1. \( K_0 \) OF A RING

Let \( R \) be an associative ring with a unit. Recall that a (left) \( R \)-module is projective if there exists a module \( Q \) such that \( P \oplus Q \) is free. Equivalently, for every diagram

\[
\begin{array}{ccc}
P & \rightarrow & M \\
\downarrow & & \downarrow \\
M & \rightarrow & N
\end{array}
\]

with \( M \to N \) surjective, there exists a map \( P \to M \) making a commutative triangle. Equivalently, \( \text{Ext}^1_R(P, \bullet) = 0 \).

The set of isomorphism classes of finitely generated projective \( R \)-modules has the structure of an abelian monoid under direct sum. Then \( K_0(R) \) is the group completion of this monoid. That is, it is the free-abelian group on the isomorphism classes of finitely generated projective \( R \)-modules mod the obvious relations: \([P \oplus Q] = [P] + [Q]\). It is not hard to check that \([P] = [P']\) in \( K_0(R) \) if and only if there exists a finitely generated projective \( R \)-module \( Q \) such that \( P \oplus Q \cong P' \oplus Q \). Even more concretely, this is equivalent to having \( P \oplus R^n \cong P' \oplus R^n \) for some \( n \geq 0 \).

**Example 1.** If \( R \) is a local ring then \( K_0(R) = \mathbb{Z} \) since finitely generated projective \( R \)-modules are free.

**Example 2.** If \( R \) is a PID then \( K_0(R) = \mathbb{Z} \) by the classification of finitely generated modules over a PID.

**Example 3.** If \( R \) is a Dedekind domain then \( K_0(R) = \mathbb{Z} \oplus \text{Cl}(R) \). To see this, note that a finitely generated projective module over \( R \) breaks up into a direct sum of fractional ideals (prove this by induction). Next note that \( I_1 \oplus I_2 \cong R \oplus I_1 I_2 \), so that if \( P \) is finitely generated and projective, \( P \cong R^{n-1} \oplus I \) for some ideal \( I \). This decomposition yields the decomposition of \( K_0(R) \).

**Example 4.** Eilenberg-Mazur Swindle: Let \( R^\infty \) be an infinitely generated free module. If \( P \oplus Q \cong R^n \) then

\[
P \oplus R^\infty \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = R^\infty
\]

Thus, if we allow non-finitely generated modules then we obtain a trivial \( K_0 \).

We record the following properties:

- \( K_0 \) is a covariant functor from rings to abelian groups;
- it respects finite direct products and filtered direct limits.
2. $K_0$ OF AN EXACT CATEGORY

**Definition 5.** An exact category is a pair $(\mathcal{C}, \mathcal{E})$ where $\mathcal{C}$ is an additive category that is a full subcategory of an abelian category $\mathcal{A}$, and $\mathcal{E}$ is the family of sequences in $\mathcal{C}$ of the form $0 \to B \to C \to D \to 0$ that are exact in $\mathcal{A}$. Further, we assume that if $B$ and $D$ lie in $\mathcal{C}$, then $C$ is also in $\mathcal{C}$ (that is, $\mathcal{C}$ is closed under extensions). We’ll usually say $\mathcal{C}$ is an exact category unless we wish to specify the exact sequences in $\mathcal{E}$.

**Definition 6.** Let $\mathcal{C}$ be a small exact category. Then $K_0(\mathcal{C})$ is the abelian group generated by the objects of $\mathcal{C}$, and with relations given by the exact sequences.

**Example 7.** Let $\mathcal{C}$ be the category of finitely generated projective $R$-modules contained in the category of all $R$-modules. Then $K_0(\mathcal{C}) = K_0(R)$ because exact sequences of projective modules split.

**Example 8.** Let $X$ be a quasi-projective scheme over a commutative ring $R$. Then one is interested in $K_0(VB(X)) = K_0(X)$, where $VB(X)$ is the exact category of vector bundles on $X$, which is a full subcategory of the category of quasicoherent modules on $X$.

**Remark 9.** If $X$ is a noetherian scheme, then let $G^1_0(X) = K_0(\text{Coh}(X))$. There exists a morphism

$$K_0(X) \to G_0(X)$$

called the Cartan homomorphism, which Serre proved is an isomorphism when $X$ is regular and quasiprojective over a noetherian ring.

**Example 10.** One can show that $K_0(\mathbb{P}^1) = \mathbb{Z}^2$. More generally, there is a surjective map

$$\text{rk} \oplus \text{det}: K_0(X) \to H^0(X, \mathbb{Z}) \oplus \text{Pic}(X)$$

which is an isomorphism for nonsingular curves.

3. $K_1$ OF A RING

As above, $R$ is an associative ring with unit.

**Definition 11.** First define $\text{GL}(R) = \lim\limits_{\longrightarrow} \text{GL}_n(R)$ where $\text{GL}_n(R) \to \text{GL}_{n+1}(R)$ is defined by $g \mapsto \left( \begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Then $K_1(R) := \text{GL}(R)/[\text{GL}(R), \text{GL}(R)]$.

