

GL-EQUIVARIANT MODULES OVER POLYNOMIAL RINGS IN INFINITELY MANY VARIABLES

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ABSTRACT. Consider the polynomial ring in countably infinitely many variables over a field of characteristic zero, together with its natural action of the infinite general linear group G . We study the algebraic and homological properties of finitely generated modules over this ring that are equipped with a compatible G -action. We define and prove finiteness properties for analogues of Hilbert series, systems of parameters, depth, local cohomology, Koszul duality, and regularity. We also show that this category is built out of a simpler, more combinatorial, quiver category which we describe explicitly.

Our work is motivated by recent papers in the literature which study finiteness properties of infinite polynomial rings equipped with group actions. (For example, the paper by Church, Ellenberg and Farb on the category of FI-modules, which is equivalent to our category.) Along the way, we see several connections with the character polynomials from the representation theory of the symmetric groups. Several examples are given to illustrate that the invariants we introduce are explicit and computable.

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INTRODUCTION

This paper concerns the algebraic and homological properties of modules over the twisted commutative algebra (tca) $A = \text{Sym}(\mathbf{C}\langle 1 \rangle)$. There are several ways to view the algebra A , but the main point of view we take in this paper is that A is the symmetric algebra on the infinite dimensional vector space $\mathbf{C}^\infty = \bigcup_{n \geq 0} \mathbf{C}^n$ equipped with the action of the general linear group $\mathbf{GL}_\infty = \bigcup_{n \geq 0} \mathbf{GL}_n$. Modules over this algebra are required to have a compatible action of \mathbf{GL}_∞ , and concepts such as “finite generation” are defined relative to this structure.

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Our study of A -modules is motivated by recent results in the literature. In [EFW] and [SW], free resolutions over A are studied. Although the terminology “twisted commutative algebra” is not mentioned there, the idea of using Schur functors without evaluating on a vector space is implicit. In [Sno], A -modules and modules over more general twisted commutative algebras are used to help establish properties of Δ -modules, which are in turn used to study syzygies of Segre embeddings. In [CEF], A -modules are studied under the name “FI-modules,” and many examples are given. Some other papers of a similar flavor are [Dra], [DK], [HM], [HS]. With this paper, we hope to initiate a systematic study of twisted commutative algebras from the point of view of commutative algebra.

0.1. Statement of results. The first difficulty one encounters when studying the category Mod_A is that it has infinite global dimension: indeed, the Koszul resolution of the residue field \mathbf{C} is unbounded, since no wedge power of \mathbf{C}^∞ vanishes. (In fact, every non-projective object of Mod_A has infinite projective dimension.) As a consequence, the Grothendieck group of Mod_A is not spanned by projective objects. In the first part of this paper, we study the structure of A -modules and establish results that allow one to deal with these difficulties. We mention a few specific results here:

1. Projective A -modules are also injective (Corollary 4.2.6).
2. Every finitely generated A -module has finite injective dimension, and, in fact, admits a finite length resolution by finitely generated injective A -modules (Theorem 4.3.1).
3. Every object M of the bounded derived category of finitely generated A -modules fits into an exact triangle of the form

$$M_t \rightarrow M \rightarrow M_f \rightarrow$$

where M_t is a finite length complex of finitely generated torsion modules and M_f is a finite length complex of finitely generated projective modules (Proposition 4.4.2).

4. The Grothendieck group of Mod_A is spanned by the classes of projective and simple modules (Proposition 4.8.1).

Our approach to studying the category Mod_A is to break it up into two pieces: the category $\text{Mod}_A^{\text{tors}}$ of torsion A -modules and the Serre quotient $\text{Mod}_K = \text{Mod}_A / \text{Mod}_A^{\text{tors}}$. The category Mod_K can be thought of as modules over the “generic point” of $\text{Proj}(A)$. In §2, we give basic structure results on Mod_K : we compute the simple and injective objects and explicitly describe the injective resolutions of simple objects (Theorem 2.3.1). We also show, somewhat surprisingly, that Mod_K is equivalent to $\text{Mod}_A^{\text{tors}}$. In §3, we refine these results and describe Mod_K and $\text{Mod}_A^{\text{tors}}$ as the category of representations of a certain quiver. This quiver has wild representation type (Remark 3.2.3), which means one cannot expect very fine results for the structure of A -modules.

In §4, we apply the results from §2 and §3 to study Mod_A . We prove the results mentioned above, as well as a few others: for instance, we show that the auto-equivalence group of Mod_A is trivial and give a complete description of $D^b(A)$ involving only the simpler category $D^b(K)$. We also introduce local cohomology and the section functor. These are adjoints to the natural functors $\text{Mod}_A^{\text{tors}} \rightarrow \text{Mod}_A$ and $\text{Mod}_A \rightarrow \text{Mod}_K$, and are important tools in establishing the results of this section.

The second part of the paper studies invariants of A -modules. Using the results from the first part, we obtain easy proofs of the following results:

5. An analogue of the Hilbert series is “rational” (Theorem 5.1.2).
6. The existence of systems of parameters is substituted by the statement that modules are annihilated by “differential operators” (Theorem 5.4.1).
7. The linear strands of a minimal projective resolution of a finitely generated A -module are finitely generated as comodules over the exterior algebra $\bigwedge \mathbf{C}^\infty$ (Theorem 6.1.2).

8. Regularity is finite, i.e., only finitely many linear strands are non-zero (Corollary 6.1.3). Together with the previous result, this gives a precise sense in which projective resolutions contain only a finite amount of data.
9. There is a well-defined notion of depth which specializes to the usual notion (Theorem 7.1.1).
10. We establish analogues of well-known relationships between local cohomology and depth and Hilbert series.

We note that some of the results in this paper, such as results 1, 2, 3, 5 mentioned above, remain interesting if we replace \mathbf{C}^∞ by a finite-dimensional \mathbf{C}^n , while others only contain content in the infinite setting (generally due to the fact that the polynomial ring in finitely many variables has finite global dimension).

0.2. Duality. Koszul duality gives a contravariant equivalence between the bounded above derived category of A -modules and that of $\bigwedge \mathbf{C}^\infty$ -comodules (equipped with a compatible group action). Results 7 and 8 above show that Koszul duality induces an equivalence of the bounded derived category of finitely generated A -modules with the bounded derived category of finitely generated $\bigwedge \mathbf{C}^\infty$ -comodules. However, the abelian category of $\bigwedge \mathbf{C}^\infty$ -comodules is equivalent to that of A -modules, using the transpose operation on partitions — this is a peculiarity of our situation due to the \mathbf{GL} -actions. Combining the two equivalences, we obtain an autoequivalence

$$\mathcal{F}: D^b(A)^{\text{op}} \rightarrow D^b(A),$$

which we call the “Fourier transform.”

A general theme of the paper is how Mod_A breaks into the two pieces $\text{Mod}_A^{\text{tors}}$ and Mod_K . This is most explicitly seen in result 3 above, but manifests in several other places, e.g., in the form of the Hilbert series and the formulas for pairings in K-theory. The Fourier transform provides a duality between these two pieces. For instance, using the notation of result 3, we have $(\mathcal{F}M)_t = \mathcal{F}(M_f)$ (Remark 6.3.2). This explains many of the symmetries seen between the two pieces throughout the paper.

0.3. Applications. The explicit resolutions we construct in Mod_K allow us to give a conceptual derivation of a formula for character polynomials (see §5.2), while our theory of local cohomology provides invariants which detect the discrepancy between the character polynomial and the actual character in low degrees. In particular, see Remark 7.4.6 which applies explicit local cohomology calculations to improve some bounds given in [CEF].

Our result on enhanced Hilbert series, and its generalization to multivariate tca’s, allows one to define and prove rationality of an enhanced Hilbert series for Δ -modules. Similarly, our result on Poincaré series, and its generalization to multivariate tca’s, allows one to prove a rationality result for Poincaré series of Δ -modules. This affirmatively answers Questions 4 and 7 from [Sno].

0.4. Analogy with $\mathbf{C}[t]$. As the notation $\text{Sym}(\mathbf{C}\langle 1 \rangle)$ is meant to suggest, we think of A as being analogous to the graded polynomial ring $\mathbf{C}[t]$. We think of A -modules as analogous to *nonnegatively* graded $\mathbf{C}[t]$ -modules. This analogy is not perfect, but is surprisingly good, and serves as something of a guiding principle: many of the results and definitions in this paper have simpler analogues in the setting of $\mathbf{C}[t]$ -modules. For example, result 1 above may seem unexpected at first, but is analogous to the fact that in the category of nonnegatively graded $\mathbf{C}[t]$ -modules, $\mathbf{C}[t]$ is injective. We point out many other instances of this analogy along the way, and encourage the reader to find more still.

Just as $\mathbf{C}[t]$ can be generalized to multivariate polynomial rings, so too can $\text{Sym}(\mathbf{C}\langle 1 \rangle)$ be generalized to multivariate tca’s: these are rings of the form $\text{Sym}(U \otimes \mathbf{C}^\infty)$, where U is a finite dimensional vector space, equipped with the obvious \mathbf{GL}_∞ -action. (Actually, these are just the polynomial tca’s generated in degree 1.) We have not yet succeeded in generalizing the results from the first part of this paper to the multivariate setting. Nonetheless, we have proved analogues

of results 5–9 listed above in this setting. The proofs of these results in the general case are significantly different (and longer), and will be treated in [SS2] and [SS3].

0.5. Roadmap. We hope the results in this paper will appeal to those interested in abelian categories, commutative algebra, and/or combinatorial representation theory. However, as some of the content in this paper may be less appealing to certain people, we provide a brief roadmap to try to indicate what might be interesting to whom.

We begin each section with a brief overview of the results that it contains. We advise that any reader of this paper look through these overviews to get a first approximation of the results contained in this paper.

The first part of the paper is largely abstract and categorical in nature. For those interested in these aspects, we highlight Theorem 3.1.4, which gives an elegant description of a natural class of abelian categories, its application to Mod_K in §3.2 and the description of the category $D^b(A)$ given in Theorem 4.6.4. For the reader mainly interested in the second part of the paper, the most important results from the first part are contained in §2.2 and §2.3; see also Proposition 4.8.2.

The second part of the paper contains the content which is more likely to be of interest to the commutative algebraist or combinatorial representation theorist. In particular, the connection with character polynomials is contained in §5.2 and §7.4. We also wish to highlight §3.2 which shows that a certain simplicial complex related to Pieri’s rule is contractible. As for the extension of the basic invariants of commutative algebra, we refer the reader to §5.1 for Hilbert series and §6 for basic properties of Koszul duality and finiteness properties for Tor. The results in §7 give analogues of the notions of depth and local cohomology, and should be of interest to both commutative algebraists and combinatorialists.

For explicit calculations, see Remark 1.0.2 for some information, as well as §6.5 and §7.4.

0.6. Notation. Throughout, \mathbf{C} denotes the complex numbers, though all results work equally well over an arbitrary field of characteristic 0. We use the symbol A for the twisted commutative algebra (tca) $\text{Sym}(\mathbf{C}\langle 1 \rangle)$. Other notation is defined in the body of the paper.

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1. BACKGROUND

We refer to [SS1] for a more thorough treatment of the material in this section.

Given a partition λ of size n (denoted $|\lambda| = n$), let \mathbf{S}_λ be the Schur functor indexed by λ and let \mathbf{M}_λ be the corresponding irreducible representation of S_n . We index them so that if $\lambda = (n)$ has one part, then $\mathbf{S}_{(n)}$ is the n th symmetric power functor, and $\mathbf{M}_{(n)}$ is the trivial representation of S_n .

Given an inclusion of partitions $\lambda \subseteq \mu$ (i.e., $\lambda_i \leq \mu_i$ for all i), we say that μ/λ is a **horizontal strip** of size $|\mu| - |\lambda|$ if $\mu_i \geq \lambda_i \geq \mu_{i-1}$ for all i . For notation, we write $\mu/\lambda \in \text{HS}_d$ (implicit in this notation is that $\lambda \subseteq \mu$) and we write $\mu/\lambda \in \text{HS}$ if there is some d for which $\mu/\lambda \in \text{HS}_d$. We recall Pieri’s formula, which states that

$$\mathbf{S}_\lambda \otimes \mathbf{S}_{(d)} = \bigoplus_{\mu/\lambda \in \text{HS}_d} \mathbf{S}_\mu.$$

We define the **transpose partition** λ^\dagger to be $\lambda_i^\dagger = \#\{j \geq i \mid \lambda_j \geq i\}$. If $\lambda \subseteq \mu$, we say that $\mu/\lambda \in \text{VS}_d$ if and only if $\mu^\dagger/\lambda^\dagger \in \text{HS}_d$, and say that μ/λ is a **vertical strip**. The notation VS is

defined similarly. The dual version of Pieri’s formula states that

$$\mathbf{S}_\lambda \otimes \bigwedge^d = \bigoplus_{\mu/\lambda \in \text{VS}_d} \mathbf{S}_\mu.$$

We will refer to the category \mathcal{V} , which has three realizations that we are interested in. Objects of \mathcal{V} may be described by sequences $(W_n)_{n \geq 0}$ where W_n is a representation of S_n . They may also be described as polynomial representations $V = \bigoplus_{n \geq 0} V_n$ of \mathbf{GL}_∞ . Alternatively, we can think of each V_n as a degree n polynomial endofunctor on the category of vector spaces. We point out that we allow infinite direct sums such as $\text{Sym}^2 \oplus \text{Sym}^2 \oplus \dots$. The relationship comes from Schur–Weyl duality:

$$V_n = \text{Hom}_{S_n}(W_n, (\mathbf{C}^\infty)^{\otimes n}), \quad W_n = \text{Hom}_{\mathbf{GL}_\infty}(V_n, (\mathbf{C}^\infty)^{\otimes n}).$$

In particular, if $W_n = \mathbf{M}_\lambda$ is irreducible, then $V_n = \mathbf{S}_\lambda$. We use \mathcal{V}_f (resp. \mathcal{V}_{gf}) to denote the full subcategory of \mathcal{V} on finite length (resp. graded-finite) objects. (“Graded-finite” means each graded piece is finite length.)

We let $(-)^{\vee}$ be the duality functor on \mathcal{V} ; it takes a sequence (W_n) as above to the sequence (W_n^*) , where W_n^* is the dual vector space. Note that in terms of representations of \mathbf{GL}_∞ , this is *not* taking the dual representation; instead, one is taking duals of the multiplicity spaces of the Schur functors.

The ring $A = \bigoplus_{d \geq 0} \text{Sym}^d$ is an algebra object in the category \mathcal{V} . An A -module is an object M of \mathcal{V} with an appropriate multiplication map $A \otimes M \rightarrow M$. A module M is finitely generated if there exists a surjection $A \otimes V \rightarrow M \rightarrow 0$ where V is a finite length object of \mathcal{V} . We denote by Mod_A the category of finitely generated A -modules. A basic result is that A is noetherian [Sno, Theorem 2.3], so Mod_A is an abelian category.

For any object $M \in \mathcal{V}$, we can decompose M as a direct sum of \mathbf{S}_λ , and we use $\ell(M)$ to denote the maximum number of rows of any λ that appears. Given $M \in \text{Mod}_A$, let $V_1 \otimes A \rightarrow V_0 \otimes A \rightarrow M \rightarrow 0$ be a minimal presentation. Then each V_i can be written as a sum of Schur functors \mathbf{S}_λ . We let $\ell_A(M)$ denote the maximum number of rows of any partition that appears in either V_0 or V_1 .

If V is a finite length object of \mathcal{V} then $A \otimes V$ is a projective object of Mod_A . An easy argument with Nakayama’s lemma shows that all projective objects are of this form. In particular, the indecomposable projectives of Mod_A are exactly the modules of the form $A \otimes \mathbf{S}_\lambda$. We have the following important result on these modules:

Proposition 1.0.1. *If $\mathbf{S}_\mu \subset \mathbf{S}_\lambda \otimes A$, then the A -submodule generated by \mathbf{S}_μ contains all $\mathbf{S}_\nu \subset \mathbf{S}_\lambda \otimes A$ such that $\mu \subseteq \nu$.*

Proof. This can be found in [Olv, §8] or [SW, Lemma 2.1]. □

Remark 1.0.2. We wish to emphasize the fact that the maps $\mathbf{S}_\mu \rightarrow \mathbf{S}_\lambda \otimes A$ given by Pieri’s rule can be made concrete. A computer implementation of these maps, based on [Olv], has been written by the first author as a Macaulay2 package [Sam]. Since all relevant calculations in this paper can be done by replacing \mathbf{C}^∞ with \mathbf{C}^n for $n \gg 0$, this means that they can be done on a computer. □

Part 1. Structure of A -modules

2. THE STRUCTURE OF Mod_K AND $\text{Mod}_A^{\text{tors}}$

The main purpose of this section is to define and study a localization functor

$$T: \text{Mod}_A \rightarrow \text{Mod}_K.$$

The definition of Mod_K is given in §2.1. The category Mod_K is analyzed and described explicitly. In particular, we classify the simple objects and the injective objects in §2.2 and construct the minimal injective resolutions of every simple object in §2.3. This implies the transition matrices

in K-theory between simple and injective objects, and it is shown that the multiplicative structure constants in K-theory are the same in both bases. Finally, in §2.5, we show that Mod_K is equivalent to the category $\text{Mod}_A^{\text{tors}}$ of torsion objects in Mod_A .

2.1. The category Mod_K . Let Mod_K be the Serre quotient of Mod_A by the Serre subcategory $\text{Mod}_A^{\text{tors}}$ of finite length objects. Recall that the objects of Mod_K are the objects of Mod_A , and that

$$\text{Hom}_{\text{Mod}_K}(M, N) = \varinjlim \text{Hom}_{\text{Mod}_A}(M', N/N')$$

where the colimit is over all submodules $M' \subseteq M$ and $N' \subseteq N$ such that M/M' and N' have finite length. Hence, if M and N are two A -modules then a map $M \rightarrow N$ in Mod_K comes from a map $M' \rightarrow N/N'$ in Mod_A , where M' and N' are submodules of M and N such that M/M' and N' have finite length. Two objects of Mod_A become isomorphic in Mod_K if and only if, loosely speaking, they differ by finite length objects. When working with an isomorphism class in Mod_K , we will always choose whichever representative is most convenient.

We can also think of Mod_K as equivariant coherent sheaves on the generic point of $\text{Proj}(A)$. This point of view is made precise as follows. Let K be the localization of $\text{Proj}(A)$ at the generic point; if $\{x_i\}$ is a basis for \mathbf{C}^∞ then K is the field $\mathbf{C}(x_i/x_j)$. A **semi-linear representation** of \mathbf{GL}_∞ over K is a K -vector space V equipped with a \mathbf{C} -linear action of \mathbf{GL}_∞ which satisfies $g(\alpha v) = (g\alpha)(gv)$ for $\alpha \in K$ and $v \in V$. We say that such a representation is “polynomial” if V contains a polynomial representation of \mathbf{GL}_∞ which spans V over K . We can then describe Mod_K as the category of finite dimensional polynomial semi-linear representations of \mathbf{GL}_∞ over K . We do not actually use this point of view in what follows, and always work with Mod_K as the Serre quotient of Mod_A ; however, the reader should keep in mind that all our results on Mod_K can be interpreted as results on semi-linear representations.

2.2. Simple and injective objects. Let λ be a partition and let $D \geq \lambda_1$ be an integer. The module $\mathbf{S}_\lambda \otimes A$ then has a unique subspace $L_\lambda^{\geq D} = \bigoplus_{d \geq D} \mathbf{S}_{(d, \lambda)}$, which is easily seen to be an A -submodule. We will also use L_λ^0 to denote $L_\lambda^{\geq \lambda_1}$.

Proposition 2.2.1. *In Mod_A there is a filtration*

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_{\lambda_1} = \mathbf{S}_\lambda \otimes A$$

such that for all $i \geq 0$,

$$F_i/F_{i-1} = \bigoplus_{\mu} L_\mu^{\geq \lambda_1}$$

where the sum is over all μ which can be obtained from λ by removing a horizontal strip of size i .

Proof. First, the space $\mathbf{S}_\lambda \otimes A$ is multiplicity-free as a representation of \mathbf{GL}_∞ by Pieri’s rule. Let F_i be the span of all \mathbf{S}_μ such that $|\lambda| - |\mu| + \mu_1 \leq i$. Again from Pieri’s rule, one can see that this is closed under multiplication and that it exhausts all summands of $\mathbf{S}_\lambda \otimes A$. \square

Proposition 2.2.2. *We have $L_\lambda^{\geq D} \cong L_\lambda^{\geq D'}$ in Mod_K . This object is simple, which we call L_λ . Furthermore, every simple object of Mod_K is isomorphic to L_λ for some λ .*

Proof. If $D \leq D'$ then $L_\lambda^{\geq D}$ contains $L_\lambda^{\geq D'}$ and the quotient is a finite length object of \mathcal{V} . Thus the two modules are isomorphic in Mod_K . Furthermore, any A -submodule of $L_\lambda^{\geq D}$ is of the form $L_\lambda^{\geq D'}$ for some $D' \geq D$. Hence the image of $L_\lambda^{\geq D}$ has no nonzero proper submodules as an object of Mod_K , so is a simple object.

For the last statement, let $L \in \text{Mod}_K$ be simple. Let \tilde{L} be a lift of L to an object in Mod_A . We may assume that \tilde{L} is a quotient of $\mathbf{S}_\lambda \otimes A$ for some partition λ . It is clear from above that any simple quotient of $\mathbf{S}_\lambda \otimes A$ must be isomorphic to L_λ after applying the localization functor T . \square

Corollary 2.2.3. *Every object of Mod_K has finite length.*

Proof. The above two results show that the image of $\mathbf{S}_\lambda \otimes A$ in Mod_K has finite length. As every object of Mod_K is a quotient of a finite direct sum of such objects, the result follows. (Note: this result is obvious if we think of Mod_K as finite dimensional semi-linear representations.) \square

Proposition 2.2.4. *The image of $\mathbf{S}_\lambda \otimes A$ in Mod_K is an injective object, which we call Q_λ .*

Proof. We will show that all injections $\mathbf{S}_\lambda \otimes A \rightarrow M$ in Mod_K split. Replacing M if necessary, we may represent such a map in Mod_A by an injection $\mathcal{S} \rightarrow M$ in Mod_A , where $\mathcal{S} \subseteq \mathbf{S}_\lambda \otimes A$ is an A -submodule such that $(\mathbf{S}_\lambda \otimes A)/\mathcal{S}$ has finite length. Furthermore, we may choose M so that M is minimally generated by a finite number of $L_\mu^{\geq d_\mu}$. Then \mathcal{S} has a filtration whose associated graded pieces are of the form $L_\mu^{\geq c_\mu}$ for various μ . Since we only need to construct the splitting in Mod_K , we may assume, by replacing \mathcal{S} and M , that if L_μ appears in both M and \mathcal{S} , then we have $c_\mu = d_\mu$ (in particular, if M contains various L_μ , then all of them can be made into the form $L_\mu^{\geq d_\mu}$ for the same value of d_μ). For notation, use \mathcal{L}_μ to denote the generators of M .

