SYZYGIES OF SEGRE EMBEDDINGS

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ABSTRACT. We study syzygies of the Segre embedding of $\mathbf{P}^{d_1} \times \cdots \times \mathbf{P}^{d_n}$, and prove two finiteness results. First, for fixed p but varying d_i and n, there are only finitely many "forms" of p-syzygies. Second, we define a power series f_p with coefficients in something like the Schur algebra, which contains essentially all the information about p-syzygies of Segre embeddings (for all n and d_i), and show that it is a rational function. The list of forms of p-syzygies and the numerator and denominator of f_p can be computed algorithmically (in theory). A key insight in this paper is that by considering all Segre embeddings at once (i.e., letting n and the d_i vary) certain structure on the space of p-syzygies emerges. We formalize this structure in the concept of a Δ-module. Many of our results on syzygies are specializations of general results on Δ-modules that we establish. Our theory also applies to certain other families of varieties, such as tangent and secant varieties of Segre embeddings.

Contents

1. Introduction

Let $d_1, \ldots, d_n$ be positive integers, and let d be their product. The Segre embedding is the map

$$\mathbf{P}^{d_1-1} \times \cdots \times \mathbf{P}^{d_n-1} \to \mathbf{P}^{d-1}$$

taking a tuple of lines to their tensor product. Despite its fundamental nature, the syzygies of this embedding are not well understood (outside of some special cases, such as when $n = 2$). The purpose of this paper is to establish a general structural theory of these syzygies. We show that, for p fixed, but n and the d_i variable, the collection of all p-syzygies can be given an algebraic structure, and that this structure has favorable finiteness properties. Our theory also applies to families of varieties that are closely related to Segre varieties, such as the tangent and secant varieties to Segre varieties.

1.1. **The spaces F_p.** Let $V_1, \ldots, V_n$ be finite dimensional complex vector spaces and let

$$X_n(V_1, \ldots, V_n) \subset \bigotimes_{i=1}^{n} V_i^*$$

be the set of pure tensors. This space is identified with the cone on the Segre variety, and will be more convenient for us than the corresponding projective variety. The construct X has three important properties:

(A1) The association $(V_1, \ldots, V_n) \mapsto X_n(V_1, \ldots, V_n)$ is functorial in the V_i.
(A2) The functor X_n is S_n-equivariant.
(A3) There is an inclusion $X_{n+1}(V_1, \ldots, V_{n+1}) \subset X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1})$.

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Let $R = R_n(V_1, \ldots, V_n)$ be the coordinate ring of $X_n(V_1, \ldots, V_n)$, and let $P = P_n(V_1, \ldots, V_n)$ be the coordinate ring of $\bigotimes V_i^*$, namely $\mathrm{Sym}(\bigotimes V_i)$. The rings R and P are graded (since the corresponding varieties have a natural $\mathbf{G}_m$-action), and there is a surjection $P \to R$. Let

$$\cdots \to M_3 \to M_2 \to P \to R \to 0$$

be a minimal free resolution of R as a P-module. The M_p are the syzygy modules we seek to understand. Note that the M_p are not canonical, but their isomorphism classes are; in other words, the terms of two minimal resolutions are non-canonically isomorphic.

Let $F_p = F_{p,n}(V_1, \ldots, V_n)$ be $M_p \otimes_P \mathbf{C}$. We have $M_p \cong P \otimes_{\mathbf{C}} F_p$ (non-canonically), so to understand the syzygy modules M_p it suffices to understand the F_p. The F_p have two advantages over the M_p. First, they are finite dimensional graded vector spaces, rather than finite rank P-modules. And second, we have a canonical identification $F_p = \mathrm{Tor}^P_{p-1}(R, \mathbf{C})$, which shows that the F_p are canonical. The F_p are the focus of this paper. We refer to elements of F_p as p-syzygies.

The F_p have three important properties:

- (B1) The association $(V_1, \ldots, V_n) \mapsto F_{p,n}(V_1, \ldots, V_n)$ is functorial in the V_i.
- (B2) The functor $F_{p,n}$ is S_n-equivariant.
- (B3) There is a natural map $F_{p,n}(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1}) \to F_{p,n+1}(V_1, \ldots, V_{n+1})$.

To see this, note that R satisfies (B1)–(B3) as a consequence of X satisfying (A1)–(A3), and that these properties carry over to F_p by the functorial nature of Tor. We can think of (B1)–(B3) as operations on syzygies: given a syzygy, they produce more syzygies.

1.2. **The first theorem.** Our first main result is the following:

Theorem A. *Fix p. There is a finite set of p-syzygies of (possibly different) Segre embeddings which gives rise to all p-syzygies of all Segre embeddings under the operations (B1)–(B3).*

We regard the finite list of syzygies provided by the above theorem as a generating set for F_p. Thus the theorem states that F_p is finitely generated. In fact, for the reader who finds the statement of the above theorem somewhat imprecise, the true statement is: "F_p is finitely generated as a Δ-module." These terms are defined in the body of the paper.

As an example, consider the case of equations ($p = 2$). It is not difficult to show that the single equation for $\mathbf{P}^1 \times \mathbf{P}^1$ inside of $\mathbf{P}^3$ generates F_2. This can be interpreted geometrically, in two stages, as follows. First, given two vector spaces V_1 and V_2, the space $X(V_1, V_2) \subset V_1^* \otimes V_2^*$ is the intersection of the inverse images of $X(\mathbf{C}^2, \mathbf{C}^2)$ as we vary over all maps $\mathbf{C}^2 \to V_1$ and $\mathbf{C}^2 \to V_2$. And second, given vector spaces $V_1, \ldots, V_n$, the space $X(V_1, \ldots, V_n)$ is the intersection of the spaces $X(V_{U_1}, V_{U_2})$ as we vary over all partitions $\{U_1, U_2\}$ of $\{1, \ldots, n\}$, and where $V_{U_1} = \bigotimes_{i \in U_1} V_i$, and similarly for V_{U_2}.

We remark that the operations (B1)–(B3) do not change the "form" of a syzygy. For example, the equation for $\mathbf{P}^1 \times \mathbf{P}^1$ inside of $\mathbf{P}^3$ can be written as a 2×2 determinant. Any element of F_2 gotten from this element via the operations (B1)–(B3) has the same property. Thus Theorem A can be interpreted as saying that there are only finitely many forms of syzygies.

Our proof of Theorem A shows that a finite generating set for F_p can be computed algorithmically (though not in a practical manner); see §4.3. One can always find generators for F_p by looking at vector spaces of dimension p, i.e., the collection of elements in $F_{p,n}(\mathbf{C}^p, \ldots, \mathbf{C}^p)$ as n varies generates F_p. For $p = 2$, we gave a generator above. For $p = 3$ there is also a single generator: any non-zero element of the two dimensional space $F_3(\mathbf{C}^3, \mathbf{C}^2)$. We do not know any generating sets beyond these two examples.

1.3. **The second theorem.** For a partition λ, let $\mathbf{S}_\lambda$ be the corresponding Schur functor. (See §2.1 for a review of this theory.) By general theory, we have a decomposition

$$F_{p,n}(V_1, \ldots, V_n) = \bigoplus_{i \in I} \mathbf{S}_{\lambda_{1,i}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{n,i}}(V_n)$$

for some index set I and partitions $\lambda_{i,j}$ (depending on n and p). We will show that I is finite. Define

$$f^*_{p,n} = \sum_{i \in I} s_{\lambda_{1,i}} \cdots s_{\lambda_{n,i}},$$

regarded as a polynomial in (commuting) formal variables s_λ. Now define

$$f_p^* = \sum_{n=1}^{\infty} f_{p,n}^*,$$

a power series in the variables s_λ. A priori, there could be infinitely many variables occurring in f_p^*; we will show that this is not the case. Our second theorem is then:

Theorem B. *The series f_p^* is a rational function of the s_λ.*

One can recover the isomorphism class of the functor $F_{p,n}$ from the series f_p^*, and so this series contains nearly all of the information about p-syzygies of Segre embeddings. (Certain information is lost, such as the data of the S_n-equivariance.) Our proof of Theorem B shows that the numerator and denominator of f_p^* can be computed algorithmically (but, again, not in a practical manner). Thus, given p, one can (in theory) perform a single computation and know essentially all there is to know about p-syzygies of all Segre embeddings.

It turns out that it is more natural to consider the series f_p whose nth term is $\frac{1}{n!} f_{p,n}^*$; of course, one can easily pass between f_p and f_p^*. Unfortunately, Theorem B does not imply that f_p has a nice form, though one might hope that f_p is a polynomial in the s_λ and $e^{\pm s_\lambda}$. The leading term of f_p is known by Lascoux's work (§4.4). We will compute a certain Euler characteristic involving the f_p's (§4.6). By known vanishing results, this allows us to compute f_2, f_3, f_4 and part of f_5 (§4.7). We have not been able to compute beyond this, however.

We now give an example to illustrate how to extract information from f_p^*. From the computations of §4.7, we obtain

$$f_2^* = \frac{1-s}{(1-s)^2 - w^2} - \frac{1}{1-s},$$

where $s = s_{(2)}$ and $w = s_{(1,1)}$. Developing the above into a power series, we obtain

$$f_2^* = w^2 + 3sw^2 + (6w^2s^2 + w^4) + \cdots$$

The order n term of this series gives the decomposition of $F_{2,n}(V_1, \ldots, V_n)$ as a representation of $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. (Note that F_2 is identified with the degree 2 piece of the ideal defining the Segre.) For example, the order two term implies that $F_{2,2}(V_1, V_2)$ is isomorphic to $\bigwedge^2 V_1 \otimes \bigwedge^2 V_2$. The order three term implies that $F_{2,3}(V_1, V_2, V_3)$ is isomorphic to

$$\big(\mathrm{Sym}^2 V_1 \otimes \textstyle\bigwedge^2 V_2 \otimes \bigwedge^2 V_3\big) \oplus \big(\bigwedge^2 V_1 \otimes \mathrm{Sym}^2 V_2 \otimes \bigwedge^2 V_3\big) \oplus \big(\bigwedge^2 V_1 \otimes \bigwedge^2 V_2 \otimes \mathrm{Sym}^2 V_3\big).$$

1.4. **Outline of proofs.** We begin by studying sequences of functors that have formal properties similar to F_p. We call such objects Δ-*modules*. We prove two key theorems about "small" Δ-modules (a certain subclass of Δ-modules): (a) a small Δ-module is noetherian; and (b) the Hilbert series of a small Δ-module is rational. ("Small" can be relaxed to "finitely generated" in the first result, we expect the same is true for the second.) Our two theorems are now formal consequences of (a) and (b) and the following observation: there is a complex of small Δ-modules (a Koszul complex) whose homology is F_p.