Note that $K_1(R)$ is an abelian group. It satisfies the following universal property: every homomorphism $\text{GL}(R) \to A$ factors through $K_1(R)$. As with $K_0$, the association $R \mapsto K_1(R)$ is functorial.

In order to understand $K_1(R)$ it’s useful to get a grip first on the commutator subgroup of $\text{GL}(R)$. Define $E_n(R) \subseteq \text{GL}_n(R)$ to be the group generated by the elementary matrices $e_{ij}(r)$ where $r \in R$, $i \neq j$, and $e_{ij}(r)$ is the usual matrix with all entries 0 save for the $(i, j)$th, which contains $r$, and the diagonal entries, which are 1. Set $E(R) = \lim\limits_{\longrightarrow} E_n(R)$. This is the same as the group generated by the images of the $e_{ij}(r)$ in $\text{GL}(R)$.

**Lemma 12 (Whitehead).** One has $E(R) = [\text{GL}(R), \text{GL}(R)]$. 

Proof. First show that $E(R) = [E(R), E(R)]$ by proving the same thing for $E_n(R)$ for $n \geq 3$ using a bunch of identities which we’ll write down:

$$e_{ij}(r)e_{ij}(s) = e_{ij}(r + s)$$

$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & j \neq k, i \neq l \\ e_{ik}(rs) & i \neq k. \end{cases}$$

If $A, B \in \text{GL}_n(R)$ then

$$\begin{pmatrix} ABA^{-1}B^{-1} \\ 0 \\ 1_n \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$  

For any $M \in \text{GL}_n(R)$,

$$\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ M^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -M^{-1} \\ M - 1 & 1 \end{pmatrix},$$

and each of these is in $E_n(R)$. So $K_1(R) = \text{GL}(R)/E(R)$. □

**Definition 13.** Define $\text{SL}(R) = \varprojlim \text{SL}_n(R)$.

One has $\text{GL}(R) = \text{SL}(R) \times \mathbb{R}^\times$ and there is a determinant map $\det: K_1(R) \to \mathbb{R}^\times$. Define $\text{SK}_1(R)$ to be the kernel of this map.

**Example 14.** If $F$ is a field then $K_1(F) = F^\times$. To see this, note that Dickson showed $\text{SL}_n(F) = [\text{GL}_n(F), \text{GL}_n(F)]$ in 1899 except for two specific cases. Or, use elementary row operators to show $E_n(F) = \text{SL}_n(F)$ for all $n \geq 1$.

**Example 15.** In 1941, Dieudonné proved that if $D$ is a division ring then $K_1(D) = D^\times/[D^\times, D^\times]$. This isomorphism is given by the so-called Dieudonné determinant $\text{GL}(D) \to (D^\times)/[D^\times, D^\times]$.

**Example 16.** Since $\text{GL}$ commutes with products, one can show $K_1(R_1 \times R_2) = K_1(R_1) \oplus K_1(R_2)$.

**Example 17.** One has $\text{GL}(R) \cong \text{GL}(M_n(R))$ so that $K_1(R) \cong K_1(M_n(R))$.

**Example 18.** Bass-Milnor-Serre proved that if $R$ is euclidean or a maximal order in a number field $K$, then $\text{SK}_1(R) = 0$ and $K_1(R) = K^\times$.

Given $a, b \in R$ with $(a, b) = 1$, choose $b$ and $c$ so that $ad - bc = 1$. Then let $[a, b]$ denote the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SK}_1(R)$. This is well-defined and one has the relations

- $[a, b] = [b, a]$,
- $[a, b] = 1$ for all $b \in R$ if $a \in R^\times$,
- $[a_1a_2, b] = [a_1, b][a_2, b]$,
- $[a, b] = [a + rb, b]$ for all $r \in R$.

These symbols generate $\text{SK}_1(R)$ under certain conditions (e.g. $R$ is noetherian of dimension $\leq 1$, plus more).

4. $K_2$ of a ring

**Definition 19.** Let $R$ be an associative unital ring and let $n \geq 3$ be an integer. The Steinberg group $\text{St}_n(R)$ is generated by symbols $x_{ij}(r)$ with $1 \leq i \leq j \leq n$, $r \in R$,.
There exists a map $\phi_n: \text{St}_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$. Define $\text{St}(R) = \lim_{\to} \text{St}_n(R)$. There is a natural map $\phi = \lim_{\to} \phi_n: \text{St}(R) \to E(R)$. Set $K_2(R) = \ker \phi$.