We claim that if λ/μ is not a horizontal strip, then the submodule of M generated by this copy of \mathcal{L}_μ does not intersect \mathcal{S} . To see this, first note that the submodule $L_\lambda \cap \mathcal{S} \subset \mathcal{S}$ is not generated by this copy of \mathcal{L}_μ by Pieri's rule. But every $L_\nu^{\geq \lambda_1} \cap \mathcal{S}$ in \mathcal{S} generates $L_\lambda \cap \mathcal{S}$, so $L_\nu^{\geq \lambda_1} \cap \mathcal{S}$ is not in the submodule generated by \mathcal{L}_μ , and hence the claim is proven.

Let M' be the submodule of M generated by all minimal generators \mathcal{L}_μ such that the submodule generated by \mathcal{L}_μ does not intersect \mathcal{S} . Then the composition $\mathcal{S} \rightarrow M \rightarrow M/M'$ is injective, and it is enough to construct a splitting $M/M' \rightarrow \mathcal{S}$. So we replace M with M/M' . The above paragraph shows that the new M is minimally generated by \mathcal{L}_μ where λ/μ is a horizontal strip.

For each ν such that λ/ν is a horizontal strip, fix a highest weight vector \mathbf{v}_ν in $\mathbf{S}_{(c_\nu, \nu)} \subset \mathcal{S}$ (since this space is multiplicity-free, this is unique up to scalar multiple). For each minimal generator \mathcal{L}_μ of M , also fix a highest weight vector \mathbf{w}_μ for its generator $\mathbf{S}_{(d_\mu, \mu)}$. By our reduction above, there is at most one \mathbf{w}_λ , and we fix it so that $\mathbf{w}_\lambda = \mathbf{v}_\lambda$. To define the splitting, we need to send \mathbf{w}_μ to some scalar multiple of \mathbf{v}_μ . To find this scalar, consider the intersection of the A -submodule generated by \mathbf{w}_μ with \mathcal{S} . We have relations of the form $\mathbf{v}_\nu = p_{\nu, \mu} \mathbf{w}_\mu$ where $p_{\nu, \mu}$ lives in the smash product of A with \mathbf{GL}_∞ , and we can choose them so that $p_{\lambda, \nu} p_{\nu, \mu} = p_{\lambda, \mu}$. Then $p_{\lambda, \mu} \mathbf{w}_\mu = \alpha_{\lambda, \mu} \mathbf{v}_\lambda$ where $\alpha_{\lambda, \mu} \in \mathbf{C} \setminus 0$, and this is the desired scalar.

Now suppose that $\mathcal{L}_\nu \subset M$ is generated by some \mathcal{L}_μ . We deduce that

$$\alpha_{\lambda, \nu}^{-1} p_{\lambda, \nu} \mathbf{w}_\nu = \alpha_{\lambda, \mu}^{-1} p_{\lambda, \mu} \mathbf{w}_\mu = \alpha_{\lambda, \mu}^{-1} p_{\lambda, \nu} p_{\nu, \mu} \mathbf{w}_\mu.$$

By the uniqueness of highest weight vectors in $A \otimes \mathbf{S}_{(d_\mu, \mu)}$, this implies that $\mathbf{w}_\nu = \alpha_{\lambda, \nu} \alpha_{\lambda, \mu}^{-1} p_{\nu, \mu} \mathbf{w}_\mu$. So the map $\mathbf{w}_\mu \mapsto \alpha_{\lambda, \mu} \mathbf{v}_\mu$ is A -linear, and this defines the desired splitting. \square

Proposition 2.2.5. (a) *Given partitions λ, μ , we have*

$$\text{Hom}_K(Q_\lambda, Q_\mu) = \begin{cases} \mathbf{C} & \text{if } \lambda/\mu \in \text{HS} \\ 0 & \text{otherwise} \end{cases}$$

(b) *The kernel of any nonzero map $Q_\lambda \rightarrow Q_\mu$ is spanned by all L_ν in Q_λ that do not appear in Q_μ , and the cokernel is spanned by all L_η in Q_μ that do not appear in Q_λ .*

Proof. Let $\nu = (\lambda_2, \lambda_3, \dots)$. Then L_ν is the last piece in the filtration of Q_λ in Proposition 2.2.1 and for any lift Q'_λ of Q_λ back to Mod_A , the subspace which is the preimage of L_ν generates Q'_λ as an A -module. Also, $[L_\nu]$ appears in $[Q_\mu]$ if and only if $\lambda/\mu \in \text{HS}$, which shows that $\text{Hom}_K(Q_\lambda, Q_\mu) = 0$ if $\lambda/\mu \notin \text{HS}$. Otherwise, we can choose a map $\mathbf{S}_\lambda \otimes A \rightarrow \mathbf{S}_\mu \otimes A$ in Mod_A , which is unique up to scalar, and this localizes to a map $Q_\lambda \rightarrow Q_\mu$. If $\mathbf{S}_\lambda \otimes A \rightarrow \mathbf{S}_\mu \otimes A$ is nonzero, then its restriction to any A -submodule generated by some $L_\nu^{\geq D}$ is also nonzero (Proposition 1.0.1). Since $\mathbf{S}_\mu \otimes A$ is multiplicity-free, and since any map $A \cdot L_\nu^{\geq D} \rightarrow \mathbf{S}_\mu \otimes A$ is determined up to scalar multiple by the

image of the highest weight vector of $\mathbf{S}_{(D,\nu)}$, it must be the restriction of a map $\mathbf{S}_\lambda \otimes A \rightarrow \mathbf{S}_\mu \otimes A$. Since $\mathbf{S}_\mu \otimes A$ has no finite length A -submodules, we have just shown that

$$\mathrm{Hom}_K(Q_\lambda, Q_\mu) = \varinjlim_{D \geq \lambda_1} \mathrm{Hom}_A(L_\nu^{\geq D}, Q_\mu) = \mathbf{C}.$$

To prove (b), we use Proposition 1.0.1: any nonzero map $Q_\lambda \rightarrow Q_\mu$ can be represented by an A -linear map $\mathbf{S}_\lambda \otimes A \rightarrow \mathbf{S}_\mu \otimes A$, and the kernel is the sum of all Schur functors which appear in $\mathbf{S}_\lambda \otimes A$, but which do not appear in $\mathbf{S}_\mu \otimes A$. \square

Corollary 2.2.6. *The object Q_λ is indecomposable and is the injective envelope of L_λ .*

Proof. This follows from $\mathrm{End}(Q_\lambda) = \mathbf{C}$. \square

Corollary 2.2.7. *Every indecomposable injective object of Mod_K is isomorphic to Q_λ for some λ . In particular, every injective object of Mod_K is a direct sum of the Q_λ .*

Proof. We just need to establish the first sentence. Let Q be an indecomposable injective object. Let $\mathrm{soc}(Q)$ be its socle, i.e., the sum of all minimal submodules of Q . Then $\mathrm{soc}(Q) = L_1 \oplus \cdots \oplus L_r$ is a direct sum of simple modules. In particular, the direct sum of the injective envelopes of the L_i is a submodule of Q , and the quotient of Q by this submodule is also injective, and hence we get a direct sum decomposition. This means that $r = 1$ and Q is the injective envelope of L_1 . Now the result follows from Proposition 2.2.2 and Corollary 2.2.6. \square

Remark 2.2.8. The results in this section imply that Mod_K is a highest weight category [CPS, Definition 3.1] over the poset of partitions under the relation of containment \subseteq . \square

2.3. Injective resolutions and consequences. We now determine the injective resolutions of simple objects.

Theorem 2.3.1. *Let λ be a partition. Define $\mathbf{I}^j = \bigoplus_{\mu, \lambda/\mu \in \mathrm{VS}_j} Q_\mu$. There are linear maps $\mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$ so that $L_\lambda \rightarrow \mathbf{I}^\bullet$ is a minimal injective resolution. In particular, the injective dimension of L_λ is $\ell(\lambda)$.*

Proof. By Proposition 2.2.5(a), the maps $\mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$ are determined by a matrix of scalars. Given $Q_\mu \subset \mathbf{I}^j$ and $Q_\nu \subset \mathbf{I}^{j+1}$, there are most two injective summands of \mathbf{I}^{j+1} which can receive nonzero maps from Q_μ and which can map to Q_ν nontrivially. If there is just 1, then $\nu/\mu \in \mathrm{HS}_2$, so the composition $Q_\mu \rightarrow Q_\nu$ is 0 regardless of choices of scalars. Otherwise, if there are two such summands, we call this quadruple a square. By reverse induction on j , it is clear that we can choose the scalars so that the image of Q_μ in Q_ν is the same along both paths of the square. So to construct the complex, we must show that there is an assignment of signs so that the images of Q_μ in Q_ν are negatives of one another in any square.

This turns into the following combinatorial problem. Consider the set of partitions μ so that $\lambda/\mu \in \mathrm{VS}$. Turn this into a poset under inclusion. For a nonnegative integer n , let $[n]$ be the totally ordered poset on the set $\{0, 1, \dots, n\}$. Then the poset above is isomorphic to the product $[m_1(\lambda)] \times [m_2(\lambda)] \times \cdots$ (starting with a vertical strip, the element in $[m_i(\lambda)]$ is how many parts of size i had a box removed from them). We need to assign signs to each covering relation in this poset so that each square contains an odd number of signs. Such an assignment can be deduced from [Zha, Proof of Lemma 5.3].

Now we prove acyclicity of the complex. Exactness at \mathbf{I}^0 follows from Proposition 2.2.5(b). Now pick an element x in the kernel of $\mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$ for $j > 0$. We wish to show that it is in the image of $\mathbf{I}^{j-1} \rightarrow \mathbf{I}^j$. Without loss of generality, we may assume that x is the highest weight vector for the generator of a simple L_η . Say that the simple L_η appears in $Q_{\mu^1}, \dots, Q_{\mu^r} \subset \mathbf{I}^j$. We have a decomposition of $x = x_1 + \cdots + x_r$ into a sum of highest weight vectors coming from these r copies of L_η . We throw out any μ^i from the list $\{\mu^1, \dots, \mu^r\}$ for which $x_i = 0$. If $r = 1$, then the fact that x is in the image of $\mathbf{I}^{j-1} \rightarrow \mathbf{I}^j$ follows from Proposition 2.2.5(b). So suppose now that $r > 1$.

If for some μ^a and μ^b , it is the case that Q_{μ^a} and Q_{μ^b} do not map to any common $Q_\nu \subset \mathbf{I}^{j+1}$, then there is some column of λ such that λ/μ^a has more than 2 more boxes than λ/μ^b (and vice versa). In particular, one cannot remove a horizontal strip from both to get a common partition, so they contain no common simple. In particular, this situation never occurs in our set $\{\mu^1, \dots, \mu^r\}$.

So for each μ^a, μ^b , we can get from one to the other by removing a box in some column and then adding a box in a different column. So $\rho = \mu^a \cup \mu^b$ is independent of a, b (for any other c , we must have $\mu^c \subset \mu^a \cup \mu^b$ or else either the pair μ^a, μ^c or μ^b, μ^c would violate the column condition). Furthermore, L_η also appears in Q_ρ . We claim there is a highest weight vector y in this L_η which maps to x . By the ‘‘exactness of squares’’, for any a, b , there is a unique y which maps to both x_a and x_b . Hence this choice of y must be the same for all pairs, and so $y \mapsto x$, and we are done. \square

The idea for the above proof was loosely modeled on [BGG, §10].

Corollary 2.3.2. $\text{Ext}_K^d(L_\lambda, L_\mu) = \begin{cases} \mathbf{C} & \text{if } \mu/\lambda \in \text{VS}_d \\ 0 & \text{otherwise} \end{cases}$

Proof. This follows from $\dim \text{Hom}_K(L_\lambda, Q_\nu) = \delta_{\lambda, \nu}$. \square

Corollary 2.3.3. *Every object in Mod_K has finite injective dimension.*

2.4. K-theory of Mod_K . We now describe the Grothendieck group of Mod_K , and several structures on it.

Proposition 2.4.1. *The group $K(\text{Mod}_K)$ has for a basis the elements $[Q_\lambda]$. As a module over $K(\mathcal{V}_f)$, it is free of rank 1 and spanned by $[Q_0]$.*

Proof. The first sentence follows immediately from Theorem 2.3.1. The second sentence follows from the first. \square

Of course, the $[L_\lambda]$ also give a basis for the Grothendieck group, so it is natural to ask for the change of basis matrices. These are given as follows:

Proposition 2.4.2. *We have the following change of basis matrices in $K(\text{Mod}_K)$:*

$$\begin{aligned} [Q_\lambda] &= \sum_{\mu, \lambda/\mu \in \text{HS}} [L_\mu] \\ [L_\mu] &= \sum_{\nu, \mu/\nu \in \text{VS}} (-1)^{|\mu| - |\nu|} [Q_\nu]. \end{aligned}$$

Proof. The first formula comes from Proposition 2.2.1. The second formula is the Euler characteristic of the injective resolution of L_λ given in Theorem 2.3.1. \square

Remark 2.4.3. We can prove directly that the change of basis matrices in Proposition 2.4.2 are inverse to one another. Given $\nu \subseteq \lambda$, this is equivalent to showing that

$$(2.4.4) \quad \sum_{\mu, \lambda/\mu \in \text{HS}, \mu/\nu \in \text{VS}} (-1)^{|\mu| - |\nu|} = \delta_{\lambda, \nu}$$

where δ means Kronecker delta. We can prove this identity as follows. Let $h_d(x) = \text{char}(\text{Sym}^d)$ and $e_d(x) = \text{char}(\wedge^d)$ and define

$$\begin{aligned} H(t) &= \sum_{d \geq 0} h_d(x) t^d = \prod_{i \geq 1} \frac{1}{1 - tx_i} \\ E(t) &= \sum_{d \geq 0} e_d(x) t^d = \prod_{i \geq 1} (1 + tx_i). \end{aligned}$$

Then clearly $H(t)E(-t) = 1$. But the left hand side of (2.4.4) is the coefficient of the Schur function s_λ in $s_\nu H(t)E(-t)$, so we are done. \square

Remark 2.4.5. In Mod_A , the simple objects are given by \mathbf{S}_λ , while the projective objects are $[\mathbf{S}_\lambda \otimes A]$, and we have the following change of basis matrices:

$$\begin{aligned} [\mathbf{S}_\lambda \otimes A] &= \sum_{\mu, \mu/\lambda \in \text{HS}} [\mathbf{S}_\mu] \\ [\mathbf{S}_\mu] &= \sum_{\nu, \nu/\mu \in \text{VS}} (-1)^{|\nu| - |\mu|} [\mathbf{S}_\nu \otimes A]. \end{aligned}$$

It is curious that the rule is exactly the same as the one in Remark 2.4.3, except that we *add* horizontal/vertical strips rather than *remove* them. In fact, the existence of such a symmetry can be deduced from properties of the Fourier transform defined in §6. \square

Remark 2.4.6 (The pairing on K-theory). There is a natural pairing

$$\langle \cdot, \cdot \rangle : \mathbf{K}(\text{Mod}_K) \otimes \mathbf{K}(\text{Mod}_K) \rightarrow \mathbf{Z}, \quad \langle [M], [N] \rangle = \chi(\text{Ext}^*(M, N)),$$

where χ denotes Euler characteristic. We have

$$\begin{aligned} \langle Q_\lambda, Q_\mu \rangle &= \begin{cases} 1 & \text{if } \lambda/\mu \in \text{HS} \\ 0 & \text{if not} \end{cases} & \langle L_\lambda, Q_\mu \rangle &= \delta_{\lambda, \mu} \\ \langle L_\lambda, L_\mu \rangle &= \begin{cases} (-1)^{|\mu| - |\lambda|} & \text{if } \mu/\lambda \in \text{VS} \\ 0 & \text{if not} \end{cases} & \langle Q_\lambda, L_\mu \rangle &= \sum_{\nu, \lambda/\nu \in \text{HS}, \mu/\nu \in \text{VS}} (-1)^{|\mu| - |\nu|} \end{aligned}$$

The top left formula follows from Proposition 2.2.5; the bottom left from Corollary 2.3.2; the top right is immediate; and the bottom right follows from Theorem 2.3.1 and Proposition 2.2.5. The sum only takes values in $\{-1, 0, 1\}$, but we will wait until Remark 7.4.5 to explain this as more combinatorial notions need to be introduced first. These formulas are invariant under switching the order of the arguments and applying the transformation $[Q_\lambda] \leftrightarrow (-1)^{|\lambda|} [L_{\lambda^\dagger}]$. In §6.4, we will see that this map on K-theory is induced by a contravariant auto-equivalence of $\mathbf{D}^b(\text{Mod}_K)$. \square

Remark 2.4.7 (The ring structure on K-theory). The tensor product on Mod_A descends to define a tensor product on Mod_K . Clearly, the injective objects of Mod_K are flat. Since every object has finite injective dimension, $\mathbf{K}(\text{Mod}_K)$ has the structure of a ring. The map $\varphi : \mathbf{K}(\mathcal{V}_f) \rightarrow \mathbf{K}(\text{Mod}_K)$ given by $\mathbf{S}_\lambda \mapsto Q_\lambda$ is a ring isomorphism.

Since φ is a ring isomorphism, the tensor product of classes of injective objects in $\mathbf{K}(\text{Mod}_K)$ is computed using the Littlewood–Richardson rule, that is

$$[Q_\lambda][Q_\mu] = \bigoplus_{\nu} c_{\lambda, \mu}^{\nu} [Q_\nu],$$

where c denotes the Littlewood–Richardson coefficient. (In fact, this even holds before passing to K-theory.) Somewhat surprisingly, the product of classes of simple objects in K-theory is also computed with the Littlewood–Richardson rule. We now show this. Since φ is a ring isomorphism, we can calculate with the $[Q_\lambda]$ using Schur functions s_λ . From Proposition 2.4.2, we see that $[L_\lambda]$ becomes $\sum_{d \geq 0} (-1)^d s_{1^d}^\perp s_\lambda$ where $s_{1^d}^\perp$ is the skewing operator (s_λ^\perp is the adjoint to multiplication by s_λ with respect to the inner product for which the Schur functions are orthonormal). In general, we have the identity

$$s_\eta^\perp(fg) = \sum_{\xi, \theta} c_{\xi, \theta}^\eta s_\xi^\perp(f) s_\theta^\perp(g)$$

[Mac, Example I.5.25(d)]. Hence we get the identity

$$[L_\lambda][L_\mu] = \left(\sum_{d \geq 0} (-1)^d s_{1^d}^\perp s_\lambda \right) \left(\sum_{d \geq 0} (-1)^d s_{1^d}^\perp s_\mu \right) = \sum_{d \geq 0} (-1)^d s_{1^d}^\perp (s_\lambda s_\mu) = \sum_{\nu} c_{\lambda, \mu}^{\nu} [L_\nu]. \quad \square$$

2.5. The equivalence of Mod_K with $\text{Mod}_A^{\text{tors}}$. We now give an equivalence between Mod_K and $\text{Mod}_A^{\text{tors}}$. For clarity, let V and W be two copies of \mathbf{C}^∞ (in particular, we have a canonical identification $V = W$), let $A = \text{Sym}(V)$ and $A' = \text{Sym}(W)$, and let Mod_K be the usual quotient of Mod_A . Put $\mathcal{K} = \text{Sym}(V) \otimes \text{Sym}(V \otimes W)$. Then \mathcal{K} is an A -module, via the first factor, and an A' -comodule, via the maps

$$\begin{aligned} \text{Sym}(V) \otimes \text{Sym}(V \otimes W) &\rightarrow \text{Sym}(V) \otimes \text{Sym}(V \otimes W) \otimes \text{Sym}(V \otimes W) \\ &\rightarrow \text{Sym}(V) \otimes \text{Sym}(V \otimes W) \otimes \text{Sym}(V) \otimes \text{Sym}(W) \\ &\rightarrow \text{Sym}(V) \otimes \text{Sym}(V \otimes W) \otimes \text{Sym}(W), \end{aligned}$$

where the first map uses the comultiplication on $\text{Sym}(V \otimes W)$, the second is the natural projection coming from the Segre embedding $\mathbf{P}(V) \times \mathbf{P}(W) \subset \mathbf{P}(V \otimes W)$, and the third uses multiplication on $\text{Sym}(V)$. To check that this is a comodule structure, we use the geometric interpretations of this map. Note that $\text{Sym}(V)$ is the ring of functions on V^* (to avoid technicalities, we note that to check the comodule structure on any given graded piece, we may replace V and W with finite dimensional vector spaces of large enough dimension, so we may use this to take duals). Then the above map corresponds to the map

$$\begin{aligned} V^* \times (V \otimes W)^* \times W^* &\rightarrow V^* \times (V \otimes W)^* \\ (x, y, z) &\mapsto (x, y + (x \otimes z)). \end{aligned}$$

The fact that we get an A' -comodule follows from the identity

$$(x, y + (x \otimes (z_1 + z_2))) = (x, y + (x \otimes z_1) + (x \otimes z_2)).$$

We obtain functors

$$\Phi': \text{Mod}_A \rightarrow \text{Mod}_{A'}, \quad M \mapsto \text{Hom}_A(M, \mathcal{K})^\vee$$

and

$$\Psi: \text{Mod}_{A'} \rightarrow \text{Mod}_A, \quad M \mapsto \text{Hom}_{A'}(M^\vee, \mathcal{K}).$$

As an A -module, \mathcal{K} is saturated. We can therefore find a functor $\Phi: \text{Mod}_K \rightarrow \text{Mod}_{A'}$ such that $\Phi' = \Phi T$. The Cauchy formula shows that

$$\mathcal{K} = \bigoplus_{\lambda} Q_{\lambda} \otimes \mathbf{S}_{\lambda}(W)$$

as a $\mathbf{GL}(W)$ -equivariant object of Mod_K . It follows (from our results about Mod_K) that Φ takes values in $\text{Mod}_{A'}^{\text{tors}}$. Of course, one can regard Ψ as a functor from $\text{Mod}_{A'}^{\text{tors}}$ to Mod_K .