We have not been able to find a direct method of studying Δ-modules. Instead, we access them through the following "ladder":

$$\{\text{graded modules}\} \leftrightarrow \{\text{modules in Sym(Vec)}\} \leftrightarrow \{\text{modules in Sym}(\mathcal{S})\} \leftrightarrow \{\Delta\text{-modules}\}$$

We define the middle two categories below. The arrows here do not mean anything precise, only that the two categories are related. We prove our results about Δ-modules by starting with the corresponding results for graded modules (which are easy and well-known) and moving up the ladder. This process is actually fairly explicit: for example, the Hilbert series of a Δ-module is identified with the (G-equivariant) Hilbert series of a module over a graded ring, which is constructed in an explicit manner from the Δ-module. This identification is the reason that the Hilbert series of a Δ-module — at least one which is described in a reasonable manner — is algorithmically computable.

1.5. **Other varieties.** Our results apply, at least in part, to families of varieties that are closely related to the Segre varieties, such as the tangent and secant varieties to Segre varieties. For such varieties, the pth syzygy space F_p is again a Δ-functor. Our theory shows that each graded piece of F_p is finitely generated and has rational Hilbert series. However, it does not show that this is the case for F_p itself: this will be the case if and only if F_p is supported in finitely many degrees. Such results must be established by other means. See §4.8 for more details.

Some pieces of our theory also apply in greater generality. For example, if X is a projective variety with an action of a reductive group G, then the syzygies of the GIT quotient $X^n//G$ (as n varies) have some of the structure that we observe here for the Segre varieties. We will return to this topic in a future paper.

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2. Δ-modules

The purpose of this section is to introduce the algebraic objects that we will use in the rest of the paper, most notably Δ-modules. We begin by quickly reviewing the theory of Schur functors and define the Schur algebra $\mathcal{S}$, which we regard as a category. We then discuss symmetric powers of semi-simple abelian categories. This operation is not strictly necessary for our purposes, but clarifies some later definitions. We then introduce the categories $\mathrm{Sym}(\mathrm{Vec})$ and $\mathrm{Sym}(\mathcal{S})$ and give explicit models for them which do not use the symmetric power construction. Finally we discuss Δ-modules, which are objects of $\mathrm{Sym}(\mathcal{S})$ with an additional piece of structure. The most important results of this section are various statements that certain types of objects are noetherian.

Algebras in $\mathrm{Sym}(\mathrm{Vec})$ are known as "twisted commutative algebras," and there is some literature on them. We have not encountered the noetherian result we prove about them in the literature, but would not be surprised if it were already known; indeed, we learned the core of the argument of Theorem 2.3 from Harm Derksen (via Ben Howard). The categories $\mathrm{Sym}(\mathcal{S})$ and Mod_Δ are a bit more esoteric, and we do not know of any occurrence of them in the literature.

2.1. **The Schur algebra.** We quickly review what we need about the Schur algebra. Further discussion can be found in [FH, §6.1]. Let Vec denote the category of complex vector spaces. A functor $\mathrm{Vec} \to \mathrm{Vec}$ is "nice" if it appears as a constituent of a functor of the form $V \mapsto V^{\otimes n}$, or a (possibly infinite) direct sum of such functors. Essentially every functor $\mathrm{Vec} \to \mathrm{Vec}$ one encounters naturally that does not use duality is nice; an example of a non-nice functor is the double dual. Let $\mathcal{S}$ denote the category of all nice functors. It is a semi-simple abelian tensor category which we call the *Schur algebra*. (By "semi-simple" here we mean that every object is a possibly infinite direct sum of simple objects.) The tensor product is the point-wise one: $(F \otimes G)(V) = F(V) \otimes G(V)$.

Let λ be a partition of n; we denote this by $|\lambda| = n$ or $\lambda \vdash n$. Let $\mathbf{M}_\lambda$ be the irreducible complex representation of the symmetric group S_n associated to λ. For a vector space V put $\mathbf{S}_\lambda(V) = (V^{\otimes n} \otimes \mathbf{M}_\lambda)_{S_n}$, where the subscript denotes co-invariants and S_n acts on $V^{\otimes n}$ by permuting tensor factors. Then $\mathbf{S}_\lambda$ belongs to $\mathcal{S}$ and is a simple object; furthermore, every simple object is isomorphic to $\mathbf{S}_\lambda$ for a unique λ. We use the convention that the partition $\lambda = (n)$ gives the functor $\mathbf{S}_\lambda = \mathrm{Sym}^n$ while the partition $\lambda = (1, \ldots, 1)$ gives the functor $\mathbf{S}_\lambda = \bigwedge^n$. We identify partitions with Young diagrams by the convention that $\lambda = (n)$ has one row and $\lambda = (1, \ldots, 1)$ has n rows.

We will also need multivariate functors of vector spaces. A functor $\mathrm{Vec}^k \to \mathrm{Vec}$ is "nice" if it appears as a constituent of a functor of the form $(V_1, \ldots, V_k) \mapsto V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}$, or a (possibly infinite) direct sum of such functors. The category of all nice functors is again a semi-simple abelian tensor category, and is naturally equivalent to $\mathcal{S}^{\otimes k}$. This means that any nice functor $F : \mathrm{Vec}^k \to \mathrm{Vec}$ can be written as

$$F(V_1, \ldots, V_k) = \bigoplus_{i \in I} \mathbf{S}_{\lambda_{1,i}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{k,i}}(V_k)$$

for some index set I and partitions $\lambda_{i,j}$. This expression is unique. We say that a partition λ *occurs* in F if it is amongst the $\lambda_{i,j}$. If $F : \mathrm{Vec} \to \mathrm{Vec}$ is a nice functor then the functors $\mathrm{Vec}^2 \to \mathrm{Vec}$ given by mapping (V, W) to $F(V \oplus W)$ and $F(V \otimes W)$ are both nice. We thus get a co-addition map a^* and a co-multiplication map m^* from $\mathcal{S}$ to $\mathcal{S}^{\otimes 2}$.

Let $\mathcal{A}$ be a $\mathbf{C}$-linear abelian tensor category. (Our tensor categories are always symmetric, i.e., there is a given involution of functors $A \otimes B \to B \otimes A$.) There is then a natural action of $\mathcal{S}$ on $\mathcal{A}$. Given a partition λ of n and an object A of $\mathcal{A}$, the object $\mathbf{S}_\lambda(A)$ is given by $(A^{\otimes n} \otimes \mathbf{M}_\lambda)_{S_n}$.

2.2. **Symmetric powers of abelian categories.** We now describe how one can form the symmetric algebra on semi-simple abelian categories. The reader who is not comfortable with this discussion need not worry: all categories that we eventually use will admit concrete descriptions. We provide this discussion only to give some context and motivation for later constructions.

Let $\mathcal{A}$ be a semi-simple $\mathbf{C}$-linear abelian category. One can then make sense of the tensor power $\mathcal{A}^{\otimes n}$. We define $\mathrm{Sym}^n(\mathcal{A})$ to be the category $(\mathcal{A}^{\otimes n})^{S_n}$, where the superscript denotes homotopy invariants (equivariant objects). Thus an object of $\mathrm{Sym}^n(\mathcal{A})$ is an object A of $\mathcal{A}^{\otimes n}$ together with an isomorphism $\sigma^* A \to A$ for each $\sigma \in S_n$, satisfying the obvious compatibility conditions. We define $\mathrm{Sym}(\mathcal{A})$ to be the sum of the categories $\mathrm{Sym}^n(\mathcal{A})$ over $n \geq 0$.

The category $\mathrm{Sym}(\mathcal{A})$ has a natural tensor structure. Multiplication involves averaging (induction). Precisely, let $\underline{\otimes}$ denote the usual concatenation tensor product $\mathcal{A}^{\otimes n} \otimes \mathcal{A}^{\otimes m} \to \mathcal{A}^{\otimes(n+m)}$. If A is an object of $(\mathcal{A}^{\otimes n})^{S_n}$ and B of $(\mathcal{A}^{\otimes m})^{S_m}$ then $A \underline{\otimes} B$ naturally has an $S_n \times S_m$ equivariance. One can then form the induction $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(A \underline{\otimes} B)$, which is an object of $(\mathcal{A}^{\otimes(n+m)})^{S_{n+m}}$. This is the product of A and B in $\mathrm{Sym}(\mathcal{A})$, which we denote by $A \otimes B$. Note that if $\mathcal{A}$ itself has a tensor structure then $(\mathcal{A}^{\otimes n})^{S_n}$ does as well, which gives an alternate tensor structure on $\mathrm{Sym}(\mathcal{A})$. When present, we will denote this tensor product by $\boxtimes$ and call it the *point-wise* tensor product.

Remark 2.1. The above definition looks more like the divided power algebra than symmetric algebra. However, one can verify it has the correct universal property. We believe that in the setting of $\mathbf{C}$-linear abelian tensor categories, the right analogue of divided powers is an action of the Schur algebra, with Sym^n taking the place of γ_n. Since all such categories have a natural action of the Schur algebra, they can all be considered to have divided powers. Thus the symmetric and divided power algebras on $\mathcal{A}$ coincide. This is a conceptual reason explaining why we can use invariants (rather than coinvariants) to form the symmetric algebra.

As $\mathrm{Sym}(\mathcal{A})$ is an abelian tensor category, we can speak of algebras in it and modules over algebras. (Algebras will always be commutative, associative and unital.) Let A be an algebra. We say that A is *finitely generated* (as an algebra) if it is a quotient of $\mathrm{Sym}(F)$ for some finite length object F of $\mathrm{Sym}(\mathcal{A})$. Similarly, we say that an A-module M is *finitely generated* if it is a quotient of $A \otimes F$ for some finite length F. We say that M is *noetherian* if every ascending chain of submodules stabilizes. We say that A is *noetherian* (as an algebra) if every finitely generated A-module is noetherian.

The category $\mathrm{Sym}(\mathcal{A})$ is again semi-simple abelian, and its simple objects can be described easily. Let A be a simple object of $\mathcal{A}$ and let $\mathbf{M}_\lambda$ be an irreducible representation of S_n. Then $\mathbf{M}_\lambda \otimes A^{\underline{\otimes} n}$ has a natural S_n-equivariance and so defines an object of $\mathrm{Sym}^n(\mathcal{A})$; in fact, this is nothing other than $\mathbf{S}_\lambda(A)$ in the category $\mathrm{Sym}(\mathcal{A})$. This object is simple, and all simple objects are of the form $\bigotimes \mathbf{S}_{\lambda_i}(A_i)$ where the A_i are mutually non-isomorphic simple objects of $\mathcal{A}$.

