y$ is a quotient of a vector bundle, and every sheaf in $\mathcal{H}(Y)$ admits a finite length resolution by vector bundles. Thus $K_i(\mathcal{H}(Y)) = K_i(\mathcal{P}(Y)) = K_i(Y)$. We have thus obtained our map $f_\ast : K_i(X) \to K_i(Y)$.

**Proposition 2** (Projection formula). Let $f$ be above. Then the equation

$$f_\ast(x \cdot f^\ast(y)) = f_\ast(x) \cdot y$$

holds in the following cases: (i) $x \in K_0(X)$, $y \in G_i(Y)$; (ii) $x \in K_0(X)$, $y \in K_i(Y)$; (iii) $x \in G_i(X)$, $y \in K_0(Y)$.

1.4. **Open and closed subschemes.**

**Proposition 3.** Let $X$ be a noetherian scheme. Let $X_{\text{red}}$ be the reduced subscheme of $X$. Then the map $f_\ast : G_i(X_{\text{red}}) \to G_i(X)$ induced by the finite map $f : X_{\text{red}} \to X$ is an isomorphism.

**Proof.** Every coherent sheaf on $X$ has a finite length filtration (given by powers of the nilradical) where the graded pieces belong to $\mathcal{M}(X_{\text{red}})$.

**Proposition 4.** Let $X$ be a noetherian scheme and let $Z$ and $Z'$ be closed subschemes with the same underlying topological space. Then there is a natural isomorphism $G_i(Z) = G_i(Z')$.

**Proof.** We have $Z_{\text{red}} = Z'_{\text{red}}$. Now use the previous proposition.

We can therefore define $G_i(Z)$ for a Zariski closed subspace $Z$ of $X$. 

Proposition 5. Let $X$ be a noetherian scheme, let $Z$ be a closed subscheme, and let $U$ be its complement. Then there is a long exact sequence

$$\cdots \to G_i(Z) \to G_i(X) \to G_i(U) \to G_{i-1}(Z) \to \cdots$$

Proof. Give $Z$ the reduced scheme structure. Let $\mathcal{M}_Z(X) \subset \mathcal{M}(X)$ be the category of sheaves supported on $Z$. Identify $\mathcal{M}(Z)$ with the subcategory of $\mathcal{M}(X)$ on objects killed by the ideal sheaf $I_Z$. Then every object of $\mathcal{M}_Z(X)$ admits a finite length filtration (by powers of $I_Z$) where the quotients belong to $\mathcal{M}(Z)$, and so $K_i(\mathcal{M}_Z(X)) = K_i(\mathcal{M}(Z)) = G_i(Z)$. It is a standard fact that $\mathcal{M}(U)$ is the Serre quotient $\mathcal{M}(X)/\mathcal{M}_Z(X)$. The result now follows from the localization sequence. □

1.5. Limits. Let $\{X_i\}_{i \in I}$ be a filtered inverse system of schemes where the transition maps $X_i \to X_j$ are affine. Then the inverse limit $X$ exists as a scheme. We have the following result in this situation:

Proposition 6. The natural map $\lim_i K_q(X_i) \to K_q(X)$ is an isomorphism. If $X$ and the $X_i$ are noetherian and the transition maps are flat, the same is true for $G_q$.

Proof. We have an equivalence $\mathcal{P}(X) = \lim_i \mathcal{P}(X_i)$, and the $Q$-construction and $\pi_i$ commute with filtered colimits. Similarly for $\mathcal{M}$. □

2. Homotopy invariance

2.1. Graded modules over polynomial rings. Let $A$ be a noetherian ring, and let $B = A[t_1, \ldots, t_n]$ be the graded ring where $\deg(t_i) = 1$. Let $\mathcal{M}_{gr}(B)$ be the category of finitely generated non-negatively graded $B$-modules. This has a functor $(-1)$ (shift the grading), which gives $K_i(\mathcal{M}_{gr}(B))$ the structure of a $\mathbb{Z}[t]$-module. There is an exact functor $\mathcal{M}(A) \to \mathcal{M}_{gr}(B)$ given by $M \mapsto M \otimes_A B$, which induces group homomorphism $G_i(A) \to K_i(\mathcal{M}_{gr}(B))$, and therefore a $\mathbb{Z}[t]$-module homomorphism $G_i(A) \otimes \mathbb{Z}[t] \to K_i(\mathcal{M}_{gr}(B))$.