Theorem 2.5.1. *The functors $\Phi: \text{Mod}_K \rightarrow \text{Mod}_A^{\text{tors}}$ and $\Psi: \text{Mod}_A^{\text{tors}} \rightarrow \text{Mod}_K$ are mutually quasi-inverse equivalences of categories.*

Proof. There are injective natural transformations $\text{id} \rightarrow \Phi\Psi$ and $\text{id} \rightarrow \Psi\Phi$, and by computing on simple objects one can show that these maps are isomorphisms. \square

Remark 2.5.2. Both $\text{Mod}_A^{\text{tors}}$ and Mod_K have tensor structures (see Remark 2.4.7), but they are not equivalent as tensor categories. To see this, simply note that no object of $\text{Mod}_A^{\text{tors}}$ is flat, but the injective objects of Mod_K are flat. \square

3. QUIVER DESCRIPTIONS OF Mod_K AND $\text{Mod}_A^{\text{tors}}$

As we have seen in §2.5, the categories Mod_K and $\text{Mod}_A^{\text{tors}}$ are equivalent. It turns out that there is a third way to describe these categories which is also convenient: as the category of representations of an explicitly described locally finite quiver with relations.

We first abstract the important properties of Mod_K in the concept of a facile abelian category. The main result of §3.1 is a complete classification of such categories. The data that controls them

is an ordered set together with a cohomology class in its nerve. We also give a general criterion for the nerve to be contractible (so that all cohomological data is trivial). This theory is applied in §3.2 to describe both Mod_K and $\text{Mod}_A^{\text{tors}}$ as representations of a combinatorially defined quiver.

3.1. Facile abelian categories. In this section, we completely classify a certain class of abelian categories. This result is purely abstract, and will be applied in the next section to the categories of interest. Fix an algebraically closed field K .

Definition 3.1.1. A K -linear abelian category is **facile** if it satisfies the following conditions:

- (a) It has enough injectives, every object has finite length and Hom spaces are finite dimensional.
- (b) Indecomposable injectives are multiplicity-free.
- (c) A simple object is a constituent of the kernel of a nonzero map of $I \rightarrow I'$ of indecomposable injectives if and only if it is a constituent of I and not I' . \square

Suppose \mathcal{A} is facile. Let $S = S(\mathcal{A})$ denote the set of isomorphism classes of simple objects (which could be infinite). For $i \in S$ write L_i for the corresponding simple and Q_i for the injective envelope of L_i . Recall that the **Cartan matrix** of \mathcal{A} is the $S \times S$ matrix whose entry at (i, j) is the multiplicity of L_i in Q_j . Since \mathcal{A} is facile, the entries of the Cartan matrix are all 0 or 1. We prefer to record this data in a slightly different form. Define $i \leq j$ if L_i is a constituent of Q_j , i.e., if the (i, j) entry of the Cartan matrix is 1. We call \leq the **Cartan relation**. It is reflexive and, as we show below, antisymmetrical (i.e., $x \leq y$ and $y \leq x$ implies $x = y$), but not necessarily transitive. It is also downwards finite, i.e., for any $j \in S$ there are only finitely many $i \in S$ for which $i \leq j$. We write $i_1 \leq \cdots \leq i_n$ to mean $i_a \leq i_b$ for all $a \leq b$.

Lemma 3.1.2. $\text{Hom}(Q_j, Q_i) = \begin{cases} K & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$

Proof. First suppose $i \leq j$. Using that L_i is a constituent of Q_j , one can construct a nonzero map $Q_j \rightarrow Q_i$. In a sense, L_i cogenerates Q_i , and any map to Q_i is determined by what it does on L_i . Since K is algebraically closed, we have $\text{End}(L_i) = K$. Using that Q_j is multiplicity-free, one sees that any two maps $Q_j \rightarrow Q_i$ are linearly dependent. Thus $\text{Hom}(Q_j, Q_i)$ is one dimensional. Now suppose $i \not\leq j$. Since L_i does not appear in Q_j , any map $Q_j \rightarrow Q_i$ is zero on the cogenerator of Q_i , and thus zero. \square

Lemma 3.1.3. *The Cartan relation is antisymmetrical.*

Proof. Suppose $i \leq j$ and $j \leq i$. We then have nonzero maps $f: Q_i \rightarrow Q_j$ and $g: Q_j \rightarrow Q_i$. Since both Q_i and Q_j have L_i and L_j as constituents, neither $\ker(f)$ nor $\ker(g)$ has L_i or L_j as a constituent. Thus $\text{image}(f)$ has an L_i in it, and so $g(\text{image}(f))$ is nonzero. We thus see the gf , and symmetrically fg , are nonzero. The third axiom implies that any nonzero endomorphism of an indecomposable injective is an isomorphism. Thus gf and fg are invertible, so f and g are isomorphisms, and $i = j$. \square

Let S be a set with a reflexive antisymmetrical downwards finite relation \leq . An **S -rigidified facile abelian category** is a facile abelian category \mathcal{A} equipped with a bijection $S \rightarrow S(\mathcal{A})$ under which \leq corresponds to the Cartan relation. An equivalence of S -rigidified facile abelian categories is required to be compatible with the rigidification. We define the **nerve** $N(S)$ of S to be the simplicial set whose n -simplices are chains $i_0 \leq \cdots \leq i_n$ in S . (See [McL, §VII.5] for some generalities on simplicial sets.) The following theorem is our main result on facile abelian categories.

Theorem 3.1.4. *Fix a set S with a reflexive antisymmetric downwards finite relation \leq .*

- (a) *The set of equivalence classes of S -rigidified facile abelian categories is canonically in bijection with $H^2(N(S), K^\times)$.*

- (b) *The auto-equivalence group of an S -rigidified facile abelian category is canonically isomorphic to $H^1(N(S), K^\times)$.*
- (c) *The automorphism group of the identity functor of an S -rigidified abelian category is canonically isomorphic to $H^0(N(S), K^\times)$.*

We require a lemma first. For an abelian category \mathcal{A} , let $\underline{\mathcal{I}}(\mathcal{A})$ (resp. $\underline{\mathcal{I}}_0(\mathcal{A})$) denote the full subcategory of \mathcal{A} on the injective objects (resp. indecomposable injective objects). Write $\text{Equiv}(\mathcal{A}, \mathcal{B})$ for the category of equivalences between two categories \mathcal{A} and \mathcal{B} .

Lemma 3.1.5. *Let \mathcal{A} and \mathcal{B} be abelian categories with enough injectives. Then the natural functor*

$$\text{Equiv}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Equiv}(\underline{\mathcal{I}}(\mathcal{A}), \underline{\mathcal{I}}(\mathcal{B}))$$

is an equivalence. If every injective object of \mathcal{A} and \mathcal{B} is a finite direct sum of indecomposable injectives then the same is true with $\underline{\mathcal{I}}_0$ in place of $\underline{\mathcal{I}}$.

Proof. One can canonically recover \mathcal{A} from $\underline{\mathcal{I}}(\mathcal{A})$ by considering 2-term complexes up to homotopy. If every injective of \mathcal{A} is a finite direct sum of indecomposable injectives then one can canonically recover $\underline{\mathcal{I}}(\mathcal{A})$ canonically from $\underline{\mathcal{I}}_0(\mathcal{A})$. \square

Proof of Theorem 3.1.4. (a) Let \mathcal{A} be an S -rigidified facile abelian category. For $i \leq j$ in S , choose a nonzero map $f_{ji}: Q_j \rightarrow Q_i$. If $i \leq j \leq k$ then

$$f_{ki} = \alpha_{kji} f_{ji} f_{kj}$$

for some $\alpha_{kji} \in K^\times$. The α_{kji} satisfy the cocycle condition, and thus define a class $[\mathcal{A}]$ in $H^2(N(S), K^\times)$, which we call the **cohomological invariant** of \mathcal{A} .

It is clear that invariants of equivalent categories are equal. Suppose now that \mathcal{A} and \mathcal{B} are S -rigidified facile abelian categories with equal invariants. We use our usual notation for \mathcal{A} and primed versions for \mathcal{B} . For $j \geq i$ in S , choose nonzero maps $f_{ji}: Q_j \rightarrow Q_i$ in \mathcal{A} and $f'_{ji}: Q'_j \rightarrow Q'_i$ in \mathcal{B} . Since the cocycles $[\mathcal{A}]$ and $[\mathcal{B}]$ are equal, we can scale the f'_{ji} so that $\alpha_{kji} = \alpha'_{kji}$. It follows that $Q_i \mapsto Q'_i$ and $f_{ji} \mapsto f'_{ji}$ defines an equivalence of categories $\underline{\mathcal{I}}_0(\mathcal{A}) \rightarrow \underline{\mathcal{I}}_0(\mathcal{B})$. Thus \mathcal{A} and \mathcal{B} are equivalent by Lemma 3.1.5.

We have thus shown that there is a natural injection from the set of equivalence classes of S -rigidified facile abelian categories into $H^2(N(S), K^\times)$. Surjectivity of this map will follow from Proposition 3.1.6.

(b) Let \mathcal{A} be an S -rigidified facile abelian category and let F be an auto-equivalence of \mathcal{A} . Since F is required to be compatible with the S -rigidification, $F(L_i)$ is isomorphic to L_i for all i . It follows that $F(Q_i)$ is isomorphic to Q_i for each i , as Q_i is the injective envelope of L_i . Choose an isomorphism $\varphi_i: Q_i \rightarrow F(Q_i)$. For $i \leq j$, we have

$$F(f_{ji}) = \alpha_{ji} \varphi_i f_{ji} \varphi_j^{-1}$$

for some $\alpha_{ji} \in K^\times$, where $f_{ij}: Q_j \rightarrow Q_i$ is a chosen nonzero map. The α_{ij} satisfy the cocycle condition, and define an element $[F] \in H^1(N(S), K^\times)$.

It is clear that $F \mapsto [F]$ is a well-defined group homomorphism $\text{Aut}(\mathcal{A}) \rightarrow H^1(N(S), K^\times)$. It is also clear that the restriction of F to $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}_0(\mathcal{A})$ is determined, up to isomorphism, by $[F]$. It follows from Lemma 3.1.5 that F is determined, up to isomorphism, by $[F]$. Now let α be a 1-cocycle on $N(S)$ with values in K^\times . Define a functor $F: \underline{\mathcal{I}}_0 \rightarrow \underline{\mathcal{I}}_0$ by $F(Q_i) = Q_i$ and $F(f_{ji}) = \alpha_{ji} f_{ji}$. The cocycle condition implies that F is actually a functor (i.e., compatible with composition), and it is clearly an equivalence. Another application of Lemma 3.1.5 shows that F extends canonically to an equivalence of \mathcal{A} , and it is clear that $[F] = \alpha$. We have thus shown that $F \mapsto [F]$ is a bijection.

(c) Let \mathcal{A} be an S -rigidified facile abelian category and let φ be an automorphism of the identity functor. Then $\varphi(L_i)$ is given by multiplication by a scalar α_i on L_i . We leave it to the reader to verify that $\varphi \mapsto \alpha$ defines an isomorphism $\text{Aut}(\text{id}_{\mathcal{A}}) \rightarrow H^0(N(S), K^\times)$. \square

We now show how to construct facile abelian categories. Let S be a set with a relation \leq which is reflexive, antisymmetrical and downwards finite and let α be a 2-cocycle representing a class in $H^2(N(S), K^\times)$. Define a quiver-with-relations \mathcal{Q}^α as follows. The vertices of \mathcal{Q}^α are the elements of S . If $i \leq j$ are elements of S then there is an arrow $\gamma_{ij}: i \rightarrow j$ in \mathcal{Q}^α . Suppose that $i \leq j$ and $j \leq k$. If $i \leq k$ then we impose the relation $\gamma_{ik} = \alpha_{kji}^{-1} \gamma_{jk} \gamma_{ij}$, while if $i \not\leq k$ then we impose the relation $\gamma_{jk} \gamma_{ij} = 0$.

Proposition 3.1.6. *The category $\text{Rep}(\mathcal{Q}^\alpha)$ is naturally an S -rigidified facile abelian category with cohomological invariant α .*

Proof. For $k \in S$, we let L_k be the representation of \mathcal{Q}^α which assigns K to the vertex k and 0 to all other vertices. Then L_k is a simple and $k \mapsto L_k$ defines a bijection $S \rightarrow S(\text{Rep}(\mathcal{Q}^\alpha))$. The injective envelope Q_k of L_k assigns to the vertex j the space K if $j \leq k$ and 0 otherwise, and to the arrow γ_{ij} the map scaling by α_{kji} , if $i \leq j \leq k$, and 0 otherwise. From this description, it is clear that the Cartan relation corresponds to \leq .

To calculate the cohomological invariants, we define $f_{ji}: Q_j \rightarrow Q_i$ when $j \geq i$ as follows: for a vector space at a vertex ℓ with $\ell \not\leq i$, we send it to 0; for a vector space at a vertex ℓ with $\ell \leq i$ and $\ell \leq j$, we define the f_{ji} to be multiplication by α_{jil} . Using these maps, if $i \leq j \leq k$, we get $f_{ki} = \alpha_{kji} f_{ji} f_{kj}$, which shows that $\text{Rep}(\mathcal{Q}^\alpha)$ has cohomological invariant α . \square

Corollary 3.1.7. *Let \mathcal{A} be an S -rigidified facile abelian category with cohomological invariant α . Then \mathcal{A} is (non-canonically) equivalent to $\text{Rep}(\mathcal{Q}^\alpha)$, with \mathcal{Q}^α as above.*

Remark 3.1.8. We can rephrase Theorem 3.1.4 neatly in the language of homotopy theory as follows. Fix S . Let \mathcal{C} be the 2-groupoid whose objects are S -rigidified facile abelian categories, whose 1-morphisms are equivalences of S -rigidified facile abelian categories and whose 2-morphisms are isomorphisms of equivalences. Then we have

$$\pi_i(\mathcal{C}, x) = H^{2-i}(N(S), K^\times)$$

for $0 \leq i \leq 2$ and for any base point x . \square

Remark 3.1.9. The theory of facile abelian categories is reminiscent of the theory of gerbes. Let X be a topological space and let \mathcal{C}' be the 2-groupoid of K^\times -gerbes on X . Then one has $\pi_i(\mathcal{C}', x) = H^{2-i}(X, K^\times)$, just as above. Given a gerbe α , one can form the abelian category of α -twisted sheaves on X , and the subcategory of such sheaves which are constructible with respect to some stratification. Not all such categories are facile, and not all facile abelian categories come from this construction, but there is significant overlap. \square

Question 3.1.10. *Let \mathcal{A} and \mathcal{B} be two facile abelian categories. What are necessary and sufficient conditions on the invariants $(S(\mathcal{A}), [\mathcal{A}])$ and $(S(\mathcal{B}), [\mathcal{B}])$ so that \mathcal{A} and \mathcal{B} are derived equivalent? How does one recover the group $\text{Aut}(\text{D}^b(\mathcal{A}))$ in terms of the invariants $(S(\mathcal{A}), [\mathcal{A}])$?*

We close this section with the following simple result, which gives a criterion for the nerve to be contractible.

Proposition 3.1.11. *Let S be a set with a reflexive antisymmetric relation \leq . Suppose there is a map $\tau: S \rightarrow S$ and a point $x_0 \in S$ satisfying the following two conditions:*

- (a) *If $x \leq y$ then $\tau y \leq x$.*
- (b) *For any $x \in S$ we have $\tau^n(x) = x_0$ for $n \gg 0$.*

Then $N(S)$ is contractible.

Proof. Let Δ be the simplex category (objects are sets $[n] = \{0, \dots, n\}$, morphisms are weakly order-preserving functions), so that a simplicial set is a contravariant functor from Δ to the category of sets. The simplicial set $N(S)$ takes $[n]$ to the set of chains $\lambda_0 \leq \dots \leq \lambda_n$. The interval I is the

simplicial set represented by the object $[1]$ of Δ ; thus $I([n])$ is the set of partitions of $[n]$ into two intervals of the form $\{0, \dots, i\} \cup \{i+1, \dots, n\}$.

Since $x \leq y$ implies $\tau x \leq \tau y$, we have a map $\tau: N(S) \rightarrow N(S)$. We claim that it is homotopic to the identity map. To see this, define a map $h: N(S) \times I \rightarrow N(S)$ as follows. An n -simplex of $N(S) \times I$ is a chain $\lambda_0 \leq \dots \leq \lambda_n$ in S together with a partition $[n] = \{0, \dots, i\} \cup \{i+1, \dots, n\}$. We map this to the n -simplex of $N(S)$ given by $\tau(\lambda_{i+1}) \leq \dots \leq \tau(\lambda_n) \leq \lambda_0 \leq \dots \leq \lambda_i$. Restricting h to the vertex 0 of I (which corresponds to partitions $[n] = [n] \cup \emptyset$), we get the identity map, while restricting h to the vertex 1 of I (which corresponds to partitions $[n] = \emptyset \cup [n]$), we get τ . This proves the claim.

Let $S_n \subset S$ be the set of elements x for which $\tau^n(x) = x_0$. Then τ induces a map $N(S_n) \rightarrow N(S_n)$, and h induces a homotopy of this map with the identity map. Since τ^n collapses $N(S_n)$ to a point, but is also homotopic to the identity map, we see that $N(S_n)$ is contractible. Finally, $N(S)$ is the filtered colimit of the spaces $N(S_n)$, and is therefore contractible as well. \square

3.2. Description of Mod_K and $\text{Mod}_A^{\text{tors}}$. Let S be the set of partitions. Define a relation \leq on S by $\mu \leq \lambda$ if $\lambda/\mu \in \text{HS}$. Let Part_{HS} be the quiver-with-relations \mathcal{Q} constructed in §3.1 from S , using the trivial cocycle. This coincides with the notion of “hypercomplex” in the sense of Olver [Olv, Definition 7.1]. The main theorem of this section is the following:

Theorem 3.2.1. *The categories Mod_K and $\text{Mod}_A^{\text{tors}}$ are equivalent to $\text{Rep}(\text{Part}_{\text{HS}})$. Furthermore, if \mathcal{C} denotes any of these categories then $\text{Aut}(\mathcal{C}) = 1$ and $\text{Aut}(\text{id}_{\mathcal{C}}) = \mathbf{C}^\times$.*

Remark 3.2.2. It is also true that there are no equivalences $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$: indeed, there are no projective objects in \mathcal{C} but there are injective objects. \square

Remark 3.2.3. We note that the quiver Part_{HS} has wild representation type. In particular, it contains a vertex with valency 5, and the representation theory of this subquiver contains as a subproblem the moduli space of 5 points in \mathbf{P}^1 , which is known to have dimension 2. General principles then imply that the representation theory of this quiver contains as a subproblem the moduli space of pairs of linear operators, which is known to be “wild”. \square

We begin with some lemmas.

Lemma 3.2.4. *The space $N(S)$ is contractible.*

Proof. Let $\tau: S \rightarrow S$ be the map which deletes the first row of a partition (and takes the empty partition to itself). If $x \leq y$ then $\tau y \leq x$. Of course, any partition is mapped to the empty partition by some iterate of τ . The result now follows from Proposition 3.1.11. \square

Lemma 3.2.5. *The structure (S, \leq) has no nontrivial automorphisms.*

Proof. The zero partition can be recovered as the unique element x of S such that $y \leq x$ implies $x = y$. Put $S_0 = \{0\}$. The set S_n of partitions with n rows can be characterized as the set of $x \in S$ such that $x \geq y$ for some $y \in S_{n-1}$, but $x \notin S_{n-1}$. Furthermore, the length of the final row of x is the maximal length of a chain $x_0 < x_1 < \dots < x_r$ with $x_0 \in S_{n-1}$; the partition obtained from x by deleting the final row is x_0 . Also, this partition can be characterized as the maximal object in S_{n-1} that is smaller than x . This allows x to be recovered inductively. We have thus shown that a partition can be recovered by how it fits into the order \leq , which shows that each partition is fixed by an automorphism of (S, \leq) . This proves the lemma. \square

Lemma 3.2.6. *One can choose for each $\mu \leq \lambda$ a projector $p_{\mu, \lambda}: \text{Sym} \otimes \mathbf{S}_\mu \rightarrow \mathbf{S}_\lambda$ such that for $\nu \leq \mu \leq \lambda$ the square*

$$\begin{array}{ccc} \text{Sym} \otimes \text{Sym} \otimes \mathbf{S}_\nu & \xrightarrow{\text{id} \otimes p_{\nu, \mu}} & \text{Sym} \otimes \mathbf{S}_\mu \\ \uparrow c \otimes \text{id} & & \downarrow p_{\mu, \lambda} \\ \text{Sym} \otimes \mathbf{S}_\nu & \xrightarrow{p_{\nu, \lambda}} & \mathbf{S}_\lambda \end{array}$$

commutes, where c denotes the comultiplication map on Sym .

Proof. Begin by choosing arbitrary projectors $p_{\mu,\lambda}$. Since $\text{Sym} \otimes \mathbf{S}_\nu$ is multiplicity-free, the above diagram commutes up to scalars; that is, we have

$$p_{\mu,\lambda}(\text{id} \otimes p_{\nu,\mu})(c \otimes \text{id}) = \alpha_{\lambda,\mu,\nu} p_{\nu,\lambda}$$

for some scalar $\alpha_{\lambda,\mu,\nu}$, which we claim is nonzero. To see this, apply the duality functor $(-)^{\vee}$. This reverses the directions of the arrows and replaces comultiplication with multiplication and projectors with injectors. Then the fact that the scalar is nonzero is the content of Proposition 1.0.1. Hence α defines a 2-cocycle on $N(S)$ with values in \mathbf{C}^{\times} . Since $N(S)$ is contractible, α is a coboundary, and so we can rescale our projectors so that $\alpha = 1$. \square

Proof of Theorem 3.2.1. We first show that Mod_K and $\text{Rep}(\text{Part}_{\text{HS}})$ are equivalent. The results of §2.2 show that Mod_K is a facile abelian category. Furthermore, we know that L_μ appears as a constituent in Q_λ in Mod_K if and only if $\mu \leq \lambda$, where \leq is the Cartan relation for Mod_K . The cohomological invariant of Mod_K vanishes, since $N(S)$ is contractible, and so Mod_K is equivalent to $\text{Rep}(\text{Part}_{\text{HS}})$.

