Gross–Zagier reading seminar

Lecture 7 • Andrew Snowden • October 28, 2014

Let x be a Heegner point on $X_0(N)$, let c be the divisor $(x)-(\infty)$, and let d be the divisor $(x)-(0)$. We need to compute the Néron height pairing $\langle c, T_m d^\sigma \rangle$, where T_m is a Hecke operator and σ is an element of $\mathrm{Gal}(H/K)$ (K being the imaginary quadratic field and H being its Hilbert class field). This height pairing factors as a sum of local height pairings. The goal of this lecture is to explain the computation of the local pairing at archimedean places.

1. Generalities on archimedean local height pairings

Let S be a compact Riemann surface. As stated in the previous lecture, there is a unique partially defined function

$$\langle -, - \rangle \colon \mathrm{Div}^0(S) \times \mathrm{Div}^0(S) \dashrightarrow \mathbf{R}_{\geq 0}$$

satisfying the following conditions:

- If $a, b \in \mathrm{Div}^0(S)$ have disjoint support then $\langle a, b \rangle$ is defined.
- The pairing $\langle -, - \rangle$ is biadditive, whenever this makes sense: i.e., given $a, a', b \in \mathrm{Div}^0(S)$ such that $\mathrm{supp}(b)$ is disjoint from $\mathrm{supp}(a)$ and $\mathrm{supp}(a')$, we have $\langle a+a', b \rangle = \langle a, b \rangle + \langle a', b \rangle$.
- If f is a meromorphic function on S and $a = \sum n_i x_i$ is a degree 0 divisor such that $\mathrm{supp}(\mathrm{div}(f))$ and $\mathrm{supp}(a)$ are disjoint then

$$\langle \mathrm{div}(f), a \rangle = \langle a, \mathrm{div}(f) \rangle = \sum_i n_i \log \vert f(x_i) \vert^2 \vert.$$

- The pairing $\langle -, - \rangle$ is continuous in each variable, i.e., for any $a \in \mathrm{Div}^0(S)$ the function

$$(S \setminus \mathrm{supp}(a)) \times (S \setminus \mathrm{supp}(a)) \to \mathbf{R}, \qquad (x, y) \mapsto \langle (x) - (y), a \rangle$$

  is continuous, and similarly with the parameters reversed.

Uniqueness of such a pairing is not difficult: the difference of two such pairings would define a biadditve continuous everywhere-defined pairing $J \times J \to \mathbf{R}$ (where J is the Jacobian of S), which must be zero since J is compact and $\mathbf{R}$ has no compact subgroups. In particular, the pairing $\langle -, - \rangle$ is necessarily symmetric.

We now sketch the proof of existence, which is more difficult. Fix distinct points x_0 and y_0 in S. For $x \neq y_0$ and $y \neq x_0$, define

$$G(x, y) = \langle (x) - (x_0), (y) - (y_0) \rangle. \tag{1}$$

Then $\langle -, - \rangle$ is completely determined by G: indeed, if $a = \sum n_i x_i$ and $b = \sum m_j y_j$ then

$$\langle a, b \rangle = \sum_{i,j} n_i m_j G(x_i, y_j). \tag{2}$$

Conversely, given a function G, we can attempt to define $\langle -, - \rangle$ using the above formula. The following result explains what we need to be successful:

Lemma 3. *The formula* (2) *defines* $\langle -, - \rangle$ *satisfying the above conditions if and only if for fixed* $x \neq y_0$*, the function* $y \mapsto G(x, y)$ *on* $S \setminus \{x, x_0\}$ *is continuous, harmonic, and has logarithmic singularities of residue* $+1$ *and* -1 *at* $y = x$ *and* $y = x_0$*, and similarly with the roles of* x *and* y *reversed.*

Remark 4. We say a function f on S has a logarithmic singularity at x_0 of residue a if $f(x) - a \log |\rho(x)|^2$ is continuous near x_0, where ρ is holomorphic function defined near x_0 and vanishing to order 1 at x_0. □

Proof. Suppose G satisfies the stated conditions. Then $\langle -, - \rangle$ is well-defined, continuous, and biadditive via (2) so long as $\mathrm{supp}(a) \cup \{x_0\}$ is disjoint from $\mathrm{supp}(b) \cup \{y_0\}$. However, we can extend by continuity to the more general case where $\mathrm{supp}(a)$ and $\mathrm{supp}(b)$ are disjoint. To explain this, it suffices to show that $G(x_1, y) - G(x_2, y)$ makes sense as y tends to x_0. This follows from the nature of the singularities of G: for $y \approx x_0$, we have

$$G(x_i, y) = -\log |\rho(y)|^2 + c_i + O(\rho(y)),$$

where ρ is holomorphic and vanishing at x_0 and c_i is constant. Thus $G(x_1, y) - G(x_2, y)$ tends to $c_1 - c_2$ as y tends to x_0, and is therefore well-defined.