Proposition 7. The canonical homomorphism $\psi: G_i(A) \otimes \mathbb{Z}[t] \to K_i(\mathcal{M}_{gr}(B))$ is an isomorphism.

Proof. Let $\mathcal{N} \subset \mathcal{M}_{gr}(B)$ be the category of modules $M$ where $\text{Tor}_i^B(M, A) = 0$ for $i > 0$. The functors $\text{Tor}_i^B(-, A) = 0$ for $i > n$ (since $A$ is the quotient of $B$ by a regular sequence of length $n$), and so $K_i(\mathcal{N}) = K_i(\mathcal{M}_{gr}(B))$.

Let $\mathcal{N}_p \subset \mathcal{N}$ be the subcategory consisting of modules generated in degrees $\leq p$. There are exact functors

$$\mathcal{M}(A) \xrightarrow{a} \mathcal{N}_p \xrightarrow{b} \mathcal{M}(A)^{p+1}$$

give by

$$a(M_0, \ldots, M_p) = \bigoplus_{i=0}^p B(-i) \otimes_A M_i$$

and

$$b(N) = ((N/B_+ N)_0, \ldots, (N/B_+ N)_p).$$

Note that $N/B_+ N = N \otimes_B A$ is exact on $\mathcal{N}$. Clearly, $b \circ a$ is the identity functor, and thus induces the identity on $K$-theory.

For a graded $B$-module $N$, let $F_q N$ be the submodule generated by elements of degree $\leq q$, and let $F_{-1} N = 0$. Regarding $F_q$ as a functor $\mathcal{N}_p \to \mathcal{N}_p$, the chain $F_0 \subset \cdots \subset F_p$
is a filtration of the identity functor \( I \). One shows that for \( N \in \mathcal{N} \), there is a canonical isomorphism
\[
F_q N / F_{q-1} N = B(-p) \otimes (N/B^+ N)_p,
\]
and so \( N \mapsto F_q N / F_{q-1} N \) is exact as well. Thus \( \sum_{p=0}^\infty (F_q / F_{q-1})_* = I_* \) induces the identity on \( K_i(\mathcal{N}_p) \). However, from the above identification, we see that \( \sum_{p=0}^\infty (F_q / F_{q-1})_* \) induces \( ab \) on \( K \)-theory. Thus \( a \) and \( b \) are isomorphisms.

Taking the limit as \( p \to \infty \), we see that the functor
\[
\mathcal{M}(A)^{\oplus \infty} \to \mathcal{N}, \quad (M_0, M_1, \ldots) \mapsto \sum_{i=0}^{\infty} B(-i) \otimes_A M_i
\]
induces an isomorphism on \( K \)-theory. But this map is exactly \( \psi \), after identifying \( K_i(\mathcal{N}) \) with \( K_i(\mathcal{M}_{gr}(B)) \).

2.2. Modules over polynomial rings. Let \( A \) be a noetherian ring. We aim to prove the following result:

**Proposition 8.** The functor \( M \mapsto M \otimes_A A[x] \) induces an isomorphism \( G_i(A) \to G_i(A[x]) \).

**Proof.** The idea is to reduce to the case of graded modules over polynomial rings. To do this, we use the following observation: an \( A[x]\)-module is the same as a \( \mathbb{G}_m \)-equivariant module on \( \mathbb{A}^2_A \setminus \mathbb{A}^1_A \), where \( \mathbb{A}^1_A \) is the \( y \)-axis in \( \mathbb{A}^2_A \). Via the localization sequence, we can understand the \( \mathbb{G}_m \)-equivariant \( K \)-theory of \( \mathbb{A}^2_A \setminus \mathbb{A}^1_A \) from that of \( \mathbb{A}^2_A \) and \( \mathbb{A}^1_A \), which is simply the \( K \)-theory of graded modules over polynomial rings over \( A \).