We now show that $\text{Mod}_A^{\text{tors}}$ and $\text{Rep}(\text{Part}_{\text{HS}})$ are equivalent. For a partition λ , put $\mathcal{K}_\lambda = A \otimes \mathbf{S}_\lambda$. The chosen projector $p_{\mu,\lambda}$ extends uniquely to a map of A -comodules $\mathcal{K}_\mu \rightarrow \mathcal{K}_\lambda$, and Lemma 3.2.6 exactly says that these maps respect the relations in Part_{HS} . We can therefore think of \mathcal{K} as a representation of Part_{HS} in the category of A -comodules. We obtain a functor

$$\Phi: \text{Mod}_A \rightarrow \text{Rep}(\text{Part}_{\text{HS}}), \quad M \mapsto \text{Hom}_A(M^{\vee}, \mathcal{K})$$

and a functor

$$\Psi: \text{Rep}(\text{Part}_{\text{HS}}) \rightarrow \text{Mod}_A, \quad M \mapsto \text{Hom}_A(M, \mathcal{K})^{\vee}.$$

Here $(-)^{\vee}$ denotes the duality functor on \mathcal{V} , which takes A to itself and A -modules to A -comodules. It is clear that Φ takes the simple A -module \mathbf{S}_λ to the simple representation L_λ of Part_{HS} , and that Ψ does the opposite. There are injective natural transformations $\text{id} \rightarrow \Phi\Psi$ and $\text{id} \rightarrow \Psi\Phi$ which are easily seen to be isomorphisms on simple objects. Since Φ and Ψ are both right exact, these maps are necessarily isomorphisms.

Finally, the results on $\text{Aut}(\mathcal{C})$ and $\text{Aut}(\text{id}_{\mathcal{C}})$ follow from the contractibility of $N(\mathcal{C})$ (Lemma 3.2.4) and the fact that (S, \leq) has no automorphisms (Lemma 3.2.5), which implies that any auto-equivalence of \mathcal{C} induces the identity map on S . \square

Remark 3.2.7. The functors Φ and Ψ in the above proof are actually very concrete. Let M be a finite length object of \mathcal{V} , and let M_λ denote the multiplicity space of \mathbf{S}_λ in M . By Pieri's rule, giving a map $A \otimes M \rightarrow M$ is the same as giving a map $f_{\mu,\lambda}: M_\mu \rightarrow M_\lambda$ for each $\mu \leq \lambda$. Thus if M is an A -module then we get something that looks like it should be a representation of Part_{HS} . In fact, the above proof shows that this thing is in fact a representation of Part_{HS} (i.e., the relations are satisfied), and this representation is none other than $\Phi(M)$. Similar comments hold in the reverse direction. \square

Remark 3.2.8. The proof of Lemma 3.2.6 demonstrates the usefulness of the cohomological approach: to prove that such projectors exist directly would likely involve detailed computations with how the Pieri rule works (but see [Olv, §8] for a more direct approach). One might object that we have used Proposition 1.0.1, which relies on some analysis of coefficients as in [Olv], but in fact, Proposition 1.0.1 is implied by the results of [EFW] (see the remark following [EFW, Theorem 3.2]), which does not require such an analysis. \square

4. THE STRUCTURE OF Mod_A

In the previous two sections, we have studied the structure of $\text{Mod}_A^{\text{tors}}$ and Mod_K . In this section, we study Mod_A and especially the manner in which it is built out of these two pieces.

We begin in §4.1 by defining a class of modules called **saturated** modules, and show that the modules L_λ^0 and $A \otimes \mathbf{S}_\lambda$ are saturated. In §4.2, we show that the localization functor $T: \text{Mod}_K \rightarrow \text{Mod}_A$ has a right adjoint, which we denote by S and call the **section functor**. It is almost formal that such an adjoint exists if we use non-finitely generated modules, but to show that the section functor takes finitely generated modules to finitely generated modules requires the special results on saturated modules. As an immediate consequence of this work, we show in §4.3 that every object of Mod_A has finite injective dimension. In §4.4, we define a right adjoint H_m^0 to the inclusion $\text{Mod}_A^{\text{tors}} \rightarrow \text{Mod}_A$ and define the **local cohomology** functors H_m^i as the derived functors of H_m^0 .

At this point, we have a diagram

$$D_{\text{tors}}^b(A) \begin{array}{c} \xleftarrow{\quad} \\ \text{R}\Gamma_m \\ \xrightarrow{\quad} \end{array} D^b(A) \begin{array}{c} \xleftarrow{T} \\ \text{R}S \\ \xrightarrow{\quad} \end{array} D^b(K)$$

In §4.5 we show that whenever one is in a situation like this, one can describe $D^b(A)$ as the category of triples (X, Y, f) with $X \in D^b(K)$, $Y \in D_{\text{tors}}^b(A)$ and $f \in \text{Hom}_{D^b(A)}(X, Y)$. This is not a completely satisfactory description of $D^b(A)$, as the data f makes reference to $D^b(A)$. In §4.6 we show that, in our specific situation, f can be defined without any reference to $D^b(A)$. This allows us to describe $D^b(A)$ purely in terms of the simpler category $D^b(K)$.

Finally, in §4.7 we show that the abelian category Mod_A is rigid, in a suitable sense, and in §4.8 we study the K-theory of Mod_A .

4.1. Saturated modules. We say that an A -module M is **saturated** if $\text{Ext}_A^i(N, M) = 0$ for $i = 0, 1$ whenever N is a torsion A -module. The relevance of this condition will be seen in the following sections.

Proposition 4.1.1. *The module L_λ^0 is saturated.*

Proof. The statement for Ext_A^0 is clear. It suffices to show that $\text{Ext}_A^1(\mathbf{S}_\mu, L_\lambda^0) = 0$ for all μ . The Koszul complex $A \otimes \mathbf{S}_\mu \otimes \bigwedge^\bullet$ gives a projective resolution of \mathbf{S}_μ . Applying $\text{Hom}_A(-, L_\lambda^0)$, we obtain a complex

$$(4.1.2) \quad \text{Hom}_V(\mathbf{S}_\mu, L_\lambda^0) \rightarrow \text{Hom}_V(\mathbf{S}_\mu \otimes \bigwedge^1, L_\lambda^0) \rightarrow \text{Hom}_V(\mathbf{S}_\mu \otimes \bigwedge^2, L_\lambda^0) \rightarrow \dots$$

which calculates $\text{Ext}_A^i(\mathbf{S}_\mu, L_\lambda^0)$. Notice that $\mathbf{S}_\mu \otimes \bigwedge^i$ is multiplicity-free and all partitions occurring in it have the same size, while L_λ^0 is also multiplicity-free but no two partitions occurring in it have the same size. It follows that each Hom space in (4.1.2) is 0 or 1 dimensional. In particular, $\text{Ext}_A^i(\mathbf{S}_\mu, L_\lambda^0)$ is 0 or 1 dimensional for all i .

Suppose that μ occurs in L_λ^0 , so that $\text{Hom}_V(\mathbf{S}_\mu, L_\lambda^0) \neq 0$. Since $\text{Hom}_A(\mathbf{S}_\mu, L_\lambda^0) = 0$, the first differential in (4.1.2) is injective. Since the second term of (4.1.2) has dimension at most 1, the first differential is an isomorphism, and so $\text{Ext}_A^1(\mathbf{S}_\mu, L_\lambda^0) = 0$.

Suppose now that μ does not occur in L_λ^0 . We claim that the second differential in (4.1.2) is injective. We may assume that $\text{Hom}_V(\mathbf{S}_\mu \otimes \bigwedge^1, L_\lambda^0) \neq 0$, otherwise there is nothing to prove. Since we assumed that μ does not occur in L_λ^0 , it must be the case that $\mu = (D, \nu)$ where $D \geq \lambda_1$, $\nu \subset \lambda$, and $|\lambda| - |\nu| = 1$. Then $\mathbf{S}_{(D, \lambda)} \subset \mathbf{S}_\mu \otimes \bigwedge^1$ and $\mathbf{S}_{(D+1, \lambda)} \subset \mathbf{S}_\mu \otimes \bigwedge^2$. An application of Proposition 1.0.1

shows that the composition of the top row of

$$\begin{array}{ccccc}
 \mathbf{S}_{(D+1,\lambda)} \otimes A & \longrightarrow & \mathbf{S}_{(D,\lambda)} \otimes A & \xrightarrow{f} & L_\lambda^0 \subset \mathbf{S}_\lambda \otimes A \\
 \downarrow & & \downarrow & \nearrow & \\
 \mathbf{S}_\mu \otimes \wedge^2 \otimes A & \longrightarrow & \mathbf{S}_\mu \otimes \wedge^1 \otimes A & &
 \end{array}$$

is nonzero when f is nonzero. To see that the image of the left map is not annihilated by the Koszul differential, we can use an exterior algebra version of Proposition 1.0.1 (either apply the transpose functor \dagger , or see [SW, Corollary 1.8]). Here the bottom row is the differential from the Koszul complex and the vertical maps are the inclusions. Hence the second differential in (4.1.2) is injective. \square

Proposition 4.1.3. *The module $A \otimes \mathbf{S}_\lambda$ is saturated.*

Proof. The statement for Ext^0 is clear. It suffices to show that $\text{Ext}_A^1(\mathbf{S}_\mu, A \otimes \mathbf{S}_\lambda) = 0$. We may suppose that some subrepresentation of $\mathbf{S}_\mu \otimes \mathbf{S}_1$ appears in $\mathbf{S}_\lambda \otimes A$ (if not, then no nontrivial extension is possible). Let $0 \rightarrow A \otimes \mathbf{S}_\lambda \rightarrow E \rightarrow \mathbf{S}_\mu \rightarrow 0$ be an extension.

First suppose that $\mu/\lambda \notin \text{HS}$. Then we can add a box to some (unique) column of μ to get a partition in $\mathbf{S}_\lambda \otimes A$; by our assumption $\mu/\lambda \notin \text{HS}$, we may add another box in the same column (and more boxes to the left of it to make the result a partition) to get ν so that $\nu/\lambda \in \text{HS}$ and $\nu/\mu \notin \text{HS}$. The subspace of E spanned by \mathbf{S}_μ must generate some other partitions that in turn generate \mathbf{S}_ν by Proposition 1.0.1, but this contradicts that $\nu/\mu \notin \text{HS}$.

Now we suppose that $\mu/\lambda \in \text{HS}$ so that E contains 2 copies of \mathbf{S}_μ . Pick two linearly independent highest weight vectors f_1, f_2 of highest weight μ . Let p be a linear combination of elements in A and \mathbf{GL}_∞ so that pf_1 is a nonzero highest weight vector of weight η different from μ . Then pf_2 must be a scalar multiple c_η of pf_1 since this representation appears with multiplicity 1. If $c_\eta \neq c_\xi$ for some η, ξ , then set $\theta_i = \max(\eta_i, \xi_i)$ so that \mathbf{S}_η and \mathbf{S}_ξ generate \mathbf{S}_θ by Proposition 1.0.1. However, this shows that f_2 generates two different scalar multiples of the highest weight vector in \mathbf{S}_θ using the same linear combination of elements in A and \mathbf{GL}_∞ , which again is a contradiction. So we can just write c for this common scalar, and the submodule generated by $cf_1 - f_2$ is isomorphic to \mathbf{S}_μ , which shows that E is a trivial extension. \square

4.2. The section functor. Recall that T denotes the natural functor $\text{Mod}_A \rightarrow \text{Mod}_K$.

Proposition 4.2.1. *The functor T has a right adjoint.*

We denote this right adjoint by S and call it the **section functor**. We need two lemmas before proving the proposition. Let $\widetilde{\text{Mod}}_A$ denote the category of all A -modules, and let $\widetilde{T}: \widetilde{\text{Mod}}_A \rightarrow \widetilde{\text{Mod}}_K$ be the quotient by the Serre subcategory $\widetilde{\text{Mod}}_A^{\text{tors}}$ of torsion A -modules.

Lemma 4.2.2. *The functor \widetilde{T} has a right adjoint \widetilde{S} . For any $M \in \widetilde{\text{Mod}}_K$, $\widetilde{S}(M)$ is a torsion-free module.*

Proof. Let M be an object of $\widetilde{\text{Mod}}_K$ and let $\mathcal{C}(M)$ be the category of pairs (N, f) where N is an object of $\widetilde{\text{Mod}}_A$ and f is a map $T(N) \rightarrow M$ in $\widetilde{\text{Mod}}_K$. There is a forgetful functor $\mathcal{C}(M) \rightarrow \widetilde{\text{Mod}}_A$ whose colimit computes $\widetilde{S}(M)$. It is therefore enough to show that this colimit exists. To see this, note that every object of $\mathcal{C}(M)$ maps to one where N is torsion-free and f is injective. The class of isomorphism classes of such objects is small, since these conditions bound the cardinality of N . Thus there is a small set of objects of $\mathcal{C}(M)$ to which every object maps, and so the colimit exists. (Note: since \mathcal{V} has all small colimits, so does $\widetilde{\text{Mod}}_A$.) Since T is exact, it preserves all small colimits, so the existence of \widetilde{S} follows from the Freyd adjoint functor theorem [McL, Theorem V.6.2].

Since $\widetilde{S}(M)$ is the universal object of $\widetilde{\text{Mod}}_A$ equipped with a map to M , it is torsion-free. \square

Lemma 4.2.3. *Let M be a saturated A -module, i.e., $\text{Ext}_A^i(N, M) = 0$ for $i = 0, 1$ whenever N is a torsion module. Then the natural map $\tilde{S}(\tilde{T}(M)) \rightarrow M$ is an isomorphism.*

Proof. Clearly M is torsion-free, and so we have a short exact sequence

$$0 \rightarrow M \rightarrow \tilde{S}(\tilde{T}(M)) \rightarrow N \rightarrow 0.$$

Since $\tilde{T}(\tilde{S}(\tilde{T}(M))) = \tilde{T}(M)$ [Gab, Prop. III.2.3], the module N is torsion. Since $\text{Ext}^1(N, M) = 0$ the sequence splits. Since $\tilde{S}(\tilde{T}(M))$ is torsion-free (Lemma 4.2.2), we get $N = 0$. \square

Proof of Proposition 4.2.1. Lemma 4.2.3 and Proposition 4.1.1 show that the natural map $L_\lambda^0 \rightarrow \tilde{S}(\tilde{T}(L_\lambda^0))$ is an isomorphism. In particular, $\tilde{S}(L_\lambda)$ is a finitely generated A -module. The functor \tilde{S} is left exact, since it is a right adjoint. It follows that \tilde{S} carries all finite length objects of $\widetilde{\text{Mod}}_K$ to finitely generated objects of $\widetilde{\text{Mod}}_A$, i.e., \tilde{S} restricts to a functor $S: \text{Mod}_K \rightarrow \text{Mod}_A$. It is clear that this functor is the right adjoint of T . \square

We now give some of the basic properties of the section functor.

Proposition 4.2.4. *In the following, M denotes an object of Mod_A and M' an object of Mod_K .*

- (a) *The functor S is left exact.*
- (b) *The functor S takes injective objects of Mod_K to injective objects of Mod_A .*
- (c) *The adjunction $T(S(M')) \rightarrow M'$ is an isomorphism.*
- (d) *The module $S(M')$ is saturated.*
- (e) *The natural map $M \rightarrow S(T(M))$ is an injection (resp. an isomorphism) if and only if M is torsion-free (resp. saturated).*

Proof. (a) This is true for every right adjoint.

(b) This is true for any functor that has an exact left adjoint.

(c) This is [Gab, Prop. III.2.3].

(d) From Lemma 4.2.2, the module $S(M')$ is torsion-free and so $\text{Ext}_A^0(N, S(M')) = 0$ if N is torsion. Suppose now that we have a short exact sequence

$$0 \rightarrow S(M') \xrightarrow{i} M'' \rightarrow N \rightarrow 0$$

with N torsion. Then $T(i)$ is an isomorphism, and so we obtain a map $T(i)^{-1}: T(M'') \rightarrow M'$. By adjunction, we obtain a map $M'' \rightarrow S(M')$, which is easily seen to split the above sequence. Thus $\text{Ext}_A^1(N, S(M')) = 0$, which shows that M' is saturated.

(e) Applying T to the map $f: M \rightarrow S(T(M))$ yields an isomorphism, and so the kernel and cokernel of f is torsion. Thus f is injective if and only if M is torsion-free. Assume now that M is torsion-free. Then we have a short exact sequence

$$0 \rightarrow M \rightarrow S(T(M)) \rightarrow N \rightarrow 0$$

with N torsion. If $N \neq 0$ then this sequence is not split (since $S(T(M))$ is torsion), and so M is not saturated. If $N = 0$ then M is saturated by (d). \square

Corollary 4.2.5. *We have $S(L_\lambda) = L_\lambda^0$ and $S(Q_\lambda) = A \otimes \mathbf{S}_\lambda$.*

Proof. These follow from Proposition 4.2.4(e) and the results of §4.1. \square

Corollary 4.2.6. *The module $A \otimes \mathbf{S}_\lambda$ is both projective and injective.*

Proof. Injectivity follows from Corollary 4.2.5 and Proposition 4.2.4(b). \square

Corollary 4.2.7. *The functor S takes injective objects of Mod_K to projective objects of Mod_A .*

Proof. This follows since every injective object of Mod_K is a direct sum of Q_λ 's. \square

Remark 4.2.8. It is possible to give a different definition of S . From Proposition 2.2.5, we see that the natural map

$$\mathrm{Hom}_A(A \otimes \mathbf{S}_\lambda, A \otimes \mathbf{S}_\mu) \rightarrow \mathrm{Hom}_K(Q_\lambda, Q_\mu)$$

is a bijection. Since every projective object of Mod_A is a direct sum of $A \otimes \mathbf{S}_\lambda$'s, and every injective object of Mod_K is a direct sum of Q_λ 's, we see that

$$T: \underline{\mathcal{P}}(\mathrm{Mod}_A) \rightarrow \underline{\mathcal{I}}(\mathrm{Mod}_K)$$

is an equivalence of categories, where $\underline{\mathcal{P}}$ (resp. $\underline{\mathcal{I}}$) denotes the full subcategory on projectives (resp. injectives). We can therefore define

$$S': \underline{\mathcal{I}}(\mathrm{Mod}_K) \rightarrow \underline{\mathcal{P}}(\mathrm{Mod}_A)$$

as the inverse of T . This functor induces an equivalence of categories

$$\mathrm{RS}': \mathrm{D}^b(\mathrm{Mod}_K) \rightarrow \mathrm{Perf},$$

where Perf is the full subcategory of $\mathrm{D}^b(\mathrm{Mod}_A)$ on complexes quasi-isomorphic to a bounded complex of projectives, and we can define S' on all of Mod_K by taking cohomology of RS' . One can show that S and S' are isomorphic. The advantage of this construction is that it is immediate and explicit: there is no question of existence, and it is more amenable to computation; for example, it is clear that $S'(Q_\lambda) = A \otimes \mathbf{S}_\lambda$. The disadvantage is that it is not clear that S' is the adjoint of T . \square

Remark 4.2.9. To give a feel for this material in a familiar setting, let us examine the case of $\mathbf{C}[t]$ -modules. Let \mathcal{A} denote the category of finitely generated nonnegatively graded $\mathbf{C}[t]$ -modules and let \mathcal{B} be the quotient of \mathcal{A} by the subcategory of torsion modules. Let $T: \mathcal{A} \rightarrow \mathcal{B}$ be the natural functor. Let F be the module $\mathbf{C}[t]$, let $F[n]$ denote the graded shifted module and put $\overline{F} = T(F)$. Then \mathcal{B} is equivalent to Vec , every object being a direct sum of copies of \overline{F} . (Note that there is an injection $F[n] \rightarrow F$ with torsion cokernel, which induces an isomorphism $T(F[n]) \rightarrow T(F) = \overline{F}$.) The right adjoint of T exists, and we call it S . We have $S(\overline{F}) = F$. In particular, this shows that F is an injective object of \mathcal{A} , which can also be verified directly. As below, we say that an object M of \mathcal{A} is ‘‘saturated’’ if the map $M \rightarrow S(T(M))$ is an isomorphism. Clearly, M is saturated if and only if it is isomorphic to a direct sum of F 's. \square

4.3. Injective resolutions. Recall that an object M of an abelian category has finite injective dimension if it admits a resolution $M \rightarrow \mathbf{I}^\bullet$ where \mathbf{I}^\bullet is a finite length complex of injectives. The main result is the following.

Theorem 4.3.1. *Every object of Mod_A has finite injective dimension.*

We first require a lemma.

Lemma 4.3.2. *An injective object of $\mathrm{Mod}_A^{\mathrm{tors}}$ remains injective in Mod_A .*

Proof. First, let N and I be objects of Mod_A such that the maximal size of a partition in I is less than the minimal size of a partition in N . Then $\mathrm{Ext}_A^1(N, I) = 0$. Indeed, suppose that

$$0 \rightarrow I \rightarrow M \rightarrow N \rightarrow 0$$

is an extension. The hypotheses imply that this extension splits uniquely in \mathcal{V} . It is easy to see that this splitting is a map of A -modules.

We now prove the lemma. Let I be an injective of $\mathrm{Mod}_A^{\mathrm{tors}}$, let n be the maximal size of a partition in I and let N be an arbitrary object of Mod_A . We have an exact sequence

$$0 \rightarrow \mathfrak{m}^{n+1}N \rightarrow N \rightarrow N/\mathfrak{m}^{n+1}N \rightarrow 0$$

and therefore an exact sequence

$$\mathrm{Ext}_A^1(N/\mathfrak{m}^{n+1}N, I) \rightarrow \mathrm{Ext}_A^1(N, I) \rightarrow \mathrm{Ext}_A^1(\mathfrak{m}^{n+1}N, I).$$

The leftmost group vanishes since I is injective in $\text{Mod}_A^{\text{tors}}$ and any extension of torsion modules is again torsion. The rightmost group vanishes by the previous paragraph. Thus the middle group vanishes, and so I is injective. \square

Proof of Theorem 4.3.1. From the equivalence $\text{Mod}_A^{\text{tors}} = \text{Mod}_K$ and our results on Mod_K , we see that $\text{Mod}_A^{\text{tors}}$ has enough injectives and every object has finite injective dimension. (This can be seen much more directly, in fact.) The lemma shows that an injective resolution in $\text{Mod}_A^{\text{tors}}$ remains an injective resolution in Mod_A , and so all torsion A -modules have finite injective dimension in Mod_A .

Now let M be an arbitrary A -module. Since every object of Mod_K has finite injective dimension, we can choose a finite injective resolution $T(M) \rightarrow \mathbf{I}^\bullet$ in Mod_K . Applying S , we obtain $S(T(M)) \rightarrow S(\mathbf{I}^\bullet)$ with $S(\mathbf{I}^\bullet)$ a complex of injectives. Since $TS = \text{id}$, the cohomology of this complex is torsion, and thus has finite injective dimension. It follows that $S(T(M))$ has finite injective dimension. Since the kernel and cokernel of $M \rightarrow S(T(M))$ are torsion, and thus of finite injective dimension, it follows that M has finite injective dimension. \square

Let I_λ be the injective envelope of \mathbf{S}_λ in $\text{Mod}_A^{\text{tors}}$.