We have thus shown that $\langle -, - \rangle$ is defined where it should be, continuous, and biadditive. We now show that it has the correct form on principal divisors. Let f be a meromorphic function on S, and write $\mathrm{div}(f) = \sum m_j y_j$. Consider the function $S \to \mathbf{R}$ defined by $x \mapsto \log |f(x)|^2 - \sum m_j G(x, y_j)$. First notice that this is actually well-defined on all of S. We have already treated the case where x approaches y_0. If x approaches some y_j then $G(x, y_j)$ behaves like $n_j \log |f(x)|$, but so does $\log |f(x)|$; thus the difference is well-defined. Furthermore, it is not difficult to show that this function is harmonic, and it is therefore constant, say equal to c. But then if $a = \sum n_i x_i$ is a degree 0 divisor with $\mathrm{supp}(a)$ disjoint from $\mathrm{supp}(\mathrm{div}(f))$, we have

$$\sum_{i,j} n_i m_j G(x_i, y_j) = \sum_i n_i (\log |f(x_i)|^2 - c) = \sum_i n_i \log |f(x_i)|^2.$$

This shows that $\langle -, - \rangle$ assumes the right value on principal divisors.

We do not prove the converse, but note that it follows from the existence of such a G and the uniqueness of the pairing. □

Remark 5. Given an appropriate G, one can define the height pairing via (2). However, (1) may not hold! One always has

$$G(x, y) = \langle (x) - (x_0), (y) - (y_0) \rangle + c$$

for some constant c. To ensure $c = 0$, one must assume that $G(x_0, y)$ vanishes for some $y \in S \setminus \{x_0\}$. □

2. Archimedean local height pairings on $X_0(N)$

We now apply the preivous section to the case $S = X_0(N) = \mathfrak{h}^*/\Gamma_0(N)$. We take $x_0 = \infty$ and $y_0 = 0$. To construct the required function G on S, it suffices to find a function $G(z, z')$ on $\mathfrak{h} \times \mathfrak{h}$ satisfying the following:

- $G(\gamma z, \gamma' z') = G(z, z')$ for all $\gamma, \gamma' \in \Gamma_0(N)$.
- $G(z, z')$ is continuous and harmonic when z stays away from the orbit of z'.

- $G(z,z') = e_z \log|z-z'|^2 + O(1)$ as $z' \to z$, with z fixed, where e_z is the order of the stabilizer of z in $\Gamma_0(N)$. (Typically $e_z = 1$.)
- For z fixed, $G(z,z') = 4\pi y' + O(1)$ as $z' \to \infty$, and $G(z,z') = O(1)$ as z' tends to any other cusp.
- For z' fixed, $G(z,z') = 4\pi \frac{y}{N|z|^2} + O(1)$ as $z \to 0$, and $G(z,z') = O(1)$ as z tends to any other cusp.

The appearance of e_z and $4\pi y'$ and $4\pi \frac{y}{N|z|^2}$ comes from the form of uniformizing parameters on $X_0(N)$ pulled back to $\mathfrak{h}$.

Suppose we had a function g on $\mathfrak{h} \times \mathfrak{h}$ satisfying the following conditions

- $g(\gamma z, \gamma z') = g(z,z')$ for all $\gamma \in \mathbf{SL}_2(\mathbf{R})$.
- $g(z,z')$ is continuous and harmonic in each variable, as long as $z \neq z'$.
- $g(z,z') = \log|z-z'|^2 + O(1)$ as $z' \to z$.

Then we could try to define G by averaging g over $\Gamma_0(N)$. In fact, it's easy to write down a function g, namely

$$g(z,z') = \log\left|\frac{z-z'}{\overline{z}-z'}\right|^2.$$

However, the average of this over $\Gamma_0(N)$ does not converge.