Let us now translate this to ring theory. We have \( \mathbb{A}^2_A = \text{Spec}(A[t, u]), \mathbb{A}^1_A = \text{Spec}(A[t]) \) and \( \mathbb{A}^2_A \setminus \mathbb{A}^1_A = \text{Spec}(A[t, u, u^{-1}]). \) We have an equivalence of categories \( \mathcal{M}_{gr}(A[t, u, u^{-1}]) = \mathcal{M}(A[x]) \), and so the localization sequence gives a long exact sequence \([\text{fix: need to use } \mathbb{Z}\text{-graded modules, not just } \mathbb{Z}_{\geq 0}\text{-graded ones. doesn’t change much}]\)
\[
\cdots \to K_i(\mathcal{M}_{gr}(A[t])) \to K_i(\mathcal{M}_{gr}(A[t, u])) \to K_i(\mathcal{M}(A[x])) \to \cdots,
\]
which translates to
\[
\cdots \to G_i(A) \otimes \mathbb{Z}[t] \to G_i(A) \otimes \mathbb{Z}[t] \to G_i(A[x]) \to \cdots,
\]
The result now follows from the following lemma. \( \square \)

**Lemma 9.** The following diagram commutes:
\[
\begin{array}{ccc}
K_i(\mathcal{M}_{gr}(A[t])) & \longrightarrow & K_i(\mathcal{M}_{gr}(A[t, u])) \\
\| & & \| \\
K_i(A) \otimes \mathbb{Z}[t] & \longrightarrow^{t-1} & K_i(A) \otimes \mathbb{Z}[t]
\end{array}
\]
Here the top horizontal map comes from treating \( A[t] \)-modules as \( A[t, u] \)-modules where \( u \) acts by \( 0 \).

**Proof.** Let \( i : \mathcal{M}(A) \to \mathcal{M}_{gr}(A[t]) \) be \( i(M) = M \otimes_A A[t] \), and let \( j : \mathcal{M}(A) \to \mathcal{M}_{gr}(A[t, u]) \) be \( j(M) = M \otimes_A A[t, u] \). Tensoring the exact sequence
\[
0 \to A[t, u](-1) \to A[t, u] \to A[t] \to 0
\]
over \( A \) with \( M \), we obtain an exact sequence of functors
\[
0 \to j(-1) \to j \to i \to 0.
\]
Thus \( i = (t - 1)j \) on \( K \)-theory.

### 2.3. Affine bundles.

**Proposition 10.** Let \( X \) be a noetherian scheme and let \( f : E \to X \) be a flat map whose fibers are affine spaces. Then \( f^* : G_i(X) \to G_i(E) \) is an isomorphism.

**Proof.** Given \( T \to X \), we say that “the proposition holds for \( T \)” if the maps \( G_i(T) \to G_i(E_T) \) are isomorphisms for all \( i \), where \( E_T = E \times_X T \). Let \( Z \) be a closed subscheme of \( X \) with complement \( U \). We then obtain a diagram

\[
\cdots \to G_i(Z) \to G_i(X) \to G_i(U) \to \cdots
\]

\[
\cdots \to G_i(E_Z) \to G_i(E) \to G_i(E_U) \to \cdots
\]