Let $K(-)$ denote the Grothendieck group of an abelian category, tensored with $\mathbf{Q}$. We have a map

$$K(\mathrm{Sym}^n(\mathcal{A})) \to \mathrm{Sym}^n(K(\mathcal{A}))$$

defined as follows: first apply the functor $\mathrm{Sym}^n(\mathcal{A}) \to \mathcal{A}^{\otimes n}$, then use the identification $K(\mathcal{A}^{\otimes n}) = K(\mathcal{A})^{\otimes n}$, then project $K(\mathcal{A})^{\otimes n} \to \mathrm{Sym}^n(K(\mathcal{A}))$ and finally divide by $n!$. For $A \in \mathrm{Sym}^n(\mathcal{A})$ we let $[A]$ denote the corresponding class in $\mathrm{Sym}^n(K(\mathcal{A}))$. We also put $[A]^* = n![A]$. The above map is not injective, but only forgets the S_n-equivariant structure: if $[A] = [B]$ then A and B have isomorphic images in $\mathcal{A}^{\otimes n}$. Summing the above maps over n, we get a map

$$K(\mathrm{Sym}(\mathcal{A})) \to \mathrm{Sym}(K(\mathcal{A})).$$

Again, for $A \in \mathrm{Sym}(\mathcal{A})$ we let $[A]$ denote the corresponding class in $\mathrm{Sym}(K(\mathcal{A}))$; we define $[A]^*$ in the obvious manner. We have $[A \otimes B] = [A][B]$; the modified class $[-]^*$ does not satisfy this. As an example, if A is an object of $\mathcal{A}$ then $[\mathrm{Sym}^n(A)] = \frac{1}{n!}[A]^n$.

We will often need to use the completion of Sym, both in the setting of $\mathbf{Z}$-modules and abelian categories. Whereas $\mathrm{Sym}(\mathbf{Z}^n)$ is the ring of polynomials in n variables, the completion is the ring of power series in n variables. We will not bother to introduce extra notation for the completion, as there should be no confusion.

2.3. **The category** Sym(Vec)**.** The following abelian tensor categories are equivalent:

(a) The category Sym(Vec).

(b) The category of functors (fs) $\to$ Vec, where (fs) is the category whose objects are finite sets and whose morphisms are bijections. Multiplication is given by convolution, using the monoidal structure on (fs) given by disjoint union. Precisely, if V and W are two functors (fs) $\to$ Vec then

$$(V \otimes W)_L = \bigoplus_{L = A \amalg B} V_A \otimes W_B,$$

where the sum is over all partitions of L into two disjoint subsets A and B. (We use subscripts to indicate the value of the functor on a set).

(c) The category of sequences $(V_n)_{n \ge 0}$, where V_n is a vector space with an action of S_n. Multiplication is given by the formula

$$(V \otimes W)_n = \bigoplus_{n=i+j} \mathrm{Ind}_{S_i \times S_j}^{S_n} (V_i \otimes W_j).$$

(d) The Schur algebra $\mathcal{S}$. Multiplication is the point-wise tensor product.

(e) The category of all positive algebraic representations of $\mathrm{GL}(\infty)$. Here $\mathrm{GL}(\infty)$ is the union of $\mathrm{GL}(n, \mathbf{C})$ for $n \ge 1$, "algebraic" means the restriction to any $\mathrm{GL}(n)$ is a (possibly infinite) direct sum of algebraic representations and "positive" means that when restricted to any $\mathrm{GL}(n)$, any highest weight appearing has non-negative entries when thought of as a non-increasing sequence of integers. Multiplication is the usual tensor product of representations.

We briefly describe the various equivalences. The equivalence between (b) and (c) is clear. Since $\mathrm{Vec}^{\otimes n} = \mathrm{Vec}$, the category of S_n-equivariant objects in $\mathrm{Vec}^{\otimes n}$ is just the representation category of S_n; this gives the equivalence between (a) and (c). The equivalence between (c) and (d) is through Schur-Weyl duality. Precisely, given a sequence (V_n) in the category (c), let $S : \mathrm{Vec} \to \mathrm{Vec}$ be the functor taking a vector space W to

$$S(W) = \bigoplus_{n \ge 0} (W^{\otimes n} \otimes V_n)_{S_n}.$$

Then $(V_n) \mapsto S$ is the equivalence. Regarding $\mathbf{M}_\lambda$ as an object of (c) supported at $n = |\lambda|$, this equivalence maps $\mathbf{M}_\lambda$ to $\mathbf{S}_\lambda$. Finally, the equivalence of (d) and (e) is given by evaluating on $\mathbf{C}^\infty$. We regard (b)–(e) as "models" for the category Sym(Vec). We name them the "standard," "sequence," "Schur," and "GL" models, respectively. We now come to an important definition:

Definition 2.2. A *twisted commutative algebra* is an algebra in the category Sym(Vec).

As always, "algebra" means commutative, associative and unital. In the standard model, a twisted commutative algebra is a functor $A : (\mathrm{fs}) \to \mathrm{Vec}$ together with a multiplication map $A_L \otimes A_{L'} \to A_{L \amalg L'}$. In the GL-model, a twisted commutative algebra is just a $\mathbf{C}$-algebra, in the usual sense, with an action of $\mathrm{GL}(\infty)$ by algebra homomorphisms (under which the algebra is a positive algebraic representation). The notion of a module over a twisted commutative algebra is evident.

Let V be an object of Sym(Vec), in the standard model. By an *element* of V we mean an element of V_L for some L; we then call $\#L$ its *order*. (It would be more natural to use the term "degree" here, but we want to reserve that term for future use.) Given a collection S of elements of V there is a unique minimal subobject of V containing S; we call it the subspace of V *generated* by S. Similarly, if A is a twisted commutative algebra and S is a collection of elements of A then one can speak of the subalgebra of A generated by S. One can do the same for modules over A. We therefore have a notion of "finitely generated" for algebras and modules; this agrees with the one defined in §2.2 using finite length objects.

Let U be a vector space. We let $U\langle 1 \rangle$ be the object of Sym(Vec) which is U in order 1 and 0 in other orders. We put $A = \mathrm{Sym}(U\langle 1 \rangle)$; this is the most important twisted commutative algebra in this paper. Here are precise descriptions of $U\langle 1 \rangle$ and A in the various models:

- In the standard model, $U\langle 1 \rangle$ assigns to a finite set L the vector space U if $\#L = 1$ and the vector space 0 otherwise. The algebra A assigns to a finite set L the vector space $U^{\otimes L}$. The multiplication map $A_L \otimes A_{L'} \to A_{L \amalg L'}$ is concatenation of tensors. (Note: $U^{\otimes L}$ is isomorphic as a vector space to $U^{\otimes n}$, where $n = \#L$, but is functorial in L. It can be defined as the universal vector space equipped with a multi-linear map from $U \times L$. We think of the factors of a pure tensor in $U^{\otimes L}$ as being labeled with elements of L.)

- In the sequence model, $U\langle 1\rangle$ is the vector space U in order 1 and 0 in other orders. The algebra A is given by $A_n = U^{\otimes n}$, with its usual S_n action; the multiplication map $A_i \otimes A_j \to A_{i+j}$ is again concatenation of tensors.
- In the Schur model, $U\langle 1\rangle$ is the functor $U \otimes \mathrm{Sym}^1$. The algebra A is given by $\mathrm{Sym}(U \otimes \mathrm{Sym}^1)$. If V is a vector space then $(U\langle 1\rangle)(V) = U \otimes V$ and $A(V) = \mathrm{Sym}(U \otimes V)$.
- In the GL-model, $U\langle 1\rangle$ is $U \otimes \mathbf{C}^\infty$. The algebra A is $\mathrm{Sym}(U \otimes \mathbf{C}^\infty)$. Thus, in this model, A is a polynomial ring in infinitely many variables with $\mathrm{GL}(\infty)$ acting by linear substitutions.

The following result underlies everything else in the paper:

Theorem 2.3. *A twisted commutative algebra finitely generated in order 1 is noetherian.*

Proof. A twisted commutative algebra which is finitely generated in order 1 is a quotient of $\mathrm{Sym}(U\langle 1\rangle)$ for some finite dimensional vector space U. Thus it suffices to show that such algebras are noetherian. Fix U and put $A = \mathrm{Sym}(U\langle 1\rangle)$. In the GL-model, A is given by $\mathrm{Sym}(U \otimes \mathbf{C}^\infty)$. Let $\dim U = d$. The following lemma shows that contraction from $\mathrm{GL}(\infty)$-stable ideals of $\mathrm{Sym}(U \otimes \mathbf{C}^\infty)$ to ideals of $\mathrm{Sym}(U \otimes \mathbf{C}^d)$ is injective. As $\mathrm{Sym}(U \otimes \mathbf{C}^d)$ is a polynomial algebra in finitely many variables, it is noetherian. We conclude that A is a noetherian module over itself. A slight modification of this argument shows that any finitely generated A-module is noetherian, which proves that A is noetherian as an algebra. $\square$

Lemma 2.4 (Weyl)**.** *Let U be a vector space of dimension d. If $W \subset \mathrm{Sym}^k(U \otimes \mathbf{C}^\infty)$ is a $\mathrm{GL}(\infty)$-stable subspace then W is generated as a $\mathrm{GL}(\infty)$-module by $W \cap \mathrm{Sym}^k(U \otimes \mathbf{C}^d)$.*

Proof. Using the formula for the symmetric power of a tensor product (see [FH, Exercise 6.11(b)]), we obtain a diagram

$$\begin{tikzcd} \mathrm{Sym}^k(U \otimes \mathbf{C}^d) \arrow[r, equal] \arrow[d, hook] & \bigoplus \mathbf{S}_\lambda(U) \otimes \mathbf{S}_\lambda(\mathbf{C}^d) \arrow[d, hook] \\ \mathrm{Sym}^k(U \otimes \mathbf{C}^\infty) \arrow[r, equal] & \bigoplus \mathbf{S}_\lambda(U) \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty) \end{tikzcd}$$

The sums are taken over the partitions λ of k. Now, let W be a $\mathrm{GL}(\infty)$-stable subspace of $\mathrm{Sym}^k(U \otimes \mathbf{C}^\infty)$. Since the $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ are mutually non-isomorphic irreducible $\mathrm{GL}(\infty)$-modules, we can write $W = \bigoplus W_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$, where W_λ is a subspace of $\mathbf{S}_\lambda(U)$. Note that if λ has more than d rows then $\mathbf{S}_\lambda(U) = 0$ and so $W_\lambda = 0$. The space $W \cap \mathrm{Sym}^k(U \otimes \mathbf{C}^d)$ is equal to $\bigoplus W_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^d)$. Since $\mathbf{S}_\lambda(\mathbf{C}^d)$ generates $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ whenever λ has at most d rows, we find that $W \cap \mathrm{Sym}^k(U \otimes \mathbf{C}^d)$ generates W. $\square$

Any twisted commutative algebra finitely generated in degree 1 has the property that the partitions appearing in it have a bounded number of rows. Most of the results we prove about algebras generated in degree 1, such as the above theorem, trivially extend to the larger class of algebras which have a bounded number of rows. Going beyond this class of algebras is harder though. For instance, we do not know how much the degree 1 hypothesis in Theorem 2.3 can be relaxed; see §5. We need one more general finiteness result about twisted commutative algebras.