Proposition 4.3.3. *Every injective object of Mod_A is a direct sum of modules of the form I_λ and $A \otimes \mathbf{S}_\lambda$.*

Proof. Let I be an injective object of Mod_A . It is clear that its maximal torsion subobject I_{tors} is an injective object of $\text{Mod}_A^{\text{tors}}$, and therefore a sum of I_λ 's. Since I_{tors} remains injective in Mod_A by Lemma 4.3.2, we have $I = I_{\text{tors}} \oplus I'$ with I' a torsion-free injective.

It thus suffices to show that a torsion-free injective I is a sum of modules of the form $A \otimes \mathbf{S}_\lambda$. Note that I is saturated (Proposition 4.1.3). For any saturated module M we have $\text{Hom}_A(N, M) = \text{Hom}_K(T(N), T(M))$, which shows that T takes saturated injective objects of Mod_A to injective objects of Mod_K . Thus $T(I)$ is injective in Mod_K , and therefore a sum of Q_λ 's. Finally, we see that $I = S(T(I))$ is a direct sum of modules of the form $S(Q_\lambda) = A \otimes \mathbf{S}_\lambda$. \square

4.4. Local cohomology. For an A -module M , define $H_m^0(M)$ to be the maximal torsion submodule of M . The functor $H_m^0: \text{Mod}_A \rightarrow \text{Mod}_A^{\text{tors}}$ is the right adjoint to the inclusion $\text{Mod}_A^{\text{tors}} \rightarrow \text{Mod}_A$. As H_m^0 is left exact, its derived functors H_m^i exist. We call H_m^\bullet the **local cohomology** functors. We write $\text{R}\Gamma_m$ for the derived functor of H_m^0 at the level of derived categories.

Proposition 4.4.1. *Let M be in Mod_A . Then $H_m^i(M)$ is a finitely generated torsion A -module for all i and zero for $i \gg 0$.*

Proof. Since M has a finite length resolution by finitely generated injectives we see that $H_m^i(M)$ is finitely generated for all i and zero for $i \gg 0$. Since H_m^0 takes values in $\text{Mod}_A^{\text{tors}}$ so do its derived functors. \square

Proposition 4.4.2. *Let M be an object of $\text{D}^b(A)$. Then we have a distinguished triangle*

$$\text{R}\Gamma_m(M) \rightarrow M \rightarrow \text{RS}(T(M)) \rightarrow$$

in $\text{D}^b(A)$, where the first two maps are the adjunctions.

Proof. It suffices to treat the case where M is a finite length complex of injectives. We have an exact sequence of complexes

$$0 \rightarrow H_m^0(M) \rightarrow M \rightarrow M' \rightarrow 0.$$

The leftmost complex is nothing other than $\text{R}\Gamma_m(M)$. By the structure of injective objects of Mod_A (Proposition 4.3.3), the natural map $M' \rightarrow S(T(M))$ is an isomorphism. Thus M' is identified with $\text{RS}(T(M))$. This proves the proposition. \square

Corollary 4.4.3. *Let M be in Mod_A and let $M' = T(M)$ be its image in Mod_K . We have an exact sequence*

$$0 \rightarrow H_m^0(M) \rightarrow M \rightarrow S(M') \rightarrow H_m^1(M) \rightarrow 0$$

and isomorphisms

$$H_m^{i+1}(M) = R^i S(M')$$

for $i > 1$.

This statement should be compared with [Ei2, Proposition A1.11] which provides a comparison between local cohomology for local rings and sheaf cohomology on the punctured spectrum.

4.5. Torsion theories and decompositions of derived categories. In this section, we give a general result on decomposing derived categories which we apply in the next section to $D^b(A)$. Let \mathcal{A} be an abelian category. A **hereditary torsion theory** is a pair $(\mathcal{T}, \mathcal{F})$ of strict full subcategories, whose objects are called torsion and torsion-free respectively, satisfying the following axioms:

- We have $\text{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- For any $M \in \mathcal{A}$ there is a short exact sequence

$$0 \rightarrow M_t \rightarrow M \rightarrow M_f \rightarrow 0$$

with $M_t \in \mathcal{T}$ and $M_f \in \mathcal{F}$.

- Any subobject of an object in \mathcal{T} belongs to \mathcal{T} .

In this situation, \mathcal{T} is a Serre subcategory of \mathcal{A} . Furthermore, the assignment $M \mapsto M_t$ is the adjoint to the right inclusion $\mathcal{T} \rightarrow \mathcal{A}$, and therefore functorial.

Fix a torsion theory $(\mathcal{T}, \mathcal{F})$ on \mathcal{A} . We assume the following two conditions hold:

- Every object of \mathcal{A} has finite injective dimension.
- The inclusion $\mathcal{T} \rightarrow \mathcal{A}$ takes injective objects to injective objects.

If I is an injective object of \mathcal{A} then I_t is automatically injective in \mathcal{T} , and thus injective in \mathcal{A} by the above axiom, and so we have a (non-canonical) splitting $I = I_t \oplus I_f$; this shows that I_f is injective as well.

Let \mathcal{D} be the bounded derived category of \mathcal{A} , which we think of as the homotopy category of finite length complexes of injectives. Let \mathcal{D}_t (resp. \mathcal{D}_f) be the full subcategory of \mathcal{D} on complexes of torsion injective objects (resp. torsion-free injective objects). Define \mathcal{D}' to be the category of triples (X, Y, f) with $X \in \mathcal{D}_f$, $Y \in \mathcal{D}_f$ and $f \in \text{Hom}_{\mathcal{D}}(X, Y)$. A morphism $(X, Y, f) \rightarrow (X', Y', f')$ in \mathcal{D}' consists of morphisms $X \rightarrow X'$ and $Y \rightarrow Y'$ in \mathcal{D} such that the obvious square commutes. The following is the main result of this section:

Proposition 4.5.1. *We have a canonical equivalence $\mathcal{D} = \mathcal{D}'$.*

Proof. We begin by defining a functor $\Phi: \mathcal{D} \rightarrow \mathcal{D}'$. Let I be an object of \mathcal{D} . We have a short exact sequence of complexes

$$0 \rightarrow I_t \rightarrow I \rightarrow I_f \rightarrow 0.$$

Our hypotheses imply that this sequence splits termwise, i.e., we can find a splitting $s: I_f \rightarrow I$, which may not be compatible with differentials. Put $f = sd - ds$. Then one readily verifies that f defines a map of complexes $f: I_f \rightarrow I_t[1]$. A different choice of s results in a homotopic f , so f as an element of $\text{Hom}_{\mathcal{D}}(I_f, I_t[1])$ depends only on I . We put $X = I_f$, $Y = I_t[1]$ and $\Phi(I) = (X, Y, f)$.

Suppose now that $\varphi: I \rightarrow I'$ is a map in \mathcal{D} . Then φ induces maps $\psi_X: X \rightarrow X'$ and $\psi_Y: Y \rightarrow Y'$. Choose termwise sections $s: I_f \rightarrow I$ and $s': I'_f \rightarrow I'$ and let f and f' be the resulting maps. Then

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \psi_X \downarrow & & \downarrow \psi_Y \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes up to the homotopy $\varphi s - s' \varphi: X \rightarrow Y'[-1]$. Thus φ induces a map $\psi: (X, Y, f) \rightarrow (X', Y', f')$ in \mathcal{D}' . If φ is replaced by a homotopic map, then so is ψ . This completes the definition of Φ .

We now define a functor $\Psi: \mathcal{D}' \rightarrow \mathcal{D}$. Thus suppose that (X, Y, f) is an object of \mathcal{D}' . Let I be the cone of f , shifted by -1 . Precisely, $I^n = X^n \oplus Y^{n-1}$, with the differential being defined in the usual manner $(x, y) \mapsto (d(x), f(x) - d(y))$. Then I is an object of \mathcal{D} , and we put $\Psi(X, Y, f) = I$.

Now suppose that $\psi: (X, Y, f) \rightarrow (X', Y', f')$ is a map in \mathcal{D}' . Choose actual maps of complexes $\tilde{\psi}_X, \tilde{\psi}_Y, \tilde{f}$ and \tilde{f}' representing the given homotopy classes of maps. Choose a map $h: X \rightarrow Y'[-1]$ such that $\tilde{\psi}_Y \tilde{f} - \tilde{f}' \tilde{\psi}_X = dh + hd$. Define a map $\varphi: M \rightarrow M'$ by letting φ^n be the map

$$X^n \oplus Y^{n-1} \rightarrow (X')^n \oplus (Y')^{n-1}, \quad (x, y) \mapsto (\tilde{\psi}_X(x), \tilde{\psi}_Y(y) + h(x)).$$

Then φ is a map of complexes. Furthermore, up to homotopy it is independent of all choices. This completes the definition of Ψ .

Now we show that Φ and Ψ are quasi-inverse to one another. Start with an object $I \in \mathcal{D}$. We choose the splitting $I = I_t \oplus I_f$. Then the differential can be written as a sum of three maps $d_t: I_t^\bullet \rightarrow I_t^{\bullet+1}$, $d_f: I_f^\bullet \rightarrow I_f^{\bullet+1}$, and $u: I_f^\bullet \rightarrow I_t^{\bullet+1}$. (Here we use that $\text{Hom}(I_t^n, I_f^m) = 0$.) The map $sd - ds$ can be taken to be u in the construction of Φ . When we apply Ψ to $(I_f, I_t[1], u)$, we get back I , so $\Psi\Phi \simeq \text{id}_{\mathcal{D}}$. Running this argument backwards shows that $\Phi\Psi \simeq \text{id}_{\mathcal{D}'}$. \square

Remark 4.5.2. The category \mathcal{D} has a standard t-structure $\mathcal{D}^{\leq 0}$ consisting of objects M for which $H^i(M) = 0$ for $i > 0$. Under the above equivalence, an object (X, Y, f) of \mathcal{D}' belongs to $\mathcal{D}^{\leq 0}$ if $H^0(f)$ is surjective and $H^i(f)$ is an isomorphism for $i > 0$. \square

Example 4.5.3. Let \mathcal{A} be the category of finitely generated nonnegatively graded $\mathbf{C}[t]$ -modules, let \mathcal{T} be the full subcategory on torsion (finite length) objects and let \mathcal{F} be the full subcategory on torsion-free (projective) objects. Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in the above sense, and satisfies the additional axioms we gave. The indecomposable torsion injective objects are of the form $\mathbf{C}[t]/t^n$. There is a single (up to isomorphism) indecomposable torsion-free injective, which is $\mathbf{C}[t]$. The category \mathcal{D}_t is equivalent to the derived category of \mathcal{T} (as is always the case), while \mathcal{D}_f is equivalent to $D^b(\text{Vec})$, the bounded derived category of vector spaces; the equivalence $D^b(\text{Vec}) \rightarrow \mathcal{D}_f$ is given by $-\otimes \mathbf{C}[t]$. If $V \in D^b(\text{Vec})$ and $M \in \mathcal{D}_t$ then giving a map $V \otimes \mathbf{C}[t] \rightarrow M$ in \mathcal{D} is the same as giving a map $V \rightarrow M_0$ in $D^b(\text{Vec})$, where M_0 is the degree 0 piece of M . Thus the category \mathcal{D}' can be described equivalently as follows: its objects are triple (V, M, f) , with $V \in D^b(\text{Vec})$, $M \in \mathcal{D}_t$ and $f \in \text{Hom}_{D^b(\text{Vec})}(V, M_0)$. The equivalence $\mathcal{D} = \mathcal{D}'$ provided by Proposition 4.5.1 gives a description of \mathcal{D} which involves only the derived category of torsion objects and the derived category of vector spaces. This is the sort of equivalence we seek for $D^b(A)$ in the following section. \square

4.6. Description of $D^b(A)$. Let $D^b(A)$, $D^b(K)$ and $D_{\text{tors}}^b(A)$ denote the bounded derived categories of Mod_A , Mod_K and $\text{Mod}_A^{\text{tors}}$ respectively. Let Perf (resp. Tors) be the full subcategory of $D^b(A)$ on objects represented by complexes of projective (resp. torsion) objects. We have functors

$$D_{\text{tors}}^b(A) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\text{RF}_m} \end{array} D^b(A) \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{\text{RS}} \end{array} D^b(K)$$

The left functors give mutually quasi-inverse equivalences between $D_{\text{tors}}^b(A)$ and Tors , while the right functors give mutually quasi-inverse equivalences between Perf and $D^b(K)$. We note that $\text{Hom}_{D^b(A)}(Y, X) = 0$ if $X \in \text{Perf}$ and $Y \in \text{Tors}$. To see this, note that X is a complex of injectives (Corollary 4.2.6), so the Hom can be computed in the category of chain complexes of A -modules, where it follows from the fact that each term of X is torsion-free. Our goal is to describe $D^b(A)$ purely in terms of $D^b(K)$.

To begin with, let \mathcal{D}' be the category whose objects are triples (X, Y, f) with $X \in \text{Perf}$, $Y \in \text{Tors}$ and $f \in \text{Hom}_{D^b(A)}(X, Y)$; morphisms are defined in the obvious manner (see §4.5).

Proposition 4.6.1. *We have an equivalence of categories $D^b(A) = \mathcal{D}'$.*

Proof. This follows immediately from Proposition 4.5.1, taking \mathcal{T} (resp. \mathcal{F}) to be the category of torsion (resp. torsion-free) objects. We have previously verified that all axioms are satisfied (the relevant results are Theorem 4.3.1 and Lemma 4.3.2). \square

Remark 4.6.2. The equivalence $D^b(A) \rightarrow \mathcal{D}'$ takes an object M to a triple (X, Y, f) . We can describe X and Y as follows:

$$X = \mathbf{R}S(T(M)), \quad Y = \mathbf{R}\Gamma_{\mathfrak{m}}(M)[1].$$

Thus “derived saturation” projects $D^b(A)$ onto Perf while local cohomology projects $D^b(A)$ onto Tors . \square

Proposition 4.6.1 is a first step towards describing $D^b(A)$, but is not a satisfactory answer, as the Hom sets in $D^b(A)$ appear in the definition of \mathcal{D}' . We now give a description of these Hom sets that is internal to $D^b(K)$. For this, let $\Phi: \text{Mod}_K \rightarrow \text{Mod}_A^{\text{tors}}$ be the equivalence of §2.5. Then:

Proposition 4.6.3. *For $M, N \in D^b(K)$ we have a natural isomorphism*

$$\text{Hom}_{D^b(A)}(\mathbf{R}S(M), \Phi(N)) = \text{Hom}_{D^b(K)}(N, M)^*.$$

Proof. Let N be an injective object of Mod_K . We begin by defining a map

$$\text{Tr}_N: \text{Hom}_A(S(N), \Phi(N)) \rightarrow \mathbf{C}.$$

Suppose first that $N = Q_\lambda$. Let \mathcal{K} be as in §2.5, and note that $\Phi = \Phi'S$. We have a natural isomorphism

$$\begin{aligned} \text{Hom}_A(S(Q_\lambda), \Phi(Q_\lambda)) &= \text{Hom}_A(S(Q_\lambda), \text{Hom}_A(S(Q_\lambda), \mathcal{K})^\vee) \\ &= \text{Hom}_A(S(Q_\lambda), \mathbf{S}_\lambda) \\ &= \text{Hom}_{\mathcal{V}}(\mathbf{S}_\lambda, \mathbf{S}_\lambda) = \mathbf{C}, \end{aligned}$$

since we have a canonical identification $S(Q_\lambda) = \mathbf{S}_\lambda \otimes A$. We define Tr_N to be the composite. Now suppose that $N = \bigoplus_\lambda U_\lambda \otimes Q_\lambda$, where the U_λ are multiplicity spaces, almost all of which are zero. We define Tr_N to be the composition

$$\text{Hom}_A(S(N), \Phi(N)) \rightarrow \bigoplus_\lambda \text{End}(U_\lambda) \otimes \text{Hom}_A(S(Q_\lambda), \Phi(Q_\lambda)) \rightarrow \mathbf{C},$$

where the second map is the trace map on $\text{End}(U_\lambda)$ combined with Tr_{Q_λ} . Since $\text{Aut}(N) \subset \mathbf{GL}(\bigoplus_\lambda U_\lambda)$ is block upper-triangular, it is clear that Tr_N is invariant under $\text{Aut}(N)$. Thus given a general injective object N of Mod_K , we can choose an isomorphism $N = \bigoplus_\lambda U_\lambda \otimes Q_\lambda$ and apply the above definition, and the result is independent of the choice of isomorphism.

Suppose now that N and M are injectives of Mod_K . We have a pairing

$$\begin{aligned} \text{Hom}_K(N, M) \otimes \text{Hom}_A(S(M), \Phi(N)) &\rightarrow \text{Hom}_A(S(N), S(M)) \otimes \text{Hom}_A(S(M), \Phi(N)) \\ &\rightarrow \text{Hom}_A(S(N), \Phi(N)) \\ &\rightarrow \mathbf{C}. \end{aligned}$$

The first map is S on the first factor, the second map is composition, the third map is the trace map. A computation using the definition of the trace map shows that this pairing is nondegenerate.

The trace map also extends to the case when N is a complex of injective objects and gives a nondegenerate pairing as above. This finishes the proof. \square

We now give our main result on $D^b(A)$. Define a category \mathcal{D}'' as follows. The objects of \mathcal{D}'' are triples (X, Y, γ) where X and Y are objects of $D^b(\text{Mod}_K)$ and γ is an element of $\text{Hom}_{D^b(K)}(Y, X)^*$.

A morphism $\varphi: (X, Y, \gamma) \rightarrow (X', Y', \gamma')$ consists of morphisms $\varphi_X: X \rightarrow X'$ and $\varphi_Y: Y \rightarrow Y'$ in $D^b(K)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y', X) & \xrightarrow{\varphi_Y^*} & \mathrm{Hom}(Y, X) \\ (\varphi_X)_* \downarrow & & \downarrow \gamma \\ \mathrm{Hom}(Y', X') & \xrightarrow{\gamma'} & \mathbf{C} \end{array}$$

commutes. We then have:

Theorem 4.6.4. *There is a natural equivalence of categories $D^b(A) = \mathcal{D}''$.*

Proof. We have an equivalence of categories

$$\mathcal{D}'' \rightarrow \mathcal{D}', \quad (X, Y, \gamma) \mapsto (\mathrm{RS}(X), \Phi(Y), f),$$

where f corresponds to γ under the isomorphism given in Proposition 4.6.3. The result now follows from Proposition 4.6.1. \square

Question 4.6.5. *Remark 4.5.2 gives a description of the standard t -structure on \mathcal{D}' . Is there a nice description of this t -structure on \mathcal{D}'' ?*

4.7. Rigidity of Mod_A . We now show that the category Mod_A has as few symmetries as possible.

Proposition 4.7.1. *The auto-equivalence group of Mod_A is trivial.*

Proof. Let \mathcal{I}_0 be the category of indecomposable injectives in Mod_A . By Proposition 4.3.3, there are two sorts of objects in this category: the injective envelopes I_λ of the simple objects \mathbf{S}_λ and the projective objects $P_\lambda = A \otimes \mathbf{S}_\lambda$. Define $\lambda \leq \mu$ if $\mu/\lambda \in \mathrm{HS}$. For $\lambda \leq \mu$ we can choose nonzero maps

$$f_{\mu,\lambda}: I_\mu \rightarrow I_\lambda, \quad g_{\mu,\lambda}: P_\mu \rightarrow P_\lambda, \quad h_{\lambda,\mu}: P_\lambda \rightarrow I_\mu$$

such that when $\lambda \leq \mu \leq \nu$ we have

$$f_{\nu,\lambda} = f_{\nu,\mu} f_{\mu,\lambda}, \quad g_{\nu,\lambda} = g_{\nu,\mu} g_{\mu,\lambda}, \quad h_{\mu,\nu} = h_{\lambda,\nu} g_{\mu,\lambda}, \quad h_{\lambda,\mu} = f_{\nu,\mu} h_{\lambda,\nu}.$$

That we can choose f and g so that the first two equations hold follows from the contractibility of $N(S)$ (see §3.2). Next, note that there is a canonical choice for $h_{\lambda,\lambda}$ and then the definition of $h_{\lambda,\mu}$ is forced by either of the third or fourth equation above. The third and fourth equations then follow in general.

Let F be an auto-equivalence of Mod_A . Then F induces an auto-equivalence of \mathcal{I}_0 and takes the I 's to I 's (since they are the finite length injectives) and the P 's to P 's. Since $\mathrm{Mod}_A^{\mathrm{tors}}$ and Mod_K have no auto-equivalences, we can assume that F is the identity on the I 's, P 's, f 's and g 's. We have $F(h_{\lambda,\lambda}) = \alpha_\lambda h_{\lambda,\lambda}$ for some scalar α_λ . Suppose $\lambda \leq \mu$. Applying F to the commutative square

$$\begin{array}{ccc} P_\lambda & \xrightarrow{h_{\lambda,\lambda}} & I_\lambda \\ g_{\mu,\lambda} \uparrow & & \uparrow f_{\mu,\lambda} \\ P_\mu & \xrightarrow{h_{\mu,\mu}} & I_\mu \end{array}$$

shows that $\alpha_\lambda = \alpha_\mu$. We conclude that $\alpha_\lambda = \alpha$ is independent of λ . We now obtain an isomorphism $\varphi: \mathrm{id} \rightarrow F$ by defining $\varphi(P_\lambda)$ to be the identity but $\varphi(I_\lambda)$ to be scaling by α . This isomorphism extends to one on all of Mod_A by Lemma 3.1.5. \square

Remark 4.7.2. It is also true that there are no equivalences $\mathrm{Mod}_A \rightarrow \mathrm{Mod}_A^{\mathrm{op}}$: indeed, every object of Mod_A has finite injective dimension, but some objects have infinite projective dimension. \square

Proposition 4.7.3. *The automorphism group of the identity functor of Mod_A is \mathbf{C}^\times .*

Proof. The proof is the same as the proof of Proposition 4.7.1. \square

4.8. **K-theory of Mod_A .** We now compute the Grothendieck group of Mod_A .

Proposition 4.8.1. *The group $K(\text{Mod}_A)$ is a free module of rank 2 over $K(\mathcal{V}_f)$ with basis $\{[\mathbf{C}], [A]\}$.*

Proof. This follows immediately from finiteness of injective dimension (Theorem 4.3.1) and the classification of injectives (Proposition 4.3.3). \square

The following proposition is similar to the above, but is more convenient in applications.