To fix this problem, we will relax the condition that g be harmonic to $\Delta g = \epsilon g$, where ϵ is small. The $\mathbf{SL}_2(\mathbf{R})$ invariance of g means that it is a function only of the hyperbolic distance between z and z', or equivalently, of the parameter $t = 1 + \frac{|z-z'|^2}{2yy'}$, which is the hyperbolic cosine of this difference. We can therefore write $g(z,z') = Q(t)$ for some function Q. The equation $\Delta g = \epsilon g$ turns into the equation

$$\left((1-t^2)\frac{d^2}{dt^2} - 2t\frac{d}{dt} + \epsilon\right) Q = 0$$

This is a Legendre equation of index $s-1$, where $\epsilon = s(s-1)$ with $s > 1$. The only solution that does not blow-up at infinity is given by

$$Q_{s-1}(t) = \int_0^\infty \left(t + \sqrt{t^2-1}\cdot\cosh u\right)^{-u} du.$$

We put

$$g_s(z,z') = -2Q_{s-1}(t), \qquad t = 1 + \frac{|z-z'|^2}{2yy'}.$$

One can check from this closed form that the analytic conditions we require of g_s are satisfied. Furthermore, one can use it to show that the average

$$G_s(z,z') = \sum_{\gamma \in \Gamma_0(N)} g_s(z, \gamma z')$$

converges. The function G_s is not harmonic in z, but satisfies

$$\Delta_z G_s(z,z') = s(s-1) G_s(z,z'),$$

and similarly for z'.

We now want to let $s \to 1$ to obtain a harmonic function. Of course, this will not work exactly. It turns out that G_s has a simple pole at $s = 1$ of residue

$$\kappa = \frac{-12}{[\Gamma(1) : \Gamma_0(N)]} = -12N^{-1} \prod_{p|N} (1 + p^{-1})^{-1}.$$

In particular, this residue is independent of z and z'. We could try to fix G_s by simply subtracting off $\kappa(s-1)^{-1}$; however, the result would not be harmonic in z because

$$\Delta_z \left(G_s(z, z') - \frac{\kappa}{s-1} \right) = s(s-1)G_s(z, z')$$

tends to κ as $s \to 1$. Instead, we should subtract from $G_s(z, z')$ a function of z having the same eigenvalue for the Laplacian and the same residue at $s = 1$. Such a function is given by $-4\pi E(z, s)$, where E is the Eisenstein series

$$E(z, s) = \sum_{\Gamma_\infty \backslash \Gamma_0(N)} \mathrm{Im}(\gamma z)^s.$$

Thus $\lim_{s\to1}(G_s(z) + 4\pi E(z, s))$ is harmonic in z as $s \to 1$. However, it is not harmonic in z'. Thus we add $4\pi E(z, s)$ to it. But now we have reintroduced a pole at $s = 1$, and so fix that by subtracting off $\frac{\kappa}{s-1}$! And actually, we use $E(wz, s)$ instead of $E(z, s)$ to put the pole in z at the correct cusp. (Here $wz = -1/Nz$ is the Atkin–Lehner involution.) We thus have

$$G(z, z') = \lim_{s\to1} \left[G_s(z, z') + 4\pi E(-1/Nz, s) + 4\pi E(z', s) + \frac{\kappa}{s-1} \right].$$

In fact, we are allowed to add to G any constant C, as this does not change the height pairing. We choose to add the unique C so that $G(z, z')$ tends to 0 as $z \to \infty$. This is computed explicitly in Gross–Zagier:

$$C = \kappa \left[2 - \log N - 2\log 2 + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} + 2\sum_{p|N} \frac{p\log(p)}{p^2 - 1} \right], \tag{6}$$

where γ is Euler's constant. We thus have:

Proposition 7. *Let x and x' be distinct non-cuspidal points of $X_0(N)$, and let z and z' in $\mathfrak{h}$ map to them. Then*

$$\langle (x) - (\infty), (x') - (0) \rangle = \lim_{s\to1} \left[G_s(z, z') + 4\pi E(wz, s) + 4\pi E(z', s) + \frac{\kappa}{s-1} \right] + C.$$