Proposition 2.5. *Let A be a noetherian twisted commutative algebra on which a reductive group G acts. Then A^G is noetherian. If M is a finitely generated A-module with a compatible action of G then M^G is a finitely generated A^G-module.*

Proof. The usual proof works. To show how it carries over, we prove that A^G is noetherian. We must therefore show that $A^G \otimes F$ is a noetherian A^G-module for any finite length object F of $\mathrm{Sym}(\mathrm{Vec})$. We have an inclusion $A^G \otimes F \subset A \otimes F$. Let G act on $A \otimes F$ by acting trivially on F. Let M be an A^G-submodule of $A^G \otimes F$ and let M' be the A-submodule of $A \otimes F$ it generates. Let y be an element of M' which is G-invariant. We can then write $y = \sum a_i x_i$ where the a_i are elements of A and the x_i are elements of M. As y and the x_i are invariant, we have $y = \mathrm{avg}(y) = \sum \mathrm{avg}(a_i) x_i$. Thus y belongs to M. This shows that $M = (M')^G$. Therefore, the map

$$\{A^G\text{-submodules of } A^G \otimes F\} \to \{A\text{-submodules of } A \otimes F\}$$

which takes an A^G-submodule to the A-submodule it generates is inclusion preserving and injective. As any ascending chain on the right side stabilizes, so too does any ascending chain on the left side. This proves that A^G is noetherian. We leave the statement about modules to the reader. $\square$

2.4. **The category** $\mathrm{Sym}(\mathcal{S})$**.** We now investigate the category $\mathrm{Sym}(\mathcal{S})$, which will feature prominently in what follows. We begin by giving a useful description of it. Let Vec^f denote the category of finite families of vector spaces. An object of Vec^f is a pair (V, L) where L is a finite set and V assigns to each element x of L a vector space V_x. A morphism $(V, L) \to (V', L')$ consists of a bijection $i : L \to L'$ and a linear map $V_x \to V'_{i(x)}$ for each $x \in L$. We now claim that the following three categories are equivalent:

(a) The category $\mathrm{Sym}(\mathcal{S})$.
(b) The category of nice functors $\mathrm{Vec}^f \to \mathrm{Vec}$.
(c) The category of sequences (F_n) where F_n is a nice S_n-equivariant functor $\mathrm{Vec}^n \to \mathrm{Vec}$.

The equivalences, as well as the definition of "nice" in (b), should be clear. Each of these categories is an abelian tensor category and the various equivalences respect this structure. The tensor structure in (b) is given by

$$(F \otimes G)(V, L) = \bigoplus_{L = A \amalg B} F(V|_A, A) \otimes G(V|_B, B).$$

The tensor structure in (c) is defined similarly. Note that since $\mathcal{S}$ is itself a tensor category we have a point-wise tensor product $\boxtimes$ on $\mathrm{Sym}(\mathcal{S})$. In (b) this product is given by

$$(F \boxtimes G)(V, L) = F(V, L) \otimes G(V, L),$$

hence the name "point-wise tensor product." This tensor product will not be used in the rest of this section but will show up later.

As $\mathrm{Sym}(\mathcal{S})$ is an abelian tensor category, we have the notions of algebras and modules over algebras in $\mathrm{Sym}(\mathcal{S})$. We note that for $A \in \mathrm{Sym}(\mathcal{S})$ giving a multiplication map $A \otimes A \to A$ amounts to giving a natural map

$$A(V, L) \otimes A(V', L') \to A(V \amalg V', L \amalg L'),$$

where $V \amalg V'$ denotes the natural map $L' \amalg L \to \mathrm{Vec}$ built out of V and V'. Define an *element* of $F \in \mathrm{Sym}(\mathcal{S})$ to be an element of $F(V, L)$ for some (V, L). We call $\#L$ the *order* of the element. (We will use the term "degree" later to reference the grading on $\mathcal{S}$.) As in the twisted commutative setting, one can then give an elemental definition for "finitely generated" and this agrees with the more general definition in terms of finite length objects given in §2.2.

We now give examples of some of the above definitions to give the reader some sense of their nature. Let F_λ be the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space 0 if $\#L \neq 1$ and the space $\mathbf{S}_\lambda(V_x)$ if $L = \{x\}$. Then $F_\lambda \otimes F_\mu$ assigns to (V, L) the space 0 if $\#L \neq 2$ and the space

$$[\mathbf{S}_\lambda(V_x) \otimes \mathbf{S}_\mu(V_y)] \oplus [\mathbf{S}_\lambda(V_y) \otimes \mathbf{S}_\mu(V_x)]$$

if $L = \{x, y\}$. As a second example, $\mathrm{Sym}(F_\lambda)$ is the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space $\bigotimes_{x \in L} \mathbf{S}_\lambda(V_x)$. Note in particular that the only partition appearing in the symmetric algebra $\mathrm{Sym}(F_\lambda)$ is λ itself. These examples underline that the product in $\mathrm{Sym}(\mathcal{S})$ is formal: the Littlewood–Richardson rule does intervene in any way. (The Littlewood–Richardson rule is used in the point-wise product $\boxtimes$ for $\mathrm{Sym}(\mathcal{S})$.)

The following is the main result we need on algebras in $\mathrm{Sym}(\mathcal{S})$.

Theorem 2.6. *An algebra in* $\mathrm{Sym}(\mathcal{S})$ *which is finitely generated in order 1 is noetherian.*

Proof. Let A be an algebra finitely generated in order 1. Let F be a finite length object of $\mathrm{Sym}(\mathcal{S})$. We must show that $A \otimes F$ is a noetherian A-module. As with any finitely generated algebra in $\mathrm{Sym}(\mathcal{S})$, the number of rows in any partition appearing in A is bounded; the same holds for $A \otimes F$. Let d be an integer such that any partition appearing in $A \otimes F$ has at most d rows, and let U be a vector space of dimension d. For a finite set L let U_L be the constant family on U, i.e., the object (V, L) of Vec^f with $V_x = U$ for all x. Then $L \mapsto U_L$ defines a functor $i : (\mathrm{fs}) \to \mathrm{Vec}^f$, which respects the monoidal structure (disjoint union) on each category. The induced functor $i^* : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathrm{Vec})$ is a tensor functor. One easily sees that if M' and M are two sub-objects of $A \otimes F$ then $M = M'$ if and only if $i^*M = i^*M'$. We have thus shown that the map

$$\{A\text{-submodules of } A \otimes F\} \to \{(i^*A)\text{-submodules of } i^*(A \otimes F)\}$$

is injective; it is obviously inclusion preserving. Now, i^*A is a twisted commutative algebra finitely generated in order 1; to see this, write A as a quotient of $\mathrm{Sym}(S)$ with $S \in \mathcal{S}$, so that i^*A is a quotient of $\mathrm{Sym}(S(U)\langle 1 \rangle)$. We thus find that i^*A is noetherian by Theorem 2.3. As $i^*(A \otimes F) = (i^*A) \otimes (i^*F)$ is a finitely generated

(i^*A)-module, it is noetherian. We thus find that every ascending chain of A-submodules of $A \otimes F$ stabilizes, and so A is noetherian. □

The following proposition is proved just like Proposition 2.5.

Proposition 2.7. *Let A be a noetherian algebra in $\mathrm{Sym}(\mathcal{S})$ on which a reductive group G acts. Then A^G is again noetherian. If M is a finitely generated A-module with a compatible action of G then M^G is a finitely generated A^G-module.*

2.5. **Δ-modules.** Let Vec^{Δ} be the category whose objects are pairs (V, L) as in Vec^f, but where now a morphism $(V, L) \to (V', L')$ consists of a surjection $L' \to L$ together with a map $V_x \to \bigotimes_{y \mapsto x} V'_y$ for each $x \in L$. We now come to a central concept in this paper:

Definition 2.8. A *Δ-module* is a nice functor $\mathrm{Vec}^{\Delta} \to \mathrm{Vec}$.

In the above definition, we say that a functor $\mathrm{Vec}^{\Delta} \to \mathrm{Vec}$ is nice if it is so when restricted to Vec^f. We denote the category of these functors by Mod_{Δ}. It is abelian, though not semi-simple. Since Vec^f is a sub-category of Vec^{Δ}, every Δ-module defines an object of $\mathrm{Sym}(\mathcal{S})$. For a Δ-module F we let $[F]$ be the class in $\mathrm{Sym}(K(\mathcal{S}))$ obtained by regarding F as an object of $\mathrm{Sym}(\mathcal{S})$.

Let L be a finite set. A *partition* of L is a subset $\mathscr{U}$ of the power set of L consisting of non-empty pairwise disjoint subsets which union to L. We call a partition $\mathscr{U}$ *proper* if it has more than one part, *discrete* if all of its parts are singletons and *little* if one of its parts has two elements and all of its other parts are singletons.

Let (V, L) be an object of Vec^f and let $\mathscr{U}$ be a partition of L. For a subset S of L put $V_S = \bigotimes_{x \in S} V_x$. Then $(V, \mathscr{U})$ is an object of Vec^f. There is a natural map $(V, \mathscr{U}) \to (V, L)$ in Vec^{Δ}, the surjection $L \to \mathscr{U}$ mapping an element of L to the part of $\mathscr{U}$ to which it belongs. Furthermore, every map in Vec^{Δ} can be factored as one of these maps followed by a map in Vec^f. In fact, we can even say a bit more. Call a map $(V, \mathscr{U}) \to (V, L)$ *little* if $\mathscr{U}$ is a little partition. One easily verifies that for any partition $\mathscr{U}$, the map $(V, \mathscr{U}) \to (V, L)$ can be factored into a sequence of little maps. Thus every map in Vec^{Δ} can be factored as a sequence of little maps followed by a map in Vec^f. This allows use to regard Δ-modules as objects of $\mathrm{Sym}(\mathcal{S})$ with a bit of extra structure; see §2.6 for the details.