Proposition 4.8.2. *Let \mathcal{P} be a property of finitely generated A -modules. Suppose that:*

(a) *Given a short exact sequence*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

in which \mathcal{P} holds on two of the modules, it holds on the third.

(b) *The property \mathcal{P} holds for all simple modules \mathbf{S}_λ .*

(c) *The property \mathcal{P} holds for all free modules $\mathbf{S}_\lambda \otimes A$.*

Then \mathcal{P} holds for all finitely generated A -modules.

Proof. Again, this follows immediately from finiteness of injective dimension (Theorem 4.3.1) and the classification of injectives (Proposition 4.3.3). \square

Part 2. Invariants of A -modules

5. HILBERT SERIES

Let M be an object of \mathcal{V}_{gf} , regarded as a sequence $M = (M_n)_{n \geq 0}$ of representations of symmetric groups S_n . We define the Hilbert series of M as

$$H_M(t) = \sum_{n \geq 0} \dim(M_n) \frac{t^n}{n!}.$$

We then have the following result [Sno, Thm. 3.1]: if M is a finitely generated module over a twisted commutative algebra generated in degree 1 then $H_M(t)$ is a polynomial in t and e^t . For finitely generated A -modules, the proof shows that only the first power of e^t can appear, i.e., the Hilbert series is of the form $p(t)e^t + q(t)$ where p and q are polynomials. Our goal in this section is to generalize this result in two very different ways.

First, we define “enhanced” Hilbert series, which not only record the dimensions of the M_n , but their structure as S_n -representations. The definition and rationality result is given in §5.1. In §5.2, we show that the “ p -part” of the enhanced Hilbert series is equivalent to the so-called “character polynomial” from the representation theory of symmetric groups. Using our theory, we give a conceptual derivation of a formula for character polynomials. In §5.3, we show that the “ q -part” of the enhanced Hilbert series is determined by local cohomology. This gives a conceptual description of the failure of the character polynomial to compute the actual character in small degrees.

Second, we introduce certain differential operators in §5.4 and show every module satisfies a differential equation. This is a sort of categorification of the rationality result from [Sno], and provides an analogue of homogeneous systems of parameters in a certain sense.

5.1. Enhanced Hilbert series. Let λ be a partition of n . We define the following notations.

- Let c_λ denote the conjugacy class of S_n corresponding to λ . We write $\text{Tr}(c_\lambda|M)$ for the trace of c_λ on M_n .
- We write t^λ for $t_1^{m_1(\lambda)} t_2^{m_2(\lambda)} \cdots$, where $m_i(\lambda)$ denotes the number of times i appears in λ .
- We write $\lambda!$ for $m_1(\lambda)! m_2(\lambda)! \cdots$.

Example 5.1.1. If $\lambda = (2, 1, 1, 1)$ then c_λ is the conjugacy class of transpositions in S_5 , $t^\lambda = t_1^3 t_2$, and $\lambda! = 3! \cdot 1! = 6$. \square

Let M be an object of \mathcal{V}_{gf} . The **enhanced Hilbert series** of M is the formal series in variables t_1, t_2, \dots given by

$$\tilde{H}_M(t) = \sum_{\lambda} \text{Tr}(c_{\lambda}|M) \frac{t^{\lambda}}{\lambda!}.$$

The isomorphism class of M as an object of \mathcal{V} is completely determined by \tilde{H}_M . The enhanced Hilbert series therefore has much more information in it than the usual Hilbert series; in fact, one can recover the usual Hilbert series directly from the enhanced Hilbert series by setting t_i to 0 for $i \geq 2$. The enhanced Hilbert series is multiplicative:

$$\tilde{H}_{M \otimes N} = \tilde{H}_M \tilde{H}_N$$

(see, for example, [Sta, Proposition 7.18.2]). Thus $M \mapsto \tilde{H}_M$ provides an isomorphism of rings $\mathbf{K}(\mathcal{V}_{\text{gf}}) \rightarrow \mathbf{Q}[[t_i]]$. Define

$$T_0 = \sum_{i \geq 1} t_i.$$

The basic fact on enhanced Hilbert series is the following theorem.

Theorem 5.1.2. *Let M be a finitely generated A -module. There is an integer n and two (unique) polynomials $p_M(t), q_M(t) \in \mathbf{Q}[t_1, \dots, t_n]$ so that $\tilde{H}_M(t) = p_M(t) \exp(T_0) + q_M(t)$.*

Proof. It is clear that \tilde{H}_M is a polynomial if $M = \mathbf{S}_{\lambda}$. Furthermore, a short calculation shows that $\tilde{H}_A(t) = \exp(T_0)$, and so multiplicativity shows that M satisfies the theorem if $M = \mathbf{S}_{\lambda} \otimes A$. Since \tilde{H} factors through K-theory, the theorem follows from Proposition 4.8.2. \square

Second proof of Theorem 5.1.2. Let n be bigger than the number of rows in any partition appearing in the presentation of M . Then $M(\mathbf{C}^n)$ has a finite $A(\mathbf{C}^n)$ -free resolution. When thought of as modules over A , the terms in this resolution are quotients of free modules $\mathbf{S}_{\lambda} \otimes A$ by the submodule spanned by all \mathbf{S}_{μ} with $\ell(\mu) > n$. So it is enough to prove that these quotients have rational enhanced Hilbert series.

If $\ell(\lambda) < n$, then there is no \mathbf{S}_{μ} in $\mathbf{S}_{\lambda} \otimes A$ with $\ell(\mu) > n$, so there is nothing to prove in this case. Otherwise, we have $\ell(\lambda) = n$. Denote this quotient by N . In this case, there exists a long exact sequence

$$0 \rightarrow N \rightarrow \mathbf{F}_{n-1} \rightarrow \mathbf{F}_{n-2} \rightarrow \dots \rightarrow \mathbf{F}_0 \rightarrow N' \rightarrow 0$$

where each \mathbf{F}_i is free over A , and N' has finite length [EFW, §3]. Since the theorem holds for free modules and finite length modules, we finish by using additivity of enhanced Hilbert series with respect to exact sequences. \square

Remark 5.1.3. We relate p to the character polynomial of M in Proposition 5.2.1, and q to the local cohomology of M in Proposition 5.3.1. In particular, under the equivalence of $\mathbf{D}^b(A)$ with the category \mathcal{D}' given in Proposition 4.6.1, the p -part of \tilde{H}_M depends only on the Perf-component of M , while the q -part depends only on the Tors-component of M . We show in Proposition 6.3.1 that the Fourier transform interchanges p and q , up to a sign, and so the two can be regarded as dual to each other. \square

Remark 5.1.4. If we define $\tilde{H}_M^*(t) = \sum_{\lambda} \text{Tr}(c_{\lambda}|M) t^{\lambda}$ and $T_0^* = \prod_{i \geq 1} (1 - t_i)^{-1}$, then the first proof of Theorem 5.1.2 shows that there are $p(t), q(t) \in \mathbf{Z}[t_1, \dots, t_n]$ so that $\tilde{H}_M^*(t) = p(t) T_0^* + q(t)$. \square

Remark 5.1.5. If we perform the substitution $t_i \mapsto ip_i$, where p_i denotes the i th power sum symmetric function $p_i = \sum_{n \geq 1} x_n^i$, then $\tilde{H}_M(t)$ becomes the Frobenius characteristic of the family of symmetric group representations $M = (M_n)_{n \geq 0}$ (see [Sta, §7.18]). \square

5.2. Character polynomials and p . Let M be a finitely generated A -module. Using the equality $\tilde{H}_M(t) = p_M(t) \exp(T_0) + q_M(t)$ afforded by Theorem 5.1.2, a simple computation shows that there is a polynomial $X_M(a_1, \dots, a_n)$ called the **character polynomial** of M , so that the quantity $\text{Tr}(c_\mu | M)$ is obtained from X_M by putting $a_i = m_i(\mu)$, at least for $|\mu|$ large enough. In fact, we have the following formula for X_M in terms of p_M . Define a linear map (“umbral substitution”)

$$\mathbf{Q}[t_i] \rightarrow \mathbf{Q}[a_i], \quad \prod_i t_i^{d_i} \mapsto \prod_i (a_i)_{d_i},$$

where $(x)_d = x(x-1)\cdots(x-d+1)$ is the falling factorial. This map is not a ring homomorphism. We denote the image of p under this map by $\downarrow p$. We then have:

Proposition 5.2.1. *In the above notation, $X_M = \downarrow p_M$.*

Observe that both X_M and p_M are unaffected if M is changed by a finite length object, and so both make sense for $M \in \text{Mod}_K$. In fact, p and X define linear maps from $\mathbf{K}(\text{Mod}_K)$ to $\mathbf{Q}[t_i]$ and $\mathbf{Q}[a_i]$, respectively. The map p is a ring homomorphism, while X is not. (To see that p is a ring homomorphism, note that $p_{A \otimes V} = \tilde{H}_V$ for $V \in \mathcal{V}$, and so p is multiplicative on the injective objects of Mod_K . Since these span the \mathbf{K} -group, the result follows.)

Write X^λ and p^λ in place of X_M and p_M when $M = L_\lambda$. Note that the polynomial X^λ describes the character of the irreducible representation $\mathbf{M}_{(N,\lambda)}$ for N sufficiently large; in fact, this is how character polynomials are usually thought of. Using our resolution for L_λ in Mod_K , we can give an explicit formula for X^λ . By Proposition 2.4.2, we have

$$p^\lambda = \sum_{\mu, \lambda/\mu \in \mathbf{VS}} (-1)^{|\lambda| - |\mu|} p_{\mathbf{S}_\mu \otimes A}.$$

Since $\tilde{H}_{\mathbf{S}_\mu \otimes A} = \tilde{H}_{\mathbf{S}_\mu} \tilde{H}_A = \tilde{H}_{\mathbf{S}_\mu} \exp(T_0)$, we have

$$p_{\mathbf{S}_\mu \otimes A} = \tilde{H}_{\mathbf{S}_\mu} = \sum_{|\mu|=n} \text{Tr}(c_\mu | \mathbf{M}_\lambda) \frac{t^\mu}{\mu!}.$$

For formulas for these traces, see [Mac, §I.7] or [Sta, §7.18]. Since $X^\lambda = \downarrow p^\lambda$, the above two formulas give an explicit formula for X^λ .

Remark 5.2.2. In fact, sharper results on character polynomials are known; see [Mac, Examples I.7.13–14] and [GG] for further details. In particular, a more efficient version of the formula for X^λ given above can be found in [GG, Proposition I.1]. We believe that our derivation of these facts from the structure of Mod_K is more conceptual than the usual approaches, which is why we have included this section despite the known results in the literature. \square

Example 5.2.3. (a) We have $[L_1] = [Q_1] - [Q_\emptyset]$. Since $Y^1(t) = t_1$ and $Y^\emptyset(t) = 1$, we get $X^1(a_1) = a_1 - 1$.

(b) We have $[L_{2,1}] = [Q_{2,1}] - [Q_2] - [Q_{1,1}] + [Q_1]$. We calculate the polynomials Y^μ :

$$Y^{2,1} = \frac{t_1^3}{3} - t_3, \quad Y^2 = \frac{t_1^2}{2} + t_2, \quad Y^{1,1} = \frac{t_1^2}{2} - t_2, \quad Y^1 = t_1.$$

Hence $X^{2,1}(a_1, a_2, a_3) = (a_1)_3/3 - a_3 - (a_1)_2 + a_1$. \square

5.3. Local cohomology and q . Recall from §4.4 that $\mathbf{H}_\mathfrak{m}^0(M)$ denotes the torsion submodule of M and $\mathbf{R}^i \Gamma_\mathfrak{m} = \mathbf{H}_\mathfrak{m}^i$ denotes the i th derived functor of $\mathbf{H}_\mathfrak{m}^0$.

Proposition 5.3.1. *Let M be an object of Mod_A . Then*

$$q_M(t) = \tilde{H}_{\mathbf{R}\Gamma_\mathfrak{m}(M)}(t) = \sum_{i \geq 0} (-1)^i \tilde{H}_{\mathbf{H}_\mathfrak{m}^i(M)}(t),$$

with q_M as in Theorem 5.1.2.

This is an analogue of [Ei2, Corollary A1.15] which expresses the difference between the Hilbert polynomial and Hilbert function of a graded module over a polynomial ring in terms of the Euler characteristic of its local cohomology modules.

Proof. Each side factors through K-theory, so it suffices to check the proposition when M is either a simple module \mathbf{S}_λ or a free module $A \otimes \mathbf{S}_\lambda$. In the first case, $H_m^0(M) = M$ while $H_m^i(M) = 0$ for $i > 0$, which proves the statement. In the second case, $H_m^i(M) = 0$ for all i , while $\tilde{H}_M(t) = \tilde{H}_{\mathbf{S}_\lambda} \exp(T_0)$, which shows that $q = 0$, and the statement follows. \square

5.4. Differential operators. The category \mathcal{V} admits a derivation $\mathbf{D}: \mathcal{V} \rightarrow \mathcal{V}$ called the ‘‘Schur derivative.’’ We describe this in the two models mentioned in §1. First suppose that F is a representation of \mathbf{GL}_∞ . Then $\mathbf{D}F$ is the representation assigning to a vector space V the subspace of $F(V \oplus \mathbf{C})$ on which \mathbf{G}_m acts through its standard character. For example,

$$\bigwedge^n (V \oplus \mathbf{C}) = \bigwedge^n (V) \oplus \bigwedge^{n-1} (V) \otimes \mathbf{C},$$

and so $\mathbf{D}(\bigwedge^n) = \bigwedge^{n-1}$. Similarly, $\mathbf{D}(\text{Sym}^n) = \text{Sym}^{n-1}$.

Now suppose that $W = (W_n)$ is a sequence of representations of symmetric groups. Then $W' = \mathbf{D}W$ is the sequence with $W'_n = W_{n+1}|_{S_n}$, that is, W'_n is obtained by restricting the representation W_{n+1} of S_{n+1} to S_n . Notice that

$$H_{W'}(t) = \sum_{n \geq 0} \dim(W'_n) \frac{t^n}{n!} = \sum_{n \geq 0} \dim(W_{n+1}) \frac{t^n}{n!} = \frac{d}{dt} H_W(t).$$

In other words,

$$H_{\mathbf{D}W}(t) = \frac{d}{dt} H_W(t),$$

that is, the Schur derivative lifts the usual derivative on Hilbert series.

Here are some basic properties of the Schur derivative:

- Leibniz rule: $\mathbf{D}(W \otimes V)$ is naturally isomorphic to $\mathbf{D}(W) \otimes V \oplus W \otimes \mathbf{D}(V)$.
- Chain rule: $\mathbf{D}(F \circ G) = (\mathbf{D}(F) \circ G) \otimes \mathbf{D}(G)$, where \circ is composition of Schur functors.

Let M be an A -module. We then have a multiplication map

$$M_n \rightarrow M_{n+1},$$

which induces a map of A -modules

$$M \rightarrow \mathbf{D}M.$$

We define $\partial(M)$ to be the two-term complex $[M \rightarrow \mathbf{D}M]$. More generally, if M is a complex of A -modules, then the above procedure defines a map $M \rightarrow \mathbf{D}M$, and we define $\partial(M)$ to be the cone of this complex. This induces a triangulated functor

$$\partial: D^b(A) \rightarrow D^b(A),$$

where $D^b(A)$ is the bounded derived category of Mod_A .

Theorem 5.4.1. *Let $M \in D^b(A)$ be a complex. Then there are nonnegative integers n_1, n_2 so that $\partial^{n_1} \mathbf{D}^{n_2} M = 0$ in $D^b(A)$.*

Proof. Represent M as a complex. The statement is true for M if we can prove it is true for each term of M , so we can reduce to the case of a module. All finite length objects are killed by some power of \mathbf{D} . So using Proposition 4.8.2, we just need to show the statement for $\mathbf{S}_\lambda \otimes A$. Using the Leibniz rule, we have

$$\mathbf{D}(\mathbf{S}_\lambda \otimes A) = \mathbf{S}_\lambda \otimes \mathbf{D}(A) \oplus \mathbf{D}(\mathbf{S}_\lambda) \otimes A = (\mathbf{S}_\lambda \oplus \mathbf{D}(\mathbf{S}_\lambda)) \otimes A,$$

and the map

$$\partial: \mathbf{S}_\lambda \otimes A \rightarrow \mathbf{D}(\mathbf{S}_\lambda \otimes A)$$

is an injection with cokernel $\mathbf{D}(\mathbf{S}_\lambda) \otimes A$. Now $\mathbf{D}(\mathbf{S}_\lambda)$ is a sum of Schur functors \mathbf{S}_μ where $|\mu| < |\lambda|$. It follows that $H^i(\partial(\mathbf{S}_\lambda \otimes A))$ is zero, except when $i = 1$, where it is a direct sum of smaller free modules. An easy inductive argument finishes the proof. \square

Remark 5.4.2. Let M be a graded $\mathbf{C}[t]$ -module in nonnegative degrees. Define $\mathbf{D}(M)$ to be the module supported in nonnegative degrees whose degree n piece is the degree $n + 1$ piece of M . Define $\partial(M)$ to be the two-term complex $[M \rightarrow \mathbf{D}M]$, where the map is multiplication by t . Then any finitely generated $\mathbf{C}[t]$ -module is annihilated by an operator of the form $\mathbf{D}^n \partial^m$ with $m = 0$ (if M is torsion) or $m = 1$ (otherwise).

Suppose M as above is non-torsion. Then the homology of $\partial(M)$ is torsion. This is a reflection of the fact that t forms a system of parameters for M . Thus Theorem 5.4.1 can be viewed as something analogous to the existence of a finite system of parameters for A -modules. \square

Remark 5.4.3. One might hope there is a common generalization of Theorems 5.1.2 and 5.4.1. We will address this in [SS3]. \square

6. KOSZUL DUALITY AND THE FOURIER TRANSFORM

In this section we introduce the Fourier transform, which is a certain modification of Koszul duality that is special to modules with \mathbf{GL}_∞ -actions. The definition and basic finiteness results are given in §6.1. In §6.2, we study the Poincaré series of modules in Mod_A . In §6.3, we explain how the Fourier transform gives a certain reciprocity between perfect and torsion complexes, and how this interchanges the p - and q -parts of enhanced Hilbert series. In §6.4, we explain how the Fourier transform can be transferred to $D^b(K)$. Finally in §6.5 we calculate some examples of this theory.

6.1. The Fourier transform. Let M be an A -module. Standard properties of the Koszul complex show that

$$\mathcal{T}_n(M) = \bigoplus_{p \geq 0} \text{Tor}_p^A(M, \mathbf{C})_{p+n}$$

is naturally a comodule over $\bigwedge \mathbf{C}\langle 1 \rangle$. It follows that $\mathcal{T}_n(M)^\vee$ is a module over $\bigwedge \mathbf{C}\langle 1 \rangle = A^\dagger$. Therefore,

$$\mathcal{F}_n(M) = (\mathcal{T}_n(M)^\vee)^\dagger$$

is a module over A . If

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is an exact sequence of A -modules, then, upon rearranging the long exact sequence in Tor , we obtain a long exact sequence

$$\cdots \rightarrow \mathcal{F}_1(M_1) \rightarrow \mathcal{F}_0(M_3) \rightarrow \mathcal{F}_0(M_2) \rightarrow \mathcal{F}_0(M_1) \rightarrow 0.$$

Standard properties of Koszul duality, reviewed in [SS2], show that all the maps in this sequence are A -linear. In fact, the \mathcal{F}_n are the homology of an equivalence of triangulated categories

$$\mathcal{F}: D(A)^{\text{op}} \rightarrow D(A)$$

which we call the **Fourier transform**.

Example 6.1.1. Let λ be a partition of size n . Using the Koszul complex, we see that $\mathcal{T}(\mathbf{S}_\lambda) = \mathcal{T}_n(\mathbf{S}_\lambda) = A^\dagger \otimes \mathbf{S}_\lambda$. Then $\mathcal{F}(\mathbf{S}_\lambda) = (A \otimes \mathbf{S}_{\lambda^\dagger})[-n]$ and $\mathcal{F}(A \otimes \mathbf{S}_\lambda) = \mathbf{S}_{\lambda^\dagger}[-n]$. (These examples suggest that $\mathcal{F}^2 = \text{id}$; in fact, this is true.) \square

We now have the following important result:

Theorem 6.1.2. *The A -module $\mathcal{F}_n(M)$ is finitely generated for all n and 0 for $n \gg 0$.*

Proof. If the statement is true for two terms in a short exact sequence then it is true for the third. Example 6.1.1 shows the theorem holds for \mathbf{S}_λ and $A \otimes \mathbf{S}_\lambda$, and so the theorem holds in general by Proposition 4.8.2. \square

Corollary 6.1.3. *Every object of Mod_A has finite regularity.*

Proof. This is just unwinding definitions: the regularity of M is the supremum of n for which $\mathcal{F}_n(M)$ is nonzero. \square

Corollary 6.1.4. *The Fourier transform gives an equivalence $\mathcal{F}: D^b(A)^{\text{op}} \rightarrow D^b(A)$.*

Proof. Theorem 6.1.2 shows that \mathcal{F} takes bounded complexes to bounded complexes. \square

Remark 6.1.5. If M has regularity d , then the truncated submodule $M_{\geq d}$ has a linear resolution, i.e., if \mathbf{F}_\bullet is its minimal free resolution, then \mathbf{F}_i is generated in degree $d + i$. This follows from the interpretation of the generators of \mathbf{F}_i as Tor modules, which can be calculated using the Koszul complex (see [EG, Proposition 1.1]). \square

Remark 6.1.6. The proof of [EFS, Theorem 3.1] can be extended to show that the linear strands of a resolution of any finitely generated A -module are eventually exact. \square

6.2. **Poincaré series.** Given an A -module M , define its **Poincaré series** by

$$P_M(t, q) = \sum_{n \geq 0} (-q)^n H_{\text{Tor}_n^A(M, \mathbf{C})}(t).$$

By setting $q = 1$ in the Poincaré series and multiplying by $H_A(t) = e^t$, one recovers the Hilbert series. Note that the Poincaré series of M has nontrivial information about the A -module structure on M , whereas the Hilbert series of M is only defined in terms of the underlying object of \mathcal{V} . The Poincaré series does *not* factor through K-theory, so it cannot be studied directly via Proposition 4.8.2. Nonetheless, as a corollary of the results in the preceding section, we have:

Proposition 6.2.1. *The Poincaré series is of the form $f(t, q) + g(t, q)e^{-tq}$ where f and g are polynomials in t and Laurent polynomials in q .*

Proof. A simple manipulation gives $P_M(t, q) = \sum_{k \geq 0} q^{-k} H_{\mathcal{F}_k(M)}(-qt)$. Since each $\mathcal{F}_k(M)$ is a finitely generated A -module, its Hilbert series is of the form $a(t) + b(t)e^t$ for polynomials a and b . As the sum is finite, the result follows. \square

Remark 6.2.2. The above proof applies equally well to the “enhanced Poincaré series.” \square

Remark 6.2.3. Although P_M only has nonnegative powers of q in its power series expansion, one may need negative powers to express P_M in terms of elementary functions. See (6.5.1) for a specific example. \square

6.3. **Reciprocity.** The following proposition describes how the Fourier transform interacts with enhanced Hilbert series.