3. Evaluation of the height pairing on Heegner points

3.1. **Background on Heegner points.** Recall that a Heegner point of $X_0(N)$ is a point corresponding to a cyclic N-isogeny $E \to E'$ where E and E' both have CM by the full order $\mathcal{O}_K$. Heegner points correspond to pairs $(\mathcal{A}, \mathfrak{n})$ where $\mathcal{A} = [\mathfrak{a}]$ is an ideal class of K and $\mathfrak{n}$ is a primitve ideal of norm N (i.e., $\mathcal{O}_K/\mathfrak{n} = \mathbf{Z}/N\mathbf{Z}$). Precisely, $(\mathcal{A}, \mathfrak{n})$ corresponds to the point $x_{\mathcal{A},\mathfrak{n}}$ of $X_0(N)$ representing the isogeny $(\mathbf{C}/\mathfrak{a} \to \mathbf{C}/\mathfrak{a}\mathfrak{n}^{-1})$. All Heegner points are defined over H, the Hilbert class field of K. If $\sigma \in \mathrm{Gal}(H/K)$ corresponds to $\mathcal{B}$ under class field theory, then $\sigma x_{\mathcal{A},\mathfrak{n}} = x_{\mathcal{B}\mathcal{A},\mathfrak{n}}$.

3.2. **Setup.** We want to compute

$$\begin{aligned}\langle (x)-(\infty),(x^\sigma)-(0)\rangle_\infty &= \sum_{v\mid\infty}\langle (x)-(\infty),(x^\sigma)-(0)\rangle_v\\ &= \sum_{\tau\in\mathrm{Gal}(H/K)}\langle (x^\tau)-(\infty),(x^{\sigma\tau})-(0)\rangle\end{aligned}$$

for a Heenger point x, where $\sigma\in\mathrm{Gal}(H/K)$. Suppose σ corresponds to $\mathcal{A}\in\mathrm{Cl}_K$ via class field theory. Then x^τ and $x^{\sigma\tau}$ vary over all $x_{\mathcal{A}_1,\mathfrak{n}}$ and $x_{\mathcal{A}_2,\mathfrak{n}}$ with $\mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}$. Thus the above pairing is equal to

$$\sum_{\mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}\langle (x_{\mathcal{A}_1,\mathfrak{n}})-(\infty),(x_{\mathcal{A}_2,\mathfrak{n}})-(0)\rangle.$$

Since $\mathfrak{n}$ is fixed in this sum, we drop it from the notation in what follows. Appealing to Proposition 7, we find that this is equal to

$$\lim_{s\to 1}\left[\gamma_s(\mathcal{A})+4\pi\sum_{\mathcal{B}\in\mathrm{Cl}_K}E(w\tau_{\mathcal{B}},s)+4\pi\sum_{\mathcal{B}\in\mathrm{Cl}_K}E(\tau_{\mathcal{B}},s)+\frac{h\kappa}{s-1}\right]+hC, \tag{8}$$

where

$$\gamma_s(\mathcal{A})=\sum_{\mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}G_s(\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2}).$$

We will study (8) in two steps, first by looking at γ_s, and then considering the remaining terms.

Remark 9. We actually want to do the computation with a Hecke operator T_m thrown into the pairing as well. We take $m=1$, so that T_m is simply the identity operator, for ease of exposition. □

3.3. **A formula for γ_s.** Let τ and τ' be points in the upper half-plane corresponding to $x_{\mathcal{A}}$ and $x_{\mathcal{B}}$. Recall that

$$g_s(\tau,\tau')=-2Q_{s-1}\left(1+\frac{|\tau-\tau'|^2}{2\,\mathrm{Im}(\tau)\,\mathrm{Im}(\tau')}\right).$$

Given $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ in $\Gamma_0(N)$, an elementary computation shows that

$$g_s(\gamma\tau,\tau')=-2Q_{s-1}\left(1+\frac{2nN}{|D|}\right) \tag{10}$$

where n is a non-negative integer. As G_s is the average of g_s over $\Gamma_0(N)$, we find

$$G_s(\tau,\tau')=-2\sum_{n=1}^{\infty}\rho(n)Q_{s-1}\left(1+\frac{2nN}{|D|}\right),$$

where $\rho(n)$ is the number of $\gamma\in\Gamma_0(N)$ giving n in the previous equation.