The forgetful functor $\mathrm{Mod}_{\Delta} \to \mathrm{Sym}(\mathcal{S})$ has a left adjoint, which we denote by Φ. Explicitly, for an object F of $\mathrm{Sym}(\mathcal{S})$ we have

$$(\Phi F)(V, L) = \bigoplus F(V, \mathscr{U}),$$

where the sum is over all partitions $\mathscr{U}$ of L. We leave it to the reader to work out the Δ-structure on $\Phi(F)$ and verify its universal property. We call $\Phi(F)$ the *free Δ-module* on F. By a *(finite) free* Δ-module we mean one isomorphic to $\Phi(F)$, where F is a (finite length) object in $\mathrm{Sym}(\mathcal{S})$. Free modules are projective since Φ is a left adjoint and $\mathrm{Sym}(\mathcal{S})$ is semi-simple.

We define an *element* of a Δ-module F to be an element of $F(V, L)$ for some (V, L). Given a collection of elements of F one can speak of the Δ-submodule that it generates. We say that F is *finitely generated* if there is a finite set of elements of F that generate it. This is equivalent to F being a quotient of a finite free Δ-module. We say that a Δ-module is *noetherian* if every ascending chain of Δ-submodules stabilizes. Note that noetherian implies finitely generated.

Let n be a positive integer. Let T_n be the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V, L) the space $V_x^{\otimes n}$ if $L = \{x\}$ and 0 otherwise. Let W_n be the symmetric algebra on T_n in the category $\mathrm{Sym}(\mathcal{S})$. It is given by

$$W_n(V, L) = \bigotimes_{x \in L} V_x^{\otimes n}$$

The multiplication map in W_n is given by concatenation of tensors. Note that S_n acts on W_n by algebra homomorphisms. We also consider the free Δ-module on T_n. It is given by

$$(\Phi T_n)(V, L) = \bigotimes_{x \in L} V_x^{\otimes n}$$

if $\#L \geq 1$, while $(\Phi T_n)(V, L) = 0$ for $\#L = 0$. Observe that $\Phi(T_n)$ is naturally a subobject of W_n in the category $\mathrm{Sym}(\mathcal{S})$, and is in fact an ideal. The following is the key result connecting Δ-modules to objects we have previously studied, the final rung of the ladder of §1.4.

Proposition 2.9. *Any Δ-submodule of $\Phi(T_n)$ is a $W_n^{S_n}$-submodule of $\Phi(T_n)$.*

Proof. Let (V, L) and (V', L') be two objects of Vec^f. Let $W = \bigotimes_{y \in L'} V'_y$, so that $W_n(V', L') = W^{\otimes n}$. Let $v \in (\Phi T_n)(V, L)$, and let $v' \in W^{\otimes n}$ be S_n-invariant. We must show that the image of $v \otimes v'$ under the multiplication map $(\Phi T_n)(V, L) \otimes W^{\otimes n} \to (\Phi T_n)(V \amalg V', L \amalg L')$ belongs to the Δ-submodule of ΦT_n generated by v. Now, $(W^{\otimes n})^{S_n}$ is spanned by nth tensor powers of elements of W. It thus suffices to treat the case $v' = w^{\otimes n}$ for some $w \in W$.

Pick an element x_0 of L. Define a map $f : (V, L) \to (V \amalg V', L \amalg L')$ in Vec^{Δ}, as follows. The map $L \amalg L' \to L$ is the identity on L and collapses all of L' to x_0. For $x \neq x_0$, the map $f_x : V_x \to V_x$ is the identity. The map $f_{x_0} : V_{x_0} \to V_{x_0} \otimes W$ is given by $\mathrm{id} \otimes w$. The map f induces a map $(\Phi T_n)(V, L) \to (\Phi T_n)(V \amalg V', L \amalg L')$, which one easily verifies is the map induced by multiplication by $w^{\otimes n}$ on W_n. Thus if v is an element of $(\Phi T_n)(V, L)$, then its product with $w^{\otimes n}$ in W_n can be computed by taking its image under $(\Phi T_n)(f)$. This shows that the product of v and v' belongs to the Δ-module generated by v, which completes the proof. □

Theorem 2.10. *The Δ-module $\Phi(T_n)$ is noetherian.*

Proof. The algebra W_n is noetherian by Theorem 2.6, and so $W_n^{S_n}$ is noetherian by Proposition 2.7. As W_n is a finite $W_n^{S_n}$-module, it is a noetherian $W_n^{S_n}$-module. The same holds for the submodule $\Phi(T_n)$. If M_i is an ascending chain of Δ-submodules of $\Phi(T_n)$ then it is an ascending chain of $W_n^{S_n}$-submodules by the previous proposition, and therefore stabilizes. Thus $\Phi(T_n)$ is a noetherian Δ-module. □

Call a Δ-module *small* if it is a subquotient of a finite direct sum of $\Phi(T_n)$'s (with the n allowed to vary). The above theorem implies that small Δ-modules are noetherian, and in particular finitely generated. We record the following result, which follows immediately from the definitions and Proposition 2.9.

Proposition 2.11. *Let F be a small Δ-module. Then there exists a finite chain $0 = F_0 \subset \cdots \subset F_r = F$ of Δ-submodules of F and integers n_i such that F_i/F_{i-1}, regarded as an object of* $\mathrm{Sym}(\mathcal{S})$*, can be given the structure of a finitely generated module over $W_{n_i}^{S_{n_i}}$.*

Remark 2.12*.* We can in fact show that all finitely generated Δ-modules are noetherian. The argument in the general case is by a Gröbner degeneration, and is much different than our argument for $\Phi(T_n)$ presented above. However, the above argument for $\Phi(T_n)$, which relates Δ-submodules to modules over $W_n^{S_n}$, is important for our later arguments with Hilbert series.

2.6. **More on Δ-modules.** Let F be a Δ-module. We define $F^{\mathrm{old}}(V, L)$ to be the space spanned by the images of the maps $F(V, \mathscr{U}) \to F(V, L)$, as $\mathscr{U}$ varies over all non-discrete partitions of L. One easily verifies that F^{old} is a Δ-submodule of F. We define a functor

$$\Psi : \mathrm{Mod}_{\Delta} \to \mathrm{Sym}(\mathcal{S}), \qquad \Psi(F) = F/F^{\mathrm{old}}.$$

Note that $\Psi(F)$ is naturally a Δ-module. However, if $(V, L) \to (V', L')$ is a map in Vec^{Δ} and $L' \to L$ is not an isomorphism, then $\Psi(F)$ applied to this map is zero; this is why we regard $\Psi(F)$ as an object of $\mathrm{Sym}(\mathcal{S})$. In fact, $\Psi(F)$ is the universal quotient of F with this property. One may thus regard $\Psi(F)$ as the maximal semi-simple quotient (i.e., cosocle) of F.

A Δ-module M is finitely generated if and only if $\Psi(M)$ is a finite length object of $\mathrm{Sym}(\mathcal{S})$; this is a version of Nakayama's lemma. In fact, M is a quotient of $\Phi(\Psi(M))$, though non-canonically. We have $\Psi(\Phi(F)) = F$. The functor Ψ is right exact, but not exact. Its left derived functors $L^i\Psi$ exist. If M is a finitely generated Δ-module then $L^i\Psi(M)$ is a finite length object of $\mathrm{Sym}(\mathcal{S})$; this can be deduced easily from the fact that finitely generated Δ-modules are noetherian. One can recover $[M]$ from $[L\Psi M]$ by applying Φ. Thus the sequence of polynomials $[L^i\Psi M]$ contains more information than the series $[M]$.

We now give an alternative definition of Δ-modules. Let $m^* : \mathcal{S} \to \mathcal{S}^{\otimes 2}$ be the co-multiplication map. It takes $F \in \mathcal{S}$ to the functor $m^*F \in \mathcal{S}^{\otimes 2}$ given by $(V, W) \mapsto F(V \otimes W)$. The functor m^*F has a natural S_2-equivariant structure and so defines an object of $\mathrm{Sym}^2(\mathcal{S})$. There is a unique extension of m^* to a derivation

$$\Delta : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathcal{S}).$$

Here by "derivation" we mean Δ satisfies the Leibniz rule and interacts correctly with divided powers (Schur functors). A Δ-module is then just an object M of $\mathrm{Sym}(\mathcal{S})$ together with a map $\Delta M \to M$ which satisfies a certain associativity condition, which we do not write out. The map $\Delta M \to M$ precisely records the functoriality of M with respect to little maps in Vec^{Δ}, and the associativity condition ensures that M

extends to a functor with respect to all maps in Vec^{Δ}. The image of the map $\Delta M \to M$ is M^{old}, and so its cokernel is $\Psi(M)$.

There is an analogy between Δ-modules and graded $\mathbf{C}[t]$-modules. The category $\mathrm{Sym}(\mathcal{S})$ is analogous to the category of graded vector spaces. The functor Φ is analogous to the functor which takes a graded vector space V to the free graded $\mathbf{C}[t]$-module $\mathbf{C}[t] \otimes V$, while the functor Ψ is analogous to the functor which takes a graded $\mathbf{C}[t]$-module M to the graded vector space $M \otimes_{\mathbf{C}[t]} \mathbf{C}$. The map $\Delta M \to M$ is analogous to multiplication by t, while the space M^{old} is analogous to the image of t.

3. Hilbert series

In this section we develop the theory of Hilbert series for certain objects of $\mathrm{Sym}(\mathrm{Vec})$, $\mathrm{Sym}(\mathcal{S})$ and Mod_{Δ}. The main results are rationality theorems. If A is a finitely generated graded ring, in the usual sense, one can prove the rationality of its Hilbert series by picking a surjection $P \to A$, where P is a polynomial ring, resolving A by free P-modules and then explicitly computing the Hilbert series of a free P-module. The key fact that makes this work is that P has finite global dimension. In the setting of twisted commutative algebras, this approach is no longer viable: the twisted commutative algebra $\mathrm{Sym}(U\langle 1\rangle)$ has infinite global dimension for any non-zero U. The reason for this is that no wedge power of $U\langle 1\rangle$ vanishes, so the Koszul complex does not terminate! We get around this problem by relating Hilbert series of twisted commutative algebras to equivariant Hilbert series of usual rings, where we can use the usual methods. To study Hilbert series of objects in $\mathrm{Sym}(\mathcal{S})$, we relate them to Hilbert series of twisted commutative algebras. Finally, to study Hilbert series of objects in Mod_{Δ} (what we ultimately care about), we relate them to Hilbert series of objects in $\mathrm{Sym}(\mathcal{S})$.