Proposition 6.3.1. *Let M be an A -module. Then $\tilde{H}_M(t) = \exp(T_0) \tilde{H}_{\mathcal{F}(M)}(-t)$. In particular, writing $\tilde{H}_M = p_M \exp(T_0) + q_M$, we have $p_{\mathcal{F}(M)}(t) = q_M(-t)$ and $q_{\mathcal{F}(M)}(t) = p_M(-t)$.*

Proof. For notational convenience, put $[M] = \tilde{H}_M$. The terms of the minimal free resolution of M are given by $A \otimes \text{Tor}_p^A(M, \mathbf{C})$, and so we have

$$[M] = \sum_{p \geq 0} (-1)^p [A][\text{Tor}_p^A(M, \mathbf{C})].$$

We thus have

$$[M] = [A] \sum_{n \geq 0} (-1)^n [\mathcal{T}_n(M)]^- = [A] \sum_{n \geq 0} (-1)^n [\mathcal{F}_n(M)]^{-, \dagger},$$

where $[-]^-$ is the map $t^\lambda \mapsto (-1)^{|\lambda|} t^\lambda$. Note that this is the same as $t_i \mapsto (-1)^i t_i$. If $V = (V_n)$ is an object of \mathcal{V} , thought of as a sequence of representations of symmetric groups, then $V^\dagger = (V_n \otimes \text{sgn})$. We have $\text{Tr}(c_\lambda | V_n \otimes \text{sgn}) = (-1)^k \text{Tr}(c_\lambda | V_n)$, where k is the number of even cycles in the cycle decomposition of λ . It follows that $[V^\dagger]$ is gotten from $[V]$ by changing t_i to $(-1)^{i+1} t_i$. Thus $[-]^{-,\dagger}$ is the operation $t_i \mapsto -t_i$, which completes the proof. \square

Remark 6.3.2. In fact, the Proposition 6.3.1 is just the shadow in K-theory of a more categorical result: we have a natural isomorphism

$$\mathcal{F}(\text{R}\Gamma_{\mathfrak{m}}(M)) = \text{RST}(\mathcal{F}(M))$$

for $M \in \text{D}^b(A)$. Note that Example 6.1.1 implies that \mathcal{F} induces an equivalence $\text{Perf}^{\text{op}} = \text{Tors}$. The above identification shows that \mathcal{F} interchanges the Perf and Tors components of M under the description of $\text{D}^b(A)$ given in Proposition 4.6.1. \square

6.4. The Fourier transform on $\text{D}^b(K)$. We have said in Remark 6.3.2 that \mathcal{F} induces an equivalence $\text{Perf}^{\text{op}} \rightarrow \text{Tors}$. On the other hand, we know that Perf is equivalent to $\text{D}^b(K)$ while Tors is equivalent to $\text{D}_{\text{tors}}^b(A)$, and that Mod_K and $\text{Mod}_A^{\text{tors}}$ are equivalent. It follows that we obtain an equivalence

$$\mathcal{F}_K: \text{D}^b(K)^{\text{op}} \rightarrow \text{D}^b(K), \quad \mathcal{F}_K(M) = \Psi(\mathcal{F}(\text{RS}(M))),$$

where Ψ is as in §2.5. One can show that if λ is a partition of size n then

$$\mathcal{F}_K(Q_\lambda) = L_{\lambda^\dagger}[-n], \quad \mathcal{F}_K(L_\lambda) = Q_{\lambda^\dagger}[-n]$$

and $\mathcal{F}_K^2 = \text{id}$. These formulas give an explanation for the symmetries in K-theory observed in Remark 2.4.6.

6.5. Example: EFW complexes. The resolutions discussed in this section are taken from [EFW, §3]. They also appeared earlier in [Olv, Theorem 8.11].

Let α be a partition and fix a positive number $e > 0$. For $i \geq 1$, define partitions $\alpha(i)$ by

$$\alpha(i)_j = \begin{cases} \alpha_1 + e & j = 1 \\ \alpha_{j-1} + 1 & 1 < j \leq i \\ \alpha_j & j > i \end{cases}$$

and set $\alpha(0) = \alpha$. By Pieri's rule, we have nonzero maps

$$d_i: \mathbf{S}_{\alpha(i)} \otimes A \rightarrow \mathbf{S}_{\alpha(i-1)} \otimes A$$

for all $i \geq 1$, which are unique up to scalar multiple, and furthermore, $d_{i-1}d_i = 0$. Set $\mathbf{F}_i = \mathbf{S}_{\alpha(i)} \otimes A$. As a consequence of [EFW, §3], the complex \mathbf{F}_\bullet is exact in positive degrees, and $M(\alpha, e) = \text{H}_0(\mathbf{F}_\bullet)$ is a finite length module. So the cokernel $\Omega^j M(\alpha, e)$ of the truncated complex $\mathbf{F}_\bullet^{\geq j}$ has depth j .

The module $M(\alpha, e)$ is generated in degree $|\alpha|$ and if $n = \ell(\alpha)$, then $M(\alpha, e)$ has regularity $|\alpha| + e - 1 + \alpha_1 - \alpha_n$.

Before we calculate the Poincaré series of $M(\alpha, e)$, we do a preliminary calculation. Let μ be a partition of size m and let (1^N) be a sequence of N 1's. The hook length formula shows that the dimension of $\mathbf{M}_{\mu+(1^N)}$ is

$$d_\mu(N) = \frac{(N+m)!}{\prod_{i=1}^m (N + \mu_i - i + 1) \cdot (N-m)! \cdot \prod_{b \in \mu} \text{hook}(b)},$$

which is a polynomial in N of degree m .

Let $\beta = \alpha(\ell(\alpha) + 1) - (1^{\ell(\alpha)+1})$. For $N > \ell(\alpha)$, we have $\alpha(N) = \beta + (1^N)$. Hence modulo the first $\ell(\alpha)$ terms, the Poincaré series of $M(\alpha, e)$ is

$$P_{M(\alpha, e)} \sim (-q)^{-|\beta|} \sum_{N > \ell(\alpha)} (-qt)^{N+|\beta|} \frac{d_\beta(N)}{(N+|\beta|)!}.$$

In particular, consider $\alpha = \emptyset$ and $e = 2$, so that $M(\emptyset, 2)$ is the quotient of A by the square of its maximal ideal. Then

$$(6.5.1) \quad P_{M(\emptyset, 2)} = 1 - q^{-1} \sum_{n \geq 2} \frac{(-qt)^n}{(n)!} (n-1) = (1 - q^{-1}) + (t + q^{-1})e^{-qt}.$$

For general $e \geq 2$, $M(\emptyset, e)$ is the quotient of A by the e th power of its maximal ideal, and

$$P_{M(\emptyset, e)} = 1 + \frac{(-q^{-1})^{e-1}}{(e-1)!} \sum_{n \geq e} \frac{(-qt)^n}{n!} (n-1)(n-2) \cdots (n-e+1)$$

Recall that the generating function of $x^d e^x$ is $\sum_{n \geq d} \frac{x^n}{n!} (n)_d$ where $(n)_d = n(n-1) \cdots (n-d+1)$ is the falling factorial. Hence if we want to simplify an expression of the form $\sum_n \frac{x^n}{n!} p(n)$ for some polynomial n , we have to convert p into the falling factorial basis.

In our situation, we have $p(n) = (n-1)_{e-1}$, which satisfies the identity

$$(n-1)_{e-1} = (n)_{e-1} - (e-1)(n-1)_{e-2}.$$

Expanding this, we can express it as

$$(n-1)_{e-1} = \sum_{i=0}^{e-1} (-1)^i (e-1)_i (n)_{e-1-i}.$$

So

$$\begin{aligned} P_{M(\emptyset, e)} &= f(t, q) + \frac{(-q^{-1})^{e-1}}{(e-1)!} \left(\sum_{i=0}^{e-1} (-1)^i (e-1)_i (-tq)^{e-1-i} \right) e^{-qt} \\ &= f(t, q) + \left(\sum_{i=0}^{e-1} \frac{t^{e-1-i} q^{-i}}{(e-1-i)!} \right) e^{-qt} \end{aligned}$$

for some Laurent polynomial $f(t, q)$ with $f(t, 1) = 0$.

7. DEPTH AND LOCAL COHOMOLOGY

In this section, we study an important homological invariant of modules: depth. We start in §7.1 by giving a definition of depth. In §7.2, we prove an analogue of Grothendieck's vanishing theorem for local cohomology. In §7.3, we make a comparison with a possible definition of local cohomology for modules over $\text{Sym}(\mathbf{C}^\infty)$ which might not have a \mathbf{GL}_∞ -equivariant structure and show that this definition agrees with the one we give for \mathbf{GL}_∞ -equivariant modules. Finally, in §7.4, we calculate the local cohomology of the modules $L_\lambda^{\geq D}$ from §2.2, and show that they lift well-known formulas in K-theory for character polynomials.

7.1. Depth. Given an A -module M , let $d_M(n)$ be the depth of $M(\mathbf{C}^n)$ with respect to the homogeneous maximal ideal $A(\mathbf{C}^n)_{>0}$. We will only consider $n \geq \ell_A(M)$.

Theorem 7.1.1. *Pick $M \in \text{Mod}_A$. If M is not of the form $V \otimes A$ for $V \in \mathcal{V}$, then $d_M(n)$ is a weakly decreasing function. In particular, the limit $\lim_{n \rightarrow \infty} d_M(n)$ exists, i.e., $d_M(n)$ is independent of n for $n \gg 0$.*

We call this limit the **depth** of M . We say that modules of the form $V \otimes A$ have infinite depth.

Proof. By the Auslander–Buchsbaum formula [Ei1, Theorem 19.9], we have

$$\text{pdim}_{A(\mathbf{C}^n)} M(\mathbf{C}^n) + d_M(n) = n,$$

so it is enough to show that $n \mapsto \text{pdim}_{A(\mathbf{C}^n)} M(\mathbf{C}^n)$ is a strictly increasing function.

Let \mathbf{K}_\bullet be the Koszul complex, i.e., $\mathbf{K}_i = \bigwedge^i \otimes A$. Then $\mathrm{Tor}_i^A(\mathbf{C}, M) = \mathrm{H}_i(\mathbf{K}_\bullet \otimes_A M)$ are Schur functors for all i , and in particular, its specialization to n variables gives $\mathrm{Tor}_i^{A(\mathbf{C}^n)}(\mathbf{C}, M(\mathbf{C}^n))$. This shows that $n \mapsto \mathrm{pdim}_{A(\mathbf{C}^n)} M(\mathbf{C}^n)$ is a weakly increasing function.

Now suppose that $\mathrm{pdim}_{A(\mathbf{C}^n)} M(\mathbf{C}^n) = \mathrm{pdim}_{A(\mathbf{C}^{n+1})} M(\mathbf{C}^{n+1})$ for some n . Let L be this common value and set $F = \ker(\mathbf{F}_{L-1} \rightarrow \mathbf{F}_{L-2})$ where \mathbf{F}_\bullet is the minimal free resolution of M over A . Then both $F(\mathbf{C}^n)$ and $F(\mathbf{C}^{n+1})$ are free modules over $A(\mathbf{C}^n)$ and $A(\mathbf{C}^{n+1})$, respectively. This means that F has the form $F' \oplus F''$ where F' is a free A -module generated by various \mathbf{S}_λ with $\ell(\lambda) < n$, and F'' is some (not necessarily free) module generated by \mathbf{S}_μ with $\ell(\mu) > n + 1$ (the restriction on the λ is a consequence of Pieri's formula).

Let C be the cokernel of $F' \rightarrow \mathbf{F}_{L-1}$. Then $\mathrm{pdim}_A C = 1$. Say that

$$0 \rightarrow \mathbf{G}_1 \rightarrow \mathbf{G}_0 \rightarrow C \rightarrow 0$$

is a minimal free resolution over A (note that $\mathbf{G}_1 = F'$ and $\mathbf{G}_0 \subseteq \mathbf{F}_{L-1}$). Then $\mathrm{rank}(\mathbf{G}_1(\mathbf{C}^n)) \leq \mathrm{rank}(\mathbf{G}_0(\mathbf{C}^n))$ for all n . However, $\dim_{\mathbf{C}} \mathbf{S}_\lambda(\mathbf{C}^n)$ is a polynomial in n of degree $|\lambda|$. Since \mathbf{G}_\bullet is minimal, there exists some generator of \mathbf{G}_1 whose partition has bigger size than a partition in any generator of \mathbf{G}_0 . So for sufficiently large n , we get $\mathrm{rank}(\mathbf{G}_1(\mathbf{C}^n)) > \mathrm{rank}(\mathbf{G}_0(\mathbf{C}^n))$, which is a contradiction.

In particular, $n \mapsto \mathrm{pdim}_{A(\mathbf{C}^n)} M(\mathbf{C}^n)$ is strictly increasing, which completes the proof. \square

Lemma 7.1.2. *Let*

$$0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$$

be an exact sequence of A -modules with F projective. Then $\mathrm{depth}(M) = \mathrm{depth}(N) + 1$.

Proof. If $\mathrm{depth}(N) = \infty$ then N is projective, and so M is projective and $\mathrm{depth}(M) = \infty$ as well. Assume now that $\mathrm{depth}(N)$ is finite. We have an exact sequence

$$0 \rightarrow M(\mathbf{C}^n) \rightarrow F(\mathbf{C}^n) \rightarrow N(\mathbf{C}^n) \rightarrow 0$$

for each n . We thus see that $d_M(n) = d_N(n) + 1$ if $n > \mathrm{depth}(N)$. The result follows. \square

7.2. Vanishing of local cohomology. The following result determines where local cohomology vanishes in general.

Proposition 7.2.1. *Let $M \in \mathrm{Mod}_A$. Then $\inf(\{d \mid \mathrm{H}_\mathfrak{m}^d(M) \neq 0\} \cup \{\infty\})$ is the depth of M , and $\sup(\{d \mid \mathrm{H}_\mathfrak{m}^d(M) \neq 0\} \cup \{0\}) - 1$ is the injective dimension of the localization $T(M) \in \mathrm{Mod}_K$.*

This is an analogue of Grothendieck's vanishing theorem [Ei2, Proposition A1.16], but we have swapped "dimension" with a different invariant. In between these two extrema, we show in Proposition 7.4.3 that the pattern of which local cohomology groups is nonzero can be anything. This can be viewed as an analogue of the corresponding fact for local cohomology of local rings (see [EP, Theorem A]).

Proof. Let M be a non-projective A -module. Say that M satisfies property (A_n) if $\mathrm{depth}(M) = n$ and property (B_n) if $\mathrm{H}_\mathfrak{m}^n(M) \neq 0$ but $\mathrm{H}_\mathfrak{m}^i(M) = 0$ for $i < n$. We show that (A_n) and (B_n) are equivalent, by induction on n .

We first consider the base case $n = 0$. If $\mathrm{H}_\mathfrak{m}^0(M) \neq 0$ then M has torsion, and so $M(\mathbf{C}^n)$ has torsion for $n \geq \ell(M)$, which implies that $M(\mathbf{C}^n)$ has depth 0 for $n \gg 0$, and so $\mathrm{depth}(M) = 0$. Thus (B_0) implies (A_0) . Conversely, suppose that $\mathrm{depth}(M) = 0$. Then $M(\mathbf{C}^n)$ has depth 0 for $n \geq N$ for some N . Let n be greater than N and $\ell(M) + 1$. As $M(\mathbf{C}^n)$ has depth 0, we can choose a nonzero \mathbf{GL}_n -stable subspace $V_0 \subset M(\mathbf{C}^n)$ which is annihilated by $\mathfrak{m}(\mathbf{C}^n)$. We have $V_0 = V(\mathbf{C}^n)$ for some unique subobject $V \subset M$ in \mathcal{V} . By hypothesis, the map $\mathfrak{m} \otimes V \rightarrow M$ is zero when evaluated on \mathbf{C}^n . Since both spaces have at most $\ell(M) + 1$ rows, the map is identically zero, and so V is annihilated by \mathfrak{m} . Thus V defines a nonzero subspace of $\mathrm{H}_\mathfrak{m}^0(M)$. Therefore (A_0) implies (B_0) .

Suppose now that M satisfies either (A_n) or (B_n) , with $n > 0$. Then M is torsion-free (this is clear if (B_n) holds; if (A_n) holds then (A_0) does not hold, and so (B_0) does not hold, and so $H_m^0(M)$ is zero). It follows that M injects into its saturation $S(T(M))$. We can choose an injection $T(M) \rightarrow I$ in Mod_K , for some injective object I . Let $F = S(I)$, a projective object of Mod_A . Since S is left exact, M injects into F . We thus have an exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0.$$

By Lemma 7.1.2, M satisfies (A_n) if and only if N satisfies (A_{n-1}) . By considering the long exact sequence in local cohomology, we also see that M satisfies (B_n) if and only if N satisfies (B_{n-1}) . Hence we finish by induction.

For the second part, the trivial cases when M is torsion or of the form $\mathbf{S}_\lambda \otimes A$ are clear from the definitions and Proposition 2.2.4. Otherwise, let \mathbf{I}^\bullet be a minimal injective resolution of $T(M)$ of length n . If $S(\mathbf{I}^{n-1}) \rightarrow S(\mathbf{I}^n)$ were surjective then it would be split, since S carries injectives to projectives; applying T we would find that $\mathbf{I}^{n-1} \rightarrow \mathbf{I}^n$ is split, contradicting minimality. Thus $S(\mathbf{I}^{n-1}) \rightarrow S(\mathbf{I}^n)$ is not surjective, and so $R^n S(T(M)) \cong H_m^{n+1}(M) \neq 0$. \square

Remark 7.2.2 (Coszygies). Let M be a finitely generated A -module which has positive finite depth d . As we saw in the above proof, we can choose an injection $M \rightarrow F$ with F projective, and we have $\text{depth}(F/M) = d - 1$. Iterating this process, we obtain a long exact sequence

$$0 \rightarrow M \rightarrow F_d \rightarrow F_{d-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M' \rightarrow 0$$

where each F_i is projective and $\text{depth}(M') = 0$. \square

7.3. Comparison with non-equivariant modules. Let A_0 denote the ring $A(\mathbf{C}^\infty) = \mathbf{C}[x_1, x_2, \dots]$, but regarded without any \mathbf{GL}_∞ -action or grading. We write \mathfrak{m}_0 for its maximal ideal. Let N be an A_0 -module. We define the local cohomology of N by

$$H_{\mathfrak{m}_0}^i(N) = \varinjlim_d \text{Ext}_{A_0}^i(A_0/\mathfrak{m}_0^d, N).$$

We define the depth of N to be the supremum of lengths of regular sequences on N . These are the usual definitions of these concepts, but they are not often applied in the setting of non-noetherian rings. Given an A -module M , we obtain an A_0 -module M_0 by simply forgetting the extra structure.

The purpose of this section is to compare the depth and local cohomology of an A -module with that of its underlying A_0 -module.

Proposition 7.3.1. *For an A -module M we have $\text{depth}(M) = \text{depth}(M_0)$.*

Before giving the proof, we require a lemma.

Lemma 7.3.2. *Let M be a \mathbf{GL}_n -equivariant module over $\text{Sym}(\mathbf{C}^n)$. The following statements are equivalent:*

- (a) *There exists a regular sequence of length d on M .*
- (b) *There exists d linearly independent elements of \mathbf{C}^n which form a regular sequence on M .*
- (c) *Every sequence of d linearly independent elements of \mathbf{C}^n forms a regular sequence on M .*

Proof. Assume $d > 0$ otherwise there is nothing to prove. The set of zerodivisors of M is a union of finitely many prime ideals, none of which is the maximal homogeneous ideal because we have a nonzerodivisor. Hence none of them contains the space of linear forms, and in particular, their union cannot contain the space of linear forms since we work over an infinite field. By induction on d , we see that (a) implies (b). Since every sequence of d linearly independent elements of \mathbf{C}^n is conjugate under \mathbf{GL}_n , we see that (b) implies (c). Finally, it is clear that (c) implies (a). \square

Proof of Proposition 7.3.1. We first show that $\text{depth}(M_0) \leq \text{depth}(M)$. Thus let (x_1, \dots, x_d) be a regular sequence on M_0 with $d = \text{depth}(M_0)$. Then each x_i belongs to $M(\mathbf{C}^n)$ for n sufficiently

large, and (x_1, \dots, x_d) forms a regular sequence on $M(\mathbf{C}^n)$. This shows that $d \leq d_M(n)$ for $n \gg 0$, and so $\text{depth}(M_0) \leq \text{depth}(M)$.

We now show that $\text{depth}(M) \leq \text{depth}(M_0)$. Let $d = \text{depth}(M)$. Let $x = (x_1, \dots, x_d)$ be a sequence of linearly independent elements of \mathbf{C}^∞ . By Lemma 7.3.2, x forms a regular sequence on all $M(\mathbf{C}^n)$ for $n \gg 0$. We claim that x forms a regular sequence on M_0 . Indeed, suppose that $x_i m = 0$ in $M_0/(x_1, \dots, x_{i-1})M_0$. The x_i 's and m belong to $M(\mathbf{C}^n)$ for $n \gg 0$, and since x is a regular sequence on $M(\mathbf{C}^n)$ for $n \gg 0$, we find $m = 0$. This proves the claim, and so $\text{depth}(M) \leq \text{depth}(M_0)$. \square

Proposition 7.3.3. *Given an A -module M , there is a natural isomorphism*

$$H_m^i(M) = H_{\mathfrak{m}_0}^i(M_0)$$

compatible with the \mathbf{GL}_∞ -action on each side.

For an analogous statement about local cohomology for local rings, see [Ei2, Proposition A1.1].