There exists an exact sequence

$$1 \to K_2(R) \to \text{St}(R) \to \text{GL}(R) \to K_1(R).$$

One can show:

**Theorem 20.** The group $K_2(R)$ is the center of $\text{St}(R)$. In particular, $K_2(R)$ is abelian.

**Proof.** If $x \in Z(\text{St}(R))$ then $\phi(x) \in Z(E(R))$, so $x \in \ker \phi$. Let $x \in K_2(R)$, so that $x \in \text{St}(R)$ and $\phi(x) = 1$. Note that for all elements $y \in \text{St}(R)$ we have $\phi([x, y]) = 1$. Choose a large $n$ such that $x$ can be written as a word in $x_{ij}(r)$s with $i, j < n$. Then for all $y = x_{kn}(s)$ with $k < n$, the Steinberg relations give allow one to write $[x, y]$ as a word in $x_{in}(r)$s for $i < n$. But the subgroup generated by the $x_{in}(r)$s with $i < n$ maps injectively by $\phi$ into $E(R)$. Since $\phi([x, y]) = 1$, it follows that $[x, y] = 1$. Hence $x$ commutes with all $x_{kn}(s)$ with $k < n$. An analogous argument allows one to show it commutes with all $x_{nk}(s)$. Then relations also allow one to show that $x$ commutes with $x_{ij}(s)$ with both $i, j < n$. This is enough to show that $x$ commutes with everything (make $n$ even larger if necessary). \qed

**Example 21.** One can show $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, $K_2(\mathbb{Z}[\sqrt{-7}]) = \mathbb{Z}/2\mathbb{Z}$, $K_2(\mathbb{Z}[i]) = 1$, and there are a bunch of other examples in the literature. But in general it’s pretty hard to compute $K_2(R)$.

**Remark 22.** The extension

$$0 \to K_2(R) \to \text{St}(R) \to E(R) \to 0$$

is the universal central extension of $E(R)$. Thus $K_2(R) = H_2(E(R), \mathbb{Z})$.

5. **Products**

If $R$ is a commutative ring then one has a product map $K_0(R) \otimes \mathbb{Z} K_0(R) \to K_0(R)$, and similarly for $K_1(R)$ and $K_2(R)$\(^1\). One even has a map

$$K_1(R) \otimes_{K_0(R)} K_1(R) \to K_2(R)$$

Map $g \otimes h \mapsto \{g, h\}$ as follows. First, suppose that $\alpha, \beta \in E(R)$ commute. Then define a product $\alpha \ast \beta \in K_2(R)$ by setting $\alpha \ast \beta = [\alpha, \beta]$ where $\bar{\alpha}, \bar{\beta}$ are lifts of $\alpha$ and $\beta$ in $\text{St}(R)$. Now regard $g \in \text{GL}_n(R)$ and $h \in \text{GL}_m(R)$. Then define

$$\{g, h\} = \begin{pmatrix} g \otimes 1_m & 0 & 0 \\ 0 & g^{-1} \otimes 1_m & 0 \\ 0 & 0 & 1_{mn} \end{pmatrix} \ast \begin{pmatrix} 1_n \otimes h & 0 & 0 \\ 0 & 1_{mn} & 0 \\ 0 & 0 & 1_n \otimes h^{-1} \end{pmatrix}$$

\(^1\)The existence of these maps is not obvious, although if one realizes $K_1(R)$ as $K_0$ of projective modules along with an automorphism, then the product in $K_1$ is given by tensor product.
Theorem 23 (Matsumoto). Let $F$ be a field. Then $K_2(F)$ is the free abelian group on the symbols $\{a, b\}$ with $a, b \in F^\times$ subject to the relations
\[
\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\},
\{a, b\} = \{b, a\}^{-1},
\{a, 1 - a\} = 1.
\]
That is, $K_2(F) \cong F^\times \otimes F^\times / \langle a \otimes (1 - a) \rangle$.

Corollary 24. One has $K_2(F_q) = 1$.

Proof. Let $x$ be a generator of $F_q^\times$. If $q$ is even then $\{x, x\} = \{x, -x\} = 1$. If $q$ is odd then $\{x, xx^{q-1}\} = \{x, -x\} = 1$. So $\{x, x\}$ has order 1 or 2 by skew symmetry. The set $F_q^\times - \{1\}$ is invariant under $z \mapsto 1 - z$. It contains $\frac{q-1}{2}$ nonsquares and $\frac{q-3}{2}$ squares. Thus, there exists a nonsquare $z$ such that $1 - z$ is also nonsquare. Write $z = x^i$ and $1 - z = x^j$ for odd $i$ and $j$. Then one checks that $1 = \{z, 1 - z\} = \{x, x\}^{ij} = \{x, x\}$ since $ij$ is odd. □

Remark 25. Milnor $K$-theory is defined using the tensor algebra and the Steinberg symbols. For a field $F$ one checks that $K_0^M(F) = \mathbb{Z}$, $K_1^M(F) = F^\times$ and $K_2^M(F) = K_2(F)$.

Remark 26. The theorem of Merkurjev-Suslin says that $K_2(F)/nK_2(F)$ is the $n$-torsion in the Brauer group of $F$. 