3.1. **Hilbert series in** $\mathrm{Sym}(\mathrm{Vec})$**.** Let M be an object of $\mathrm{Sym}(\mathrm{Vec})$, taken in the sequence model. We assume each M_n is finite dimensional. We define the *Hilbert series* H_M of M by:

$$H_M(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (\dim M_n)\, t^n.$$

Of course, $H_M(t)$ is simply the element $[M]$ of $\mathrm{Sym}(K(\mathrm{Vec})) = \mathbf{Q}[\![t]\!]$. Our goal in this section is to demonstrate the following theorem:

Theorem 3.1. *Let A be a twisted commutative algebra finitely generated in order 1 and let M be a finitely generated A-module. Then $H_M(t)$ is a polynomial in t and e^t.*

Define $H^*_M(t)$ similarly to $H_M(t)$, but without the factorial. The theorem is equivalent to the following statement, which is what we actually prove: we have

$$H^*_M(t) = \sum_{k=0}^{d} \frac{p_k(t)}{(1-kt)^{a_k}}$$

for some integer d, polynomials $p_k(t)$ and non-negative integers a_k. Note that for a module over a graded ring, in the usual sense, the Hilbert series only has a pole at $t = 1$, while our Hilbert series for modules over twisted commutative algebras can have poles at $t = 1/k$ for any non-negative integer k.

The above theorem only applies to modules over twisted commutative algebras generated in order 1, and is false more generally. For example, let $M = A = \mathrm{Sym}((\mathbf{C}^{\infty})^{\otimes 2})$, a twisted commutative algebra generated in order 2. Then $H_M(t) = e^{t^2}$. Although this is not a polynomial in t and e^t, it is a very reasonable function, and one can hope that there is a nice generalization of Theorem 3.1.

Before getting into the proof of Theorem 3.1 we introduce equivariant Hilbert series. Say a group G acts on an object M of $\mathrm{Sym}(\mathrm{Vec})$. We define its *$G$-equivariant Hilbert series* $H^*_{M,G}$ by:

$$H^*_{M,G}(t) = \sum_{n=0}^{\infty} [M_n] t^n$$

where $[M_n]$ denotes the class of M_n in the Grothendieck group $K(G)$. Thus $H^*_{M,G}$ is a power series with coefficients in the ring $K(G)$. We will need to use these Hilbert series in our proof of Theorem 3.1 and we will also need a generalization of Theorem 3.1 to the equivariant setting.

We now begin the proof of Theorem 3.1. Thus let A and M be given. Since A is finitely generated in order one, it is a quotient of $\mathrm{Sym}(U\langle 1\rangle)$ for some finite dimensional vector space U. Of course, M is a finitely generated module over $\mathrm{Sym}(U\langle 1\rangle)$. It thus suffices to consider the case where $A = \mathrm{Sym}(U\langle 1\rangle)$. Now, regard A and M in the Schur model. If $\mathbf{S}_\lambda$ occurs in A then λ has at most $\dim U$ rows. Since M is finitely generated, it is a quotient of $A \otimes S$ for some finite length object S of $\mathcal{S}$; it follows that there is an integer d such that only those $\mathbf{S}_\lambda$ for which λ has at most d rows appear in M. We therefore do not lose information by considering $M(\mathbf{C}^d)$ with its $\mathrm{GL}(d)$ action. In fact, we can even consider $M(\mathbf{C}^d)$ with its T action without losing information, where T is the diagonal torus in $\mathrm{GL}(d)$. The main idea of the proof of Theorem 3.1 is to relate H^*_M to $H^*_{M(\mathbf{C}^d),T}$, prove that the latter is of a specific form and then deduce from this the rationality of H^*_M. (One can regard any graded $\mathbf{C}$-algebra as a twisted commutative algebra. Thus $H^*_{M(\mathbf{C}^d),T}$ makes sense. In fact, it agrees with the usual T-equivariant Hilbert series of $M(\mathbf{C}^d)$.)

We need to introduce a bit of notation related to T. We let $\alpha_1, \ldots, \alpha_d$ be the standard projections $T \to \mathbf{G}_m$. We define an involution of the coordinate ring of T, denoted with an overline, by $\overline{\alpha}_i = \alpha_i^{-1}$, and we write $|x|^2$ for $x\overline{x}$. We let $\Delta(\alpha)$ be the discriminant $\prod_{i<j}(\alpha_i - \alpha_j)$. For a character χ of T we define $\int_T \chi(\alpha) d\alpha$ to be 1 if χ is trivial and 0 otherwise, and we extend $\int_T d\alpha$ linearly to all functions on T. (The symbol $\int_T d\alpha$ is just notation and does not indicate actual integration.) If χ_1 and χ_2 are characters of irreducible representations of $\mathrm{GL}(d)$ then Weyl's integration formula (see [FH, §26.2]), stated in our language, reads

$$\frac{1}{d!}\int_T \chi_1(\alpha)\chi_2(\overline{\alpha})|\Delta(\alpha)|^2 d\alpha = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{if } \chi_1 \neq \chi_2 \end{cases}$$

We identify $K(T)$ with $\mathbf{Q}[\alpha_i, \alpha_i^{-1}]$ so that a T-equivariant Hilbert series can be identified with a power series in t whose coefficients are Laurent polynomials in the α_i. The following is a key step in our understanding of H^*_M:

Lemma 3.2. *We have*

$$H^*_M(t) = \frac{1}{d!}\int_T H^*_{M(\mathbf{C}^d),T}(t;\alpha)\frac{|\Delta(\alpha)|^2}{1-\sum \overline{\alpha}_i} d\alpha.$$

Proof. Write $M = \bigoplus \mathbf{S}_\lambda^{\oplus m_\lambda}$, the sum taken over λ. We then have:

$$H^*_M(t) = \sum_\lambda m_\lambda \cdot (\dim \mathbf{M}_\lambda) \cdot t^{|\lambda|}.$$

On the other hand

$$H^*_{M(\mathbf{C}^d),T}(t;\alpha) = \sum_\lambda m_\lambda \cdot (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^d)) \cdot t^{|\lambda|}.$$

Put

$$f(\alpha) = \sum_\lambda (\text{the character of } \mathbf{S}_\lambda(\mathbf{C}^d)) \cdot \dim \mathbf{M}_\lambda.$$

Weyl's integration formula now gives us

$$H^*_M(t) = \frac{1}{d!}\int_{T'} H^*_{M(\mathbf{C}^d),T}(t;\alpha) f(\overline{\alpha})|\Delta(\alpha)|^2 d\alpha.$$

We must compute $f(\alpha)$. Observe:

$$\bigoplus_{k=0}^{\infty} (\mathbf{C}^d)^{\otimes k} = \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^d) \otimes \mathbf{M}_\lambda.$$

The character of the right side is $f(\alpha)$. The character of the left side is

$$\sum_{k=0}^{\infty}\left(\sum_{i=1}^{d} \alpha_i\right)^k = \frac{1}{1-\sum \alpha_i}.$$

This yields the stated formula. □

We have thus related H^*_M, what we care about, to $H^*_{M(\mathbf{C}^d),T}$, which should be easier to understand since $M(\mathbf{C}^d)$ is a finitely generated module over the polynomial ring $A(\mathbf{C}^d)$. We now see that $H^*_{M(\mathbf{C}^d),T}$ is indeed easy to understand:

Lemma 3.3. *We have*

$$H^*_{M(\mathbf{C}^d),T}(t;\alpha) = \frac{p(t;\alpha)}{\prod_{i=1}^d (1-\alpha_i t)^n}$$

where $p(t;\alpha)$ is a polynomial and $n = \dim U$.

Proof. The terms of the minimal resolution for $M(\mathbf{C}^d)$ over $A(\mathbf{C}^d)$ are $A(\mathbf{C}^d) \otimes E_\bullet$ where

$$E_i = \mathrm{Tor}_i^{A(\mathbf{C}^d)}(M(\mathbf{C}^d), \mathbf{C}).$$

Since Tor is functorial, each E_i carries an action of $\mathrm{GL}(d)$ (and therefore T), and the minimal resolution is equivariant (or rather, can be taken to be so). Thus the T-equivariant Hilbert series for M is the alternating sum of those for $A(\mathbf{C}^d) \otimes E_i$; of course, each of these is the product of those for $A(\mathbf{C}^d)$ and E_i. Since E_i is a finite dimensional representation of T its Hilbert series is a polynomial. Thus the lemma is reduced to the case $M = A$. Now,

$$A(\mathbf{C}^d) = \mathrm{Sym}(U \otimes \mathbf{C}^d) = \mathrm{Sym}(\mathbf{C}^d \oplus \cdots \oplus \mathbf{C}^d) = \mathrm{Sym}(\mathbf{C}^d) \otimes \cdots \otimes \mathrm{Sym}(\mathbf{C}^d)$$

where $\mathbf{C}^d$ is summed with itself $n = \dim U$ times. We thus find that $H_{A(\mathbf{C}^d),T}$ is the nth power of $H^*_{\mathrm{Sym}(\mathbf{C}^d),T}$. Similarly, $\mathrm{Sym}(\mathbf{C}^d) = \mathrm{Sym}(\mathbf{C} \oplus \cdots \oplus \mathbf{C})$, where there are d copies of $\mathbf{C}$ and T acts on the ith one by the character α_i. Thus $H^*_{\mathrm{Sym}(\mathbf{C}^d),T} = \prod (1-\alpha_i t)^{-1}$. This proves the lemma. □

Combining the two lemmas, we obtain an expression

$$H^*_M(t) = \int_T \frac{p(t;\alpha)}{\prod (1-\alpha_i t)^n} \frac{1}{1 - \sum \overline{\alpha}_i} d\alpha$$

where $p(t;\alpha)$ is a polynomial in t, the α_i and the α_i^{-1}. (We have absorbed the $1/d!$ and $|\Delta(\alpha)|^2$ into p.) Expanding the integrand into a power series, we find

$$H^*_M(t) = \int_T \left[\sum_{k,\ell} \binom{k}{n} \alpha^k \left(\sum \overline{\alpha_i} \right)^\ell p(t;\alpha) t^{|k|} \right] d\alpha$$

where the sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$ and $\ell \in \mathbf{Z}_{\geq 0}$. Here

$$\binom{k}{n} = \binom{k_1}{n} \cdots \binom{k_d}{n}, \qquad \alpha^k = \alpha_1^{k_1} \cdots \alpha_d^{k_d} \qquad \text{and} \qquad |k| = k_1 + \cdots + k_d.$$

We must show that this is a rational function in t. It suffices, by linearity, to treat the case where $p(t;\alpha) = t^{e_0} \alpha_1^{e_1} \cdots \alpha_d^{e_d}$ where the e_i are integers. Of course, the t^{e_0} factor does not really affect anything, so we leave it out. We are thus reduced to showing that

$$\int_T \left[\sum_{k,\ell} \binom{k}{n} \alpha^{k+e} \left(\sum \overline{\alpha_i} \right)^\ell t^{|k|} \right] d\alpha$$

is rational. By degree considerations, the (k,ℓ) term in the above sum integrates to zero unless $\ell = |k+e|$. Furthermore, when $\ell = |k+e|$ only one monomial in $(\sum \overline{\alpha_i})^\ell$ contributes a non-zero quantity, namely the one where $\overline{\alpha}_i$ has exponent $k_i + e_i$. We therefore find that the above is equal to

$$\sum_k \binom{k}{n} C_{k+e} t^{|k|}$$

where

$$C_k = \frac{(|k|)!}{(k_1)! \cdots (k_d)!}$$

is the multinomial coefficient. (We use the convention that $C_k = 0$ if any of the coordinates of k are negative.) Theorem 3.1 now follows from the following lemma:

Lemma 3.4. *Let d be a positive integer, let $e \in \mathbf{Z}^d$ and let p be a polynomial of d variables. Then*

$$\sum p(k) C_{k+e} t^{|k|}$$

is a rational function of t, with poles only at $t = 1/a$ where $1 \leq a \leq d$ is an integer. (The sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$.)