To get a feel for $\rho(n)$, let us consider the simplest case where $K=\mathbf{Q}(i)$ and $N=1$ and $\tau=\tau'=i$. (In fact, $G_s(\tau,\tau')=\infty$ since $\tau=\tau'$, but the coefficients $\rho(n)$ still make sense.) We have

$$\frac{|\gamma\tau-\tau'|}{2\,\mathrm{Im}(\tau)\,\mathrm{Im}(\tau')}=(b+c)^2+(a-d)^2,$$

and so (10) holds with $n = (b+c)^2 + (a-d)^2$ (note $|D| = 4$). Thus $\rho(n)$ counts the number of solutiosn to $n = (a-d)^2 + (b+c)^2$ with $ad - bc = 1$. The second equation, in the presence of the first, is equivalent to $(a+b) + (b-c)^2 = n+4$, and so $\rho(n)$ roughly counts the number of ways n and $n+4$ can be written as a sum of two squares. (Roughly because there are some mod 2 congruences involved.)

The answer in general is similar: $\rho(n)$ counts the number of pairs of elements in certain fractional ideals of K with given norms. By examining the particular formula and summing it over ideal classes, one arrives at the following formula:

$$\gamma_s(\mathcal{A}) = -2u^2 \sum_{n=1}^{\infty} \delta(n) R_{\{\mathcal{A}\mathfrak{n}\}}(n) r_{\mathcal{A}}(nN + |D|) Q_{s-1}\left(1 + \frac{2nN}{|D|}\right). \tag{11}$$

Here u is the half number of units in K, $\delta(n)$ is 2^t, where t is the number of distinct primes dividing (n, D), $R_{\{\mathcal{A}\mathfrak{n}\}}(n)$ is the number of integral ideals of K of norm n and genus $\{\mathcal{A}\mathfrak{n}\}$, and $r_{\mathcal{A}}(n)$ is the number of integral ideals of K of norm n and class $\mathcal{A}$.

3.4. **The remaining terms.** We now look at the remaining terms in (8). We begin with the following lemma.

Lemma 12. *We have*

$$\sum_{\mathcal{B}\in \mathrm{Cl}_K} E(\tau_{\mathcal{B}}, s) = N^{-s} \prod_{p|N} (1 + p^{-s})^{-1} \cdot 2^{-s} |D|^{s/2} u \zeta(2s)^{-1} \zeta_K(s),$$

and similarly if we replace $\tau_{\mathcal{B}}$ by $w\tau_{\mathcal{B}}$ on the left side.

Proof. Let

$$E'(z, s) = \sum_{\gamma \in \Gamma(1)} \mathrm{Im}(\gamma z)^s$$

be the Eisenstein series for $\Gamma(1)$. Then

$$E(z, s) = N^{-s} \prod_{p|N} (1 - p^{-2s})^{-1} \cdot \sum_{d|N} \frac{\mu(d)}{d^s} E'(Nz/d, s).$$

We thus have

$$\sum_{\mathcal{B}\in \mathrm{Cl}_K} E(\tau_{\mathcal{B}}, s) = N^{-s} \prod_{p|N} (1 - p^{-2s})^{-1} \cdot \sum_{d|N} \frac{\mu(d)}{d^s} \sum_{\mathcal{B}\in \mathrm{Cl}_K} E'(N\tau_B/d, s).$$

It is not difficult to show that the inner sum on the right side is independent of d. As

$$\sum_{d|N} \frac{\mu(d)}{d^s} = \prod_{p|N} (1 - p^{-s}),$$

we find

$$\sum_{\mathcal{B}\in \mathrm{Cl}_K} E(\tau_{\mathcal{B}}, s) = N^{-s} \prod_{p|N} (1 + p^{-s})^{-1} \cdot \sum_{\mathcal{B}\in \mathrm{Cl}_K} E'(\tau_B, s).$$

Let

$$\zeta_K(\mathcal{A}, s) = \sum_{[\mathfrak{a}]=\mathcal{A}} \frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

be the partial Dedekind zeta function, where the sum is over integral ideals $\mathfrak{a}$ of K of class $\mathcal{A}$. Then we have the straightforward formula

$$E'(\tau_{\mathcal{B}}, s) = 2^{-s}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(\mathcal{B}, s),$$

and so

$$\sum_{\mathcal{B}\in \mathrm{Cl}_K} E(\tau_{\mathcal{B}}, s) = N^{-s}\prod_{p|N}(1+p^{-s})^{-1}\cdot 2^{-s}|D|^{s/2}u\zeta(2s)^{-1}\zeta_K(s).$$