Thus if the proposition holds for two of \( Z, U, \) or \( X \), then it holds for the third as well. By noetherian induction, we can assume that the proposition holds for \( E_Z \to Z \) for all proper closed subschemes \( Z \) of \( X \). If \( X \) is reducible, say \( X = Z_1 \cup Z_2 \), then the proposition holds for \( Z_1 \) and \( Z_2 \) and \( Z_1 \cap Z_2 \), and thus for \( X \setminus Z_1 = Z_2 \setminus (Z_1 \cap Z_2) \), and therefore for \( X \). We can therefore assume \( X \) is irreducible. Since \( G_i \) is insensitive to nilpotents, we can assume \( X \) is integral. Now take the direct limit of the above diagram over all proper closed subschemes of \( X \), to obtain a diagram

\[
\cdots \to \lim G_i(Z) \to G_i(X) \to \lim G_i(U) \to \cdots
\]

\[
\cdots \to \lim G_i(E_Z) \to G_i(E) \to \lim G_i(E_U) \to \cdots
\]

It thus suffices to show that the map

\[
(11) \quad \lim G_i(U) \to \lim G_i(E_U)
\]

is an isomorphism. We have

\[
\lim G_i(U) = G_i(\lim U) = G_i(K),
\]

where \( K \) is the function field of \( X \). Similarly,

\[
\lim G_i(E_U) = G_i(\lim E_U) = G_i(K[x_1, \ldots, x_n]).
\]

Thus (11) is an isomorphism by Proposition 8. \( \square \)

### 3. Filtration by codimension and the BGQ spectral sequence

#### 3.1. Preliminaries.

If \( X \to Y \) is a map of topological spaces with homotopy fiber \( F \) then there is a long exact sequence of homotopy groups

\[
\cdots \to \pi_i(F) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to \cdots
\]

If \( \mathcal{A} \) is an abelian category and \( \mathcal{B} \) is a Serre subcategory then the map \( N(Q(\mathcal{A})) \to N(Q(\mathcal{A}/\mathcal{B})) \) has homotopy fiber \( N(Q(\mathcal{B})) \), and the resulting long exact sequence is the localization sequence in \( K \)-theory.
One can think of $B \subset A$ as a 1-step filtration of $A$. There is a version of localization for longer filtrations, where the long exact sequence is replaced by a spectral sequence. We now explain how this works.

First, suppose that we have maps of topological spaces

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_0$$

Let $X_0 = Y_0$ and for $1 \leq i \leq n$ let $X_i$ be the homotopy fiber of $Y_i \to Y_{i-1}$. One would like to say that there is a spectral sequence with $E_1^{p,q} = \pi_{p-q}(X_q)$ that converges to $\pi_{p-q}(Y)$. This is essentially the case, except for the fact that $\pi_0$ and $\pi_1$ cause problems (because they’re not abelian groups). However, if the $Y_i$’s are all H-spaces, and the maps are maps of H-spaces, then this problem goes away, and there is indeed such a spectral sequence.

Now suppose that $\mathcal{A}$ is an abelian category and

$$0 = F^n \mathcal{A} \subset \cdots \subset F^1 \mathcal{A} \subset \mathcal{A}$$

is a decreasing filtration by Serre subcategories. For $1 \leq i \leq n$ put $B_i = F^{i-1}\mathcal{A}/F^i \mathcal{A}$, and let $B_0 = \mathcal{A}/F^0 \mathcal{A}$. For $0 \leq i \leq n$, let $Y_i = N(Q(A/F^i \mathcal{A}))$. Then for $1 \leq i \leq n$ the map $Y_i \to Y_{i-1}$ has homotopy fiber $X_i = N(Q(B_i))$, and $X_0 = Y_0 = N(Q(B_0))$. We thus have a spectral sequence with $E_1^{p,q} = \pi_{p-q}(X_q) = K_{p-q-1}(B_q)$ that converges to $\pi_{p-q}(Y) = K_{p-q-1}(\mathcal{A})$.

4. Severi–Brauer varieties and projective bundles

[to add]