Proof. Observe that the formula

$$k_1 C_{k_1,\dots,k_d} = |k| C_{k_1-1,\dots,k_d}$$

is valid for any tuple of integers $k \in \mathbf{Z}^d$. To prove the lemma, it suffices to treat the case where p is a monomial. Thus, if p is not a constant, we can write $p = k_i p'$ for some index i and some smaller monomial p'. We therefore have

$$\begin{aligned}\sum p(k) C_{k+e} t^{|k|} &= \sum k_i p'(k) C_{k+e} t^{|k|} \\ &= \sum p'(k)(k_i + e_i - e_i) C_{k+e} t^{|k|} \\ &= \sum p'(k)|k+e| C_{k+e'} t^{|k|} - e_i \sum p'(k) C_{k+e} t^{|k|}\end{aligned}$$

where e' is obtained from e by replacing e_i with $e_i - 1$. In the right term we have replaced p with a lower degree polynomial. In the left term we have replaced p with an equal degree polynomial, but one that is of the form $p'(k)|k|$ where p' has smaller degree. (Note that $|k+e| = |k| + |e|$.) It follows that by repeatedly applying the above process, we can reduce to the case where p is a function of $|k|$. Now, note that

$$\sum |k|^n C_{k+e} t^{|k|} = \left(t \frac{d}{dt}\right)^n \sum C_{k+e} t^{|k|}.$$

It thus suffices to show that $\sum C_{k+e} t^{|k|}$ is a rational function. We have

$$\sum_{k \ge 0} C_{k+e} t^{|k|} = \sum_{k \ge e} C_k t^{|k-e|} = t^{-|e|} \sum_{k \ge e} C_k t^{|k|}$$

where $k \ge e$ means $k_i \ge e_i$ for each i. Now, the terms in the right sum for which some k_i is negative are zero and therefore do not contribute. We can thus assume that each e_i is non-negative. We can also ignore the $t^{-|e|}$ factor. Now, write $k = (k_1, k')$ where k' is a $d-1$ tuple, and do similarly for e. Then

$$\sum_{k \ge e} C_k t^{|k|} = \sum_{k_1 \ge e_1, k' \ge e'} C_k t^{|k|} = \sum_{k_1 \ge 0, k' \ge e'} C_k t^{|k|} - \sum_{k_1=0}^{e_1-1} \sum_{k' \ge e'} C_k t^{|k|}$$

Now, for k_1 fixed, we have

$$\sum_{k' \ge e'} C_k t^{|k|} = \frac{t^{k_1}}{k_1!} \sum_{k' \ge e'} (|k'| + k_1) \cdots (|k'| + 1) C_{k'} t^{|k'|}.$$

Thus each of the sums on the right in the previous expression is of the general form that we are considering in this lemma but with a smaller d. We can therefore assume that they are each rational by induction. By repeating this procedure, we can thus reduce to the case $e = 0$. The identity

$$\sum_{|k|=n} C_k = d^n$$

now gives

$$\sum_{k \ge 0} C_k t^{|k|} = \frac{1}{1 - dt}.$$

This completes the proof. □

3.2. **Equivariant Hilbert series in** Sym(Vec)**.** Unfortunately, Theorem 3.1 is too weak for our eventual applications. Before stating the result we need, we make a definition for the sake of clarity:

Definition 3.5. Let A be a ring. A series $f \in A[\![t]\!]$ is *rational* if there exists a polynomial $q \in A[t]$ with $q(0) = 1$ such that qf is a polynomial.

Note that it could be that $f \in A[\![t]\!]$ is not rational, but that there is an extension $A \subset B$ so that f is rational when regarded as an element of $B[\![t]\!]$. Thus a bit of care needs to be taken with the definition. However, we do have the following simple result, the proof of which is left to the reader:

Lemma 3.6. *Let $A \subset B$ be an inclusion of rings and let f be an element of $A[\![t]\!]$ such that f is rational when regarded as an element of $B[\![t]\!]$. Then f itself is rational in the following cases: (1) there is a finite group G acting on B such that $A = B^G$; (2) A and B are fields.*

The main result of this section is the following:

Theorem 3.7. *Let G be a connected reductive group, let Γ be a finite group, let A be a twisted commutative algebra finitely generated in order 1 on which $G \times \Gamma$ acts and let M be a finitely generated A-module with a compatible action of $G \times \Gamma$. Then the G-equivariant Hilbert series $H^*_{M^\Gamma,G}$ of M^Γ, regarded as an element of the power series ring $K(G)[\![t]\!]$, is a rational function.*

Most likely, this proposition could be generalized by replacing $\Gamma \subset G \times \Gamma$ with an arbitrary normal algebraic subgroup of an arbitrary algebraic group. We do not need this more general result, and so only prove the special one, which allows for some simplifications in the proof. With some book-keeping, one can also show that the denominator of $H^*_{M^\Gamma,G}$ has a particular form, but we do not do this. The theorem implies a certain result about $H_{M^\Gamma,G}$, but one that is not so elegant: exponentials of algebraic functions (roots of polynomials over $K(G)$) appear. We prove this proposition following the same plan as the proof of last one, after some preliminary reductions.

First, we observe that it suffices to prove that $H^*_{M,\Gamma\times G}$, an element of $K(\Gamma \times G)[\![t]\!]$, is a rational function. To see this, assume we have an equation $(1 + tq)H^*_{M,\Gamma\times G} = p$ with p and q in $K(\Gamma \times G)[\![t]\!]$. Now, observe that $K(\Gamma)$ can be thought of as the ring of class functions on Γ (at least, after an extension of scalars, which does not affect rationality by Lemma 3.6) and so we may write

$$H^*_{M,\Gamma\times G} = \sum H_i\delta_i, \qquad q = \sum q_i\delta_i, \qquad p = \sum p_i\delta_i$$

where H_i, q_i and t_i belong to $K(G)[\![t]\!]$ and the δ_i are characteristic functions of conjugacy classes in Γ. Since the δ_i are orthogonal idempotents, we have

$$(1 + tq)H^*_{M,\Gamma\times G} = \sum (1 + tq_i)H_i\delta_i = \sum p_i\delta_i$$

and so $(1+tq_i)H_i = p_i$ holds for each i. Thus each H_i is a rational function in $K(G)[\![t]\!]$. Since $H^*_{M^\Gamma,G} = \sum H_i$, it follows that it too is a rational function. This establishes our claim.

Now, we can think of the coefficients of $H^*_{M,\Gamma\times G}$ as class functions on Γ. The series $H^*_{M,\Gamma\times G}$ defines a rational element of $K(\Gamma \times G)[\![t]\!]$ if and only if the series $H^*_{M,\gamma,G}$ obtained by evaluating on the element $\gamma \in \Gamma$ is a rational element of $K(G)[\![t]\!]$, for each γ. Let Γ' be the cyclic subgroup generated by some $\gamma \in \Gamma$. By the same reasoning, $H^*_{M,\gamma,G}$ will be a rational element of $K(G)[\![t]\!]$ if $H^*_{M,\Gamma'\times G}$ is a rational element of $K(\Gamma' \times G)[\![t]\!]$. Thus it suffices to show that for each cyclic subgroup Γ' of Γ, the series $H^*_{M\Gamma'\times G}$ is a rational element of $K(\Gamma' \times G)[\![t]\!]$. In other words, we may assume from the outset that Γ is cyclic.

We make another reduction. We have $K(G) = K(H)^W$ where H is a maximal torus in G and W is its Weyl group. By Lemma 3.6, a power series with coefficients in $K(G)$ is rational if and only if it is so when regarded with $K(H)$ coefficients. Thus it suffices to show that $H^*_{M,\Gamma\times H}$ is rational. In other words, we may as well assume from the outset that G is a torus.

Finally, as in the proof of the Theorem 3.1, we may assume $A = \mathrm{Sym}(U\langle 1\rangle)$, where U is a finite dimensional representation of $\Gamma \times G$. Since $\Gamma \times G$ is a commutative reductive group, we can write $U = \bigoplus_{j=1}^n \mathbf{C}\psi_j$ where the ψ_j are characters of $\Gamma \times G$. Pick d large compared to the number of rows appearing in M and let $T \subset \mathrm{GL}(d)$ be the diagonal torus. We now have:

Lemma 3.8. *We have*

$$H^*_{M(\mathbf{C}^d),\Gamma\times G\times T}(t;\alpha) = \frac{p(t;\alpha)}{\prod_{ij}(1 - \alpha_i\psi_j t)}.$$

Here p belongs to $K(\Gamma\times G\times T)[t] = K(\Gamma\times G)[t, \alpha_i]$. The product is taken over $1 \le i \le d$ and $1 \le j \le \dim U$.