Proof. Put $G^i(M) = H_{\mathfrak{m}_0}^i(M_0)$. Then G^\bullet defines a δ -functor on Mod_A and G^0 coincides with H_m^0 . To prove the proposition, it suffices to show that $G^i(I) = 0$ for $i > 0$ whenever I is an indecomposable injective object of Mod_A . There are two kinds of indecomposable injectives of Mod_A : the injective envelopes of simple modules and modules of the form $\mathbf{S}_\lambda \otimes A$. Since G^\bullet is just the usual local cohomology of the ring A , the higher functors vanish on all torsion modules. This takes care of the first sort of indecomposable injective. We now observe that we have $G^i(\mathbf{S}_\lambda \otimes A) = \mathbf{S}_\lambda \otimes G^i(A)$, since the \mathbf{GL}_∞ -structure does not affect the computation of G^i . Therefore, it is enough to show that $G^i(A) = 0$ for $i > 0$.

This vanishing follows from the fact that A_0 has infinite depth. Since our situation is somewhat exotic, we give a proof. Let A_n be the quotient of A_0 by the ideal (x_1, \dots, x_n) . We have an exact sequence

$$0 \rightarrow A_n \xrightarrow{x_{n+1}} A_n \rightarrow A_{n+1} \rightarrow 0$$

and an associated long exact sequence after applying $H_{\mathfrak{m}_0}^\bullet$. This shows that if $H_{\mathfrak{m}_0}^i(A_n) = 0$ for $i < k$ but $H_{\mathfrak{m}_0}^k(A_n) \neq 0$ then $H_{\mathfrak{m}_0}^{k-1}(A_{n+1}) \neq 0$, if $k > 0$. (Note: local cohomology is always torsion, and so multiplication by an element of the maximal ideal on a nonzero local cohomology group is never injective.) In particular, if $H_{\mathfrak{m}_0}^k(A_0) \neq 0$ then $H_{\mathfrak{m}_0}^0(A_k) \neq 0$. However, none of the A_k have any torsion, and so $H_{\mathfrak{m}_0}^0(A_k) = 0$ for all k . Thus $G^i(A) = H_{\mathfrak{m}_0}^i(A_0) = 0$ for all i . \square

Remark 7.3.4. Propositions 7.2.1, 7.3.1, and 7.3.3 show that if N is a module over A_0 then $H_{\mathfrak{m}_0}^i(N) = 0$ for $i < \text{depth}(N)$, provided N can be endowed with a \mathbf{GL}_∞ -equivariant structure. \square

Remark 7.3.5. There are a few reasons we defined local cohomology as the derived functors of H_m^0 instead of first passing to A_0 and then using $H_{\mathfrak{m}_0}^i$. First, it does not seem to be clear that $H_{\mathfrak{m}_0}^i(M_0)$ belongs to \mathcal{V} , i.e., that the \mathbf{GL}_∞ -action on it is polynomial: on the one hand, an equivariant injective resolution of M_0 as an A_0 -module will involve non-polynomial modules; on the other, while one can equivariantly resolve A_0/\mathfrak{m}_0^d using polynomial modules, applying $\text{Hom}_{A_0}(-, M_0)$ destroys this property. Second, it is not clear $H_{\mathfrak{m}_0}^i(M_0) = 0$ for $i \gg 0$. And finally, it is not clear that $M \mapsto H_{\mathfrak{m}_0}^i(M_0)$ is the derived functor of $M \mapsto H_m^0(M_0)$. \square

Remark 7.3.6. The functors H_m^i and $\varinjlim \text{Ext}_A^i(A/\mathfrak{m}^d, -)$ are not isomorphic: they even differ for $i = 0$. \square

Remark 7.3.7. Consider the category $\text{Mod}_{\mathbf{C}[t]}$ of nonnegatively graded $\mathbf{C}[t]$ -modules. One can define H_m^0 on this category to be the maximal torsion submodule, as we did above. Its derived functors exist. However, the analogue of Proposition 7.3.3 is false in this setting. Indeed, $\mathbf{C}[t]$ is an injective object of $\text{Mod}_{\mathbf{C}[t]}$, and so $H_m^1(\mathbf{C}[t]) = 0$, while the usual theory of local cohomology shows

that $\varinjlim \text{Ext}_{\mathbf{C}[t]}^1(\mathbf{C}[t]/\mathfrak{m}^n, \mathbf{C}[t])$ is nonzero. This problem disappears if one considers all graded $\mathbf{C}[t]$ -modules. \square

7.4. Local cohomology and character polynomials. Consider the module $L_\lambda^0 = \bigoplus_{d \geq \lambda_1} \mathbf{S}_{(d, \lambda_1)}$ from §2.2 and its enhanced Hilbert series

$$\tilde{H}_{L_\lambda^0}(t) = p_{L_\lambda^0}(t) \exp(T_0) + q_{L_\lambda^0}(t).$$

We discussed the character polynomial X^λ in §5.2. In particular, we have that

$$X^\lambda(\mu) := X^\lambda(a_1, \dots, a_n) = \text{Tr}(c_\mu | \mathbf{M}_{(N-|\lambda|, \lambda)})$$

for any partition $\mu \vdash N$ with $m_i(\mu) = a_i$ and $N \gg 0$. By the determinantal expression for $\text{Tr}(c_\mu | \mathbf{M}_{(N-|\lambda|, \lambda)})$ discussed in [Mac, Example I.7.14], we see that it in fact holds for $N \geq \lambda_1 + |\lambda|$.

Actually, this determinantal expression implies more. Namely, it gives an interpretation for $X^\lambda(\mu)$ when $|\mu| < \lambda_1 + |\lambda|$. Note that the coefficient of t^μ in $\tilde{H}_{L_\lambda^0}$ vanishes, so this value of the character polynomial is related to the coefficient of t^μ in $q_{L_\lambda^0}$.

First, define the sequence $\rho = (-1, -2, -3, \dots)$. Given a permutation $w \in S_\infty$ and a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ with finitely many nonzero terms, define $w \bullet \alpha = w(\alpha + \rho) - \rho$ where the right-hand side is the usual permutation action on sequences. There are two mutually exclusive cases:

- (a) There is a non-identity permutation w such that $w \bullet \alpha = \alpha$ (in which case we say that α is **singular**), or
- (b) There is a unique permutation w such that $w \bullet \alpha$ is weakly decreasing (in which case we write $w \bullet \alpha \geq 0$).

Now we go back to character polynomials. Set $\alpha = (|\mu| - |\lambda|, \lambda)$. In case (a), $X^\lambda(\mu) = 0$. In case (b), let w be the unique permutation such that $w \bullet \alpha$ is weakly decreasing (and hence is a partition). Then

$$(7.4.1) \quad X^\lambda(\mu) = (-1)^{\ell(w)} \text{Tr}(c_\mu | \mathbf{M}_{w \bullet \alpha})$$

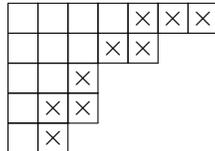
where $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$ is the number of inversions of w . In this case, we will write

$$\alpha \xrightarrow{\ell(w)} w \bullet \alpha.$$

This gives us a formula for the Euler characteristic of $\mathbf{H}_m^\bullet(L_\lambda^0)$ by Proposition 5.3.1. Since L_λ^0 is saturated by Proposition 4.1.1, we always have $\mathbf{H}_m^0(L_\lambda^0) = \mathbf{H}_m^1(L_\lambda^0) = 0$, so by Corollary 4.4.3, this is the Euler characteristic of S applied to the minimal injective resolution of Theorem 2.3.1. We will refine this statement in Proposition 7.4.3.

Since X_M is in general a linear combination of the X^λ , similar remarks apply to any finitely generated module M in place of L_λ^0 .

Now we give a pictorial description of (7.4.1). First, given a partition λ , consider its Young diagram. (The following terminology is not standard, but we will only need it in the formulation of Proposition 7.4.3.) A **border strip** B of its Young diagram is a set of boxes which do not contain a 2×2 square, and such that no boxes of λ in the complement of B lie strictly below B , and such that $\lambda \setminus B$ is the Young diagram of a partition. Equivalently, this is a set of boxes which can be obtained by first removing a vertical strip from λ , and then removing a horizontal strip from the result. For example, for $\lambda = (7, 5, 3, 3, 2)$, we have marked a border strip with 2 connected components with \times :



We say that a border strip is **aligned** if it contains the last box in the first row of λ and if it is connected. The rightmost 5 boxes in the above example form an aligned border strip. Then aligned border strips are determined by their size. We denote the aligned border strips of λ by $\text{BS}(\lambda)$. The **height** of a border strip, denoted $\text{ht}(B)$, is the number of rows that it occupies. Then the $w \bullet \alpha$ in (7.4.1) becomes the partition obtained by removing the aligned border strip of size $\lambda_1 - |\mu| + |\lambda|$ from (λ_1, λ) (if possible) and $\ell(w)$ becomes the height of this border strip. If it is not possible to remove this aligned border strip, then the right-hand side of (7.4.1) is 0.

To see the equivalence of the rules, start with the sequence $(N - |\lambda|, \lambda)$ which we visualize as a modified Young diagram (if $N - |\lambda| < 0$, we can think about the first row going to the left). In the process of straightening out this sequence with a permutation, we will move the first entry past i rows, add i to it, and subtract 1 from each of these rows that we moved past. A moment's thought shows that all we have done is removed a border strip from (λ_1, λ) and that this gives us a bijection between modified sequences and removing border strips.

Example 7.4.2. (a) The simplest case above is $\lambda = (1)$. Then $(N - 1, 1)$ is a partition whenever $N \geq 2$ and for $N = 0, 1$, we get $(-1, 1) \xrightarrow{1} (0, 0)$ and $(0, 1)$ is singular. Applying S to the minimal injective resolution of L_1 , we get

$$\mathbf{S}_1 \otimes A \rightarrow A$$

whose kernel is L_1^0 and whose cokernel is \mathbf{C} . Hence $\mathbf{H}_m^2(L_1^0) = \mathbf{C}$ and this accounts for the fact that $(-1, 1) \xrightarrow{1} (0, 0)$.

(b) Now consider the case $\lambda = (2)$. Then $(N - 2, 2)$ is a partition whenever $N \geq 4$ and for $N = 0, 1, 2, 3$, we get $(-2, 2)$ is singular, $(-1, 2) \xrightarrow{1} (1, 0)$, $(0, 2) \xrightarrow{1} (1, 1)$, and $(1, 2)$ is singular. Applying S to the minimal injective resolution of L_2 , we get

$$\mathbf{S}_2 \otimes A \rightarrow \mathbf{S}_1 \otimes A$$

whose kernel is L_2^0 and whose cokernel is $\mathbf{H}_m^2(L_2^0) = \mathbf{S}_1 \oplus \mathbf{S}_{1,1}$ (which could be deduced from Proposition 1.0.1). Note that we are only treating $\mathbf{H}_m^2(L_2^0)$ as an object of \mathcal{V} here.

(c) For a slightly more involved example, consider $\lambda = (2, 1)$. Then $(N - 3, 2, 1)$ is a partition whenever $N \geq 5$, and this sequence is singular for $N = 0, 2, 4$. Otherwise, we have $(-2, 2, 1) \xrightarrow{2} (1)$ and $(0, 2, 1) \xrightarrow{1} (1, 1, 1)$. The corresponding border strips are

$$\begin{array}{|c|c|} \hline \square & \times \\ \hline \times & \times \\ \hline \times & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \times \\ \hline \square & \times \\ \hline \square & \\ \hline \end{array}$$

Applying S to the minimal injective resolution of $L_{2,1}$, we get

$$\mathbf{S}_{2,1} \otimes A \rightarrow (\mathbf{S}_{1,1} \oplus \mathbf{S}_2) \otimes A \rightarrow \mathbf{S}_1 \otimes A.$$

Again from Proposition 1.0.1, we deduce that the cokernel of the last map is $\mathbf{H}_m^3(L_{2,1}^0) = \mathbf{S}_1$. Since the kernel of the first map is $L_{2,1}^0$, we can take the Euler characteristic (either by hand using Pieri's rule, or using the discussion in this section) to get that the middle homology is $\mathbf{H}_m^2(L_{2,1}^0) = \mathbf{S}_{1,1,1}$. \square

Here is the general situation for $L_\lambda^{\geq D}$, which upgrades the K-theory formula (7.4.1).

Proposition 7.4.3. *For $i \geq 1$, the local cohomology of $L_\lambda^{\geq D}$ is given by*

$$\mathbf{H}_m^i(L_\lambda^0) = \bigoplus_{\substack{B \in \text{BS}((D, \lambda)), \\ \text{ht}(B) = i}} \mathbf{S}_{(D, \lambda) \setminus B}$$

as an object of \mathcal{V} . As an A -module, it is generated by the smallest partition in $\{(D, \lambda) \setminus B\}$ with $\text{ht}(B) = i$.

Proof. Since the saturation of $L_\lambda^{\geq D}$ is L_λ^0 , the formula is correct for $i = 1$ by Corollary 4.4.3. Hence we may assume that $D = \lambda_1$ without loss of generality. To check it for $i > 1$, we use the injective resolution of L_λ in Mod_K given in Theorem 2.3.1, apply the section functor S to it, and calculate cohomology. By Proposition 4.1.3, we get a complex \mathbf{I}^\bullet where $\mathbf{I}^j = \bigoplus_{\mu, \lambda/\mu \in \text{VS}_j} \mathbf{S}_\lambda \otimes A$.

First, we consider how to count the occurrences of $L_\nu^{\geq D}$ in \mathbf{I}^j . The only such ν that appear are obtained by first removing a vertical strip from λ to get some partition μ , and then removing a horizontal strip from μ . In other words, the ν are those partitions we can get by removing a border strip B from λ . In this case, we will have $D = \mu_1$, and $L_\nu^{\geq \mu_1}$ appears as a constituent of $\mathbf{I}^{|\lambda| - |\mu|}$. Given a connected border strip, there are exactly two ways to get it from removing a vertical strip and then a horizontal strip: the box that lies in the top-right can be removed at either step, but the order in which the other boxes are removed is forced.

Now consider the border strip B removed from λ to get ν . If B does not contain the top-right box of λ , then the description above shows that all instances of $L_\nu^{\geq D}$ have $D = \lambda_1$, and the proof of Theorem 2.3.1 shows that the subcomplex (in the category \mathcal{V}) consisting of them is exact.

So we assume that B contains the top-right box of λ now. Let r be the number of connected components of B , and order them from right to left. Then there are 2^r ways to obtain B by first removing a vertical strip and then removing a horizontal strip. We encode these with a bitstring $\underline{a} = (a_1, \dots, a_r)$ with $a_i \in \{0, 1\}$, where $a_i = 1$ if and only if we include the top-right box of the i th connected component in the vertical strip. Let f be the number of boxes which are forced to be included in the vertical strip. Then the bitstring \underline{a} contributes a copy of $L_\nu^{\lambda_1 - a_1}$ to $\mathbf{I}_{f+a_1+\dots+a_r}$. In particular, if $r \geq 2$, we see that the number of copies of $L_\nu^{\lambda_1 - 1}$ and $L_\nu^{\lambda_1}$ are evenly distributed, and in fact, that the subcomplex given by all of them is exact.

Finally, we deal with the case $r = 1$, which corresponds in a natural way to removing an aligned border strip from (λ_1, λ) . In this case, the subcomplex given by the $L_\nu^{\geq D}$ looks like

$$L_\nu^{\geq \lambda_1} \rightarrow L_\nu^{\geq \lambda_1 - 1}$$

in cohomological degrees f and $f + 1$, respectively. Since these are the unique ways to get L_ν , there are no other terms in cohomological degrees $f, f + 1$ that interfere with this map, i.e., the cokernel of this map contributes to $\text{R}^{f+1}S(L_\lambda^0) \cong \text{H}_m^{f+2}(L_\lambda^0)$. The cokernel of this map is $\mathbf{S}_{(\lambda_1 - 1, \nu)}$, which is exactly obtained from (λ_1, λ) by removing the unique aligned border strip of size $f + 2$.

Hence we have calculated the local cohomology as elements of \mathcal{V} . The last statement on the A -module structure follows from the fact that $\text{H}_m^i(L_\lambda^0)$ is a subquotient of $\mathbf{S}_\eta \otimes A$ where η is the partition obtained from λ by removing the f forced vertical strip boxes mentioned above, plus the last box in the first row of λ . \square

Corollary 7.4.4. *The depth of $L_\lambda^{\geq D}$ is the multiplicity of D in the partition (D, λ) . In particular, if $D > \lambda_1$, the depth is 1, and otherwise, the depth of L_λ^0 is 1 more than the multiplicity of λ_1 in λ .*

Proof. This follows directly from Propositions 7.2.1 and 7.4.3. \square

Remark 7.4.5. Now we have introduced enough language to simplify the calculation of $\langle Q_\lambda, L_\mu \rangle$ from Remark 2.4.6. We have

$$\langle Q_\lambda, L_\mu \rangle = \langle T(A \otimes \mathbf{S}_\lambda), L_\mu \rangle = \langle A \otimes \mathbf{S}_\lambda, \text{RS}(L_\mu) \rangle$$

by adjointness. Since $\mathbf{S}_\lambda \otimes A$ is projective and RSL_μ is essentially the local cohomology of L_μ^0 (Corollary 4.4.3), we see that this pairing is $m_0 - \sum_{i>0} (-1)^i m_i$ where m_0 denotes the multiplicity of λ in L_μ^0 (which is either 0 or 1) and m_i for $i > 0$ denotes the multiplicity of λ in $\text{H}_m^i(L_\mu^0)$. From the discussion in this section, we have $\sum_{i \geq 0} m_i \leq 1$, so in particular, the pairing takes values in $\{-1, 0, 1\}$. The exact value can be calculated using the combinatorics of border strips (or using the shifted symmetric group action from above). \square

Remark 7.4.6. For an A -module M , let $d_i(M)$ be the maximum size of a partition appearing in $H_m^i(M)$. Then by Proposition 5.3.1, we have

$$\deg(q_M) \leq \max(d_i(M)).$$

For $i \geq 2$, the quantity $H_m^i(M)$ depends only on the image $T(M)$ of M in Mod_K . Let $\Lambda(M)$ be the set of partitions λ for which L_λ is a constituent of $T(M)$. By filtering $T(M)$ and looking at various long exact sequences, we see that

$$d_i(M) \leq \max_{\lambda \in \Lambda(M)} d_i(L_\lambda^0)$$

for $i \geq 2$. For a nonzero partition λ , let $d(\lambda) = |\lambda| + \lambda_1 - n - 1$, where n is the multiplicity of λ_1 in λ ; put $d(\emptyset) = 0$. Then Proposition 7.4.3 shows that $d_i(L_\lambda^0) \leq d(\lambda)$. Combining all of this, we see that

$$\deg(q_M) \leq \max\left(d_0(M), d_1(M), \max_{\lambda \in \Lambda(M)} d(\lambda)\right).$$

Recall that the character polynomial of M computes the character of the representation M_n for $n > N$, for some integer N . It is desirable to know the optimal value of N . In fact, the optimal value of N is exactly equal to the degree of q , and so the above inequality gives an explicit bound on the optimal value of N . This bound is an improvement on the one given in [CEF, Thm. 2.65]: note that $d_0(M)$ and $d_1(M)$ are both \leq the stability degree, while $\max_{\lambda \in \Lambda(M)} d(\lambda)$ is \leq the weight (and this inequality is often strict). \square

Example 7.4.7. We will say a little bit about calculating the depth of the modules $L_\lambda^{\geq D}$ using the Auslander–Buchsbaum formula. This depth was obtained using local cohomology in Corollary 7.4.4.

Let $\{\mu^1, \dots, \mu^r\}$ be the partitions obtained by adding a single box to (D, λ) anywhere except the first row. Then the presentation of $L_\lambda^{\geq D}$ as an A -module is

$$\bigoplus_{i=1}^r \mathbf{S}_{\mu^i} \otimes A \rightarrow \mathbf{S}_{(D, \lambda)} \otimes A \rightarrow L_\lambda^{\geq D} \rightarrow 0.$$

If $\lambda = \emptyset$, then $L_\emptyset^{\geq D} = \Omega M(\emptyset, D)$ is the D th power of the maximal ideal of A , and was discussed in the previous section §6.5. The minimal free resolution in general was constructed in [SW, Corollary 2.10].

We will consider the case when $\lambda = (n)$ has a single part. If $D = n$, then

$$L_{(n)}^{\geq n} = \Omega^2 M((n-1, n-1), 1),$$

so again, was discussed in the previous section §6.5.

The basic idea of [SW, §2.3] is to use mapping cones. In our situation, we have the presentation

$$(\mathbf{S}_{(D, n+1)} \oplus \mathbf{S}_{(D, n, 1)}) \otimes A \rightarrow \mathbf{S}_{(D, n)} \otimes A \rightarrow L_{(n)}^{\geq D} \rightarrow 0.$$

Define A -modules N, N' using the presentations

$$\begin{aligned} \mathbf{S}_{(D, n, 1)} \otimes A &\rightarrow \mathbf{S}_{(D, n)} \otimes A \rightarrow N \rightarrow 0 \\ \mathbf{S}_{(D, n+1, 1)} \otimes A &\rightarrow \mathbf{S}_{(D, n+1)} \otimes A \rightarrow N' \rightarrow 0. \end{aligned}$$

Then $N = \Omega^2 M(n-1, D-n+1)$ and $N' = \Omega^2 M(n, D-n)$, so both have depth 2. Also, we have a short exact sequence

$$0 \rightarrow N' \rightarrow N \rightarrow L_{(n)}^{\geq D} \rightarrow 0.$$

We can construct a free resolution for $L_{(n)}^{\geq D}$ by taking a mapping cone on the minimal free resolutions of N' and N . All of the partitions that appear will be distinct, so in fact it will be minimal. In

particular, the minimal free resolution \mathbf{F}_\bullet of $L_{(n)}^{\geq D}$ has the terms

$$\mathbf{F}_i = (\mathbf{S}_{(D,n,1^i)} \oplus \mathbf{S}_{(D,n+1,1^{i-1})}) \otimes A \quad (i > 0),$$

so we see, by the Auslander–Buchsbaum formula, that $\text{depth } L_{(n)}^{\geq D} = 1$ when $D > n$. This is in contrast with the fact that $\text{depth } L_{(n)}^{\geq n} = 2$.

In general, [SW, Corollary 2.10] implies that $\text{depth } L_\lambda^{\geq D}$ is the number of times that D occurs in the partition (D, λ) , which agrees with Corollary 7.4.4. Again, the idea is to relate $L_\lambda^{\geq D}$ to simpler modules, constructed like N and N' above, and then to use a mapping cone construction. In general, N and N' are not of the form $\Omega^j M(\mu, e)$. \square

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