It's not hard to show the same if we replace $\tau_{\mathcal{B}}$ with $w\tau_{\mathcal{B}}$. ◻

Applying the lemma to (8) yields

$$\lim_{s\to 1}\left[\gamma_s(\mathcal{A}) + \frac{8\pi u(2N)^{-s}|D|^{s/2}}{\prod_{p|N}(1+p^{-s})}\frac{\zeta_K(s)}{\zeta(2s)} + \frac{h\kappa}{s-1}\right] + hC, \tag{13}$$

We have $\zeta_K(s) = \zeta(s)L(s,\epsilon)$. Write

$$\frac{8\pi u(2N)^{-s}|D|^{s/2}}{\prod_{p|N}(1+p^{-s})}\frac{L(s,\epsilon)}{\zeta(2s)} = c_1 + c_2(s-1) + O((s-1)^2),$$

where

$$c_1 = \left[\frac{8\pi u(2N)^{-s}|D|^{s/2}}{\prod_{p|N}(1+p^{-s})}\frac{L(s,\epsilon)}{\zeta(2s)}\right]_{s=1}$$

and

$$c_2 = \frac{d}{ds}\left[\frac{8\pi u(2N)^{-s}|D|^{s/2}}{\prod_{p|N}(1+p^{-s})}\frac{L(s,\epsilon)}{\zeta(2s)}\right]_{s=1}.$$

We have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1),$$

and so

$$\frac{8\pi u(2N)^{-s}|D|^{s/2}}{\prod_{p|N}(1+p^{-s})}\frac{\zeta_K(s)}{\zeta(2s)} = \frac{c_1}{s-1} + c_1\gamma + c_2 + O(s-1).$$

From the class number formula $L(1,\epsilon) = \pi h/u\sqrt{|D|}$, we find $c_1 = -2\kappa h$. Thus (13) yields

$$\lim_{s\to 1}\left[\gamma_s(\mathcal{A}) - \frac{h\kappa}{s-1}\right] + c_1\gamma + c_2 + hC, \tag{14}$$

We have

$$c_2 = c_1\left[-\log(2N) + \tfrac{1}{2}\log|D| + \sum_{p|N}\frac{\log p}{p+1} + \frac{L'(1,\epsilon)}{L(1,\epsilon)} - 2\frac{\zeta'(2)}{\zeta(2)}\right],$$

and so, using (6),

$$c_1\gamma + c_2 + hC = \kappa h\left[\log(N/|D|) + 2\sum_{p|N}\frac{\log p}{p^2-1} + 2\frac{\zeta'(2)}{\zeta(2)} - 2\frac{L'(1,\epsilon)}{L(1,\epsilon)}\right],$$

Applying this to (14), we reach our end result:

Proposition 15. *Let x be a Heegner point, and let $\sigma \in \mathrm{Gal}(H/K)$ correspond to $\mathcal{A} \in \mathrm{Cl}_K$. Then*

$$\langle (x) - (\infty), (x^\sigma) - (0) \rangle_\infty = \lim_{s \to 1} \left[\gamma_s(\mathcal{A}) - \frac{h\kappa}{s-1} \right] + \kappa h \left[\log(N/|D|) + 2 \sum_{p \mid N} \frac{\log p}{p^2 - 1} + 2 \frac{\zeta'(2)}{\zeta(2)} - 2 \frac{L'(1, \epsilon)}{L(1, \epsilon)} \right],$$

where $\gamma_s(\mathcal{A})$ is given by (11).

3.5. **Dealing with Hecke operators.** The above work has computed $\langle c, d^\sigma \rangle_\infty$. However, we really need to compute $\langle c, T_m d^\sigma \rangle_\infty$ for all Hecke operators T_m. When the divisors c and $T_m d^\sigma$ have disjoint support, this is not significantly more difficult than what we have done above. When the supports overlap, there are additional complications, the first being that the height pairing is not even defined. Nonetheless, one can still carry out a meaningful calculation. However, Nekovar found a trick that shows, for the ultimate applications to L-values, one does not need to know anything about the case where the supports overlap. Thus what we have done is representative of the general case.