Proof. As before, we can reduce to the case $M = A$ by considering the minimal resolution of M. An easy computation, similar to the previous one, gives $H^*_{A(\mathbf{C}^d),\Gamma\times G\times T} = \prod_{ij}(1 - \alpha_i\psi_j t)^{-1}$. $\square$

Lemma 3.2 carries over exactly to the present situation. Combining it with the previous lemma yields

$$H^*_{M,\Gamma\times G}(t) = \int_T \frac{p(t;\alpha)}{\prod_{ij}(1 - \alpha_i\psi_j t)} \frac{1}{1 - (\sum \overline{\alpha}_i)} d\alpha. \tag{1}$$

For i fixed, we have

$$\frac{1}{\prod_j(1 - \alpha_i\psi_j t)} = \sum \alpha_i^{|a|}\psi^a t^{|a|}$$

where the sum is taken over $a \in \mathbf{Z}^n_{\geq 0}$, with $n = \dim U$, $|a|$ is defined as $a_1 + \cdots + a_n$ and ψ^a is defined as $\psi_1^{a_1} \cdots \psi_n^{a_n}$. Define

$$\begin{bmatrix} k \\ n \end{bmatrix}_\psi = \sum_{|a|=k} \psi^a,$$

where a belongs to $\mathbf{Z}^n_{\geq 0}$, so that

$$\frac{1}{\prod_j (1 - \alpha_i \psi_j t)} = \sum_{k=0}^{\infty} \begin{bmatrix} k \\ n \end{bmatrix}_\psi \alpha_i^k t^k.$$

With this notation in hand, we now expand the integrand of (1) into a series. We find

$$H^*_{M,\Gamma\times G}(t) = \int_T \left[\sum_{k,\ell} \begin{bmatrix} k \\ n \end{bmatrix}_\psi \alpha^k \left(\sum \overline{\alpha_i} \right)^\ell p(t;\alpha) t^{|k|} \right] d\alpha.$$

The sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$ and $\ell \in \mathbf{Z}_{\geq 0}$ and our notation is as before; we mention

$$\begin{bmatrix} k \\ n \end{bmatrix}_\psi = \begin{bmatrix} k_1 \\ n \end{bmatrix}_\psi \cdots \begin{bmatrix} k_d \\ n \end{bmatrix}_\psi.$$

We must show that the previous equation is rational in t. By linearity, it suffices to treat the case where $p(t;\alpha)$ is of the form $xt^{e_0}\alpha_1^{e_1} \cdots \alpha_d^{e_d}$ where x belongs to $K(\Gamma \times G)$ and the e_i are integers. Of course, xt^{e_0} pulls out of the integral, and can thus be safely ignored. Hence, it suffices to consider the case where p is α^e, with $e = (e_1, \ldots, e_d)$. As before, the (k,ℓ) term only contributes if $\ell = |k + e|$ and then only one term of $(\sum \overline{\alpha}_i)^\ell$ contributes. The previous integral thus evaluates to:

$$\sum_k \begin{bmatrix} k \\ n \end{bmatrix}_\psi C_{k+e} t^{|k|}.$$

Now, for a single integer k the expression $\begin{bmatrix} k \\ n \end{bmatrix}_\psi$ is of the form $\sum_i a_i \psi_i^k$ where a_i is a rational function of the ψ. It follows that for $k \in \mathbf{Z}^d_{\geq 0}$ the expression $\begin{bmatrix} k \\ n \end{bmatrix}_\psi$ is of the form $\sum_i a_i \psi_i^k$ where the sum is taken over tuples $i \in \{1, \ldots, n\}^d$ and ψ_i^k denotes $\psi_{i_1}^{k_1} \cdots \psi_{i_d}^{k_d}$; again the a_i are rational functions in the ψ. It thus suffices to show that for each such tuple i, the expression

$$\sum_k \psi_i^k C_{k+e} t^{|k|}$$

is rational in t. This is accomplished in the following lemma, which completes the proof of Theorem 3.7:

Lemma 3.9. *Keep the above notation and let p be a polynomial. Then*

$$\sum p(k) \psi_i^k C_{k+e} t^{|k|}$$

is a rational function of t. (The sum is taken over $k \in \mathbf{Z}^d_{\geq 0}$.) Furthermore, the only denominators which appear are of the form $1 - at$ where a is a sum of at most d of the ψ's.

Proof. As in the proof of Lemma 3.4, we reduce to the case where $p = 1$. We then change k to $k - e$ and pull monomials out of the sum, to obtain an expression of the form

$$\sum_{k \geq e} \psi_i^k C_k t^{|k|}.$$

The difference

$$\sum_{k \geq 0} \psi_i^k C_k t^{|k|} - \sum_{k \geq e} \psi_i^k C_k t^{|k|}$$

is a finite sum of sums of the form considered in the lemma, but with a smaller value of d. (The terms which appear may no longer have $p = 1$; this is the reason for including p in the general form of the sum we consider.) It thus suffices to show that the first sum above is rational in t. We have

$$\sum_{|k|=n} \psi_i^k C_k = (\psi_{i_1} + \cdots + \psi_{i_d})^n = a^n$$

by the multinomial theorem. Thus the first sum in the previous expression is $(1-at)^{-1}$, which completes the proof. □

3.3. **Hilbert series in** $\mathrm{Sym}(\mathcal{S})$**.** Let M be an object of $\mathrm{Sym}(\mathcal{S})$. We can regard M as a sequence of equivariant functors $M_n : \mathrm{Vec}^n \to \mathrm{Vec}$. Write

$$M_n(V_1, \dots, V_n) = \bigoplus_{i \in I_n} \mathbf{S}_{\lambda_{i,1}}(V_1) \otimes \cdots \otimes \mathbf{S}_{\lambda_{i,n}}(V_n)$$

for some index set I_n and partitions $\lambda_{i,j}$ (both depending on n). We assume I_n is finite for each n. Put

$$m_n = \sum_{i \in I_n} s_{\lambda_{i,1}} \cdots s_{\lambda_{i,n}}, \qquad H_M^*(t) = \sum_{n=0}^{\infty} m_n t^n.$$

We regard m_n as an element of the polynomial ring $\mathbf{Q}[s_\lambda]$ and $H_M^*(t)$ as an element of the power series ring $\mathbf{Q}[s_\lambda][\![t]\!]$. The variable t is basically superfluous, since the power of t can be obtained from the order of the polynomial m_n. When t is omitted (or set to 1), H_M^* agrees with $[M]^*$. One can also define $H_M(t)$, which is analogous to $[M]$. Our main result concerning these series is the following:

Theorem 3.10. *Let A be an algebra in $\mathrm{Sym}(\mathcal{S})$ finitely generated in order 1 on which a finite group Γ acts and let M be a finitely generated A-module with a compatible action of Γ. Then $H^*_{M^\Gamma}$ is a rational function of the s_λ.*

As with Theorem 3.7, this result does not translate to an elegant statement about H_{M^Γ}. We now explain the basic strategy of our proof, in the case where Γ is trivial. Let M be a finitely generated A-module. We would like to relate H_M^* to the Hilbert series of an object of $\mathrm{Sym}(\mathrm{Vec})$, so that we can apply the results we have established in that case. The most obvious way to obtain an object of $\mathrm{Sym}(\mathrm{Vec})$ is to pick a vector space U, let $i : (\mathrm{fs}) \to \mathrm{Vec}^f$ be the functor assigning to L the constant family U_L and then consider i^*M. Unfortunately, H_M^* cannot be recovered from $H^*_{i^*M}$, or even $H^*_{i^*M,\mathrm{GL}(U)}$. Thus the most obvious approach fails. However, a slight modification works: instead of picking just one vector space U we pick finitely many $U_1, \dots, U_r$ and build from M an object of $\mathrm{Sym}(\mathrm{Vec})$ with an action of $\mathrm{GL}(U_1) \times \cdots \times \mathrm{GL}(U_r)$. It turns out that, due to the form of M, we can take r large enough so that information is not lost — this is essentially the content of Proposition 3.14 below. Before proving that result, we need some lemmas, the first two of which are left to the reader.

Lemma 3.11. *Let $f : \mathbf{C}^n \to \mathbf{C}^m$ be a polynomial map whose components are homogeneous of positive degree and whose image is not contained in any linear subspace of $\mathbf{C}^m$. Then for $r \gg 0$ any element of $\mathbf{C}^m$ can be expressed as a sum of r elements of the image of f.*

Lemma 3.12. *Let K be a field and let (f_i) be a sequence of elements in K. Assume that there exists $m > 0$ such that for all $k_1, \dots, k_m$ sufficiently large the $m \times m$ matrix $(f_{k_i - j})$ has determinant zero. Then $\sum_{i \geq 0} f_i t^i$ can be expressed in the form a/b where a and b belong to $K[t]$ and $\deg b \leq m-1$.*

Lemma 3.13. *Let A be a UFD and let (f_i) be a sequence of elements of A. Assume that there exists an integer m and elements $\alpha_1, \dots, \alpha_m$ of A such that*

$$f_k = \sum_{i=1}^{m} \alpha_i f_{k-i}$$

holds for all $k \gg 0$. Amongst all such expressions choose one with m minimal. Then given any N there exist $k_1, \dots, k_m > N$ such that the determinant of the matrix $(f_{k_i - j})$ is non-zero.

Proof. Put $f = \sum f_i t^i$, an element of $A[\![t]\!]$, and $q = 1 - \sum_{i=1}^m \alpha_i t^i$, an element of $A[t]$. We have that qf belongs to $A[t]$; write $p = qf$ so that $f = p/q$. Note that $A[t]$ is also a UFD and that the minimality assumption on m implies that p and q are coprime. Indeed, say p and q are both divisible by some non-unit $r \in A[t]$. Then we can write $p = rp'$ and $q = rq'$. Evaluating the second expression at $t = 0$ gives $1 = r(0)q'(0)$ so that $r(0)$ and $q'(0)$ both belong to $A^\times$. We can therefore scale r so that $r(0) = 1$, in which case $q'(0) = 1$ as well. Since r is assumed to be a non-unit and $r(0)$ is a unit, it follows that r has degree at least 1. We then have $f = p'/q'$ with $q'(0) = 1$ and $\deg q' < m$, contradicting the minimality of m.

Now, assume for the sake of contradiction that $\det(f_{k_i - j}) = 0$ for all sufficiently large k_i. By Lemma 3.12 we have $f = g/h$ where g and h belong to $K[t]$ and $\deg h < m$. Here K is the field of fractions of A. Pick

$a \in A$ non-zero so that ag and ah belong to $A[t]$. We then have $(ah)p = (ag)q$. Since p is coprime to q it follows that ah is divisible by q. However, this contradicts h having smaller degree than q. We thus conclude that $\det(f_{k_i-j})$ cannot vanish for all sufficiently large k_i. $\square$

Proposition 3.14. *Let W be a finite dimensional vector space and let V be a finite dimensional subspace of $\mathrm{Sym}(W)$ spanned by homogeneous elements. For a positive integer r, let $i_r : \mathrm{Sym}(V) \to \mathrm{Sym}(W)^{\otimes r}$ be the ring homomorphism which on V is given by $x \mapsto \sum\{x\}_i$, where $\{-\}_i : \mathrm{Sym}(W) \to \mathrm{Sym}(W)^{\otimes r}$ is the ring homomorphism given by inclusion into the ith factor. Then for $r \gg 0$ we have:*

(a) *The map i_r is injective.*
(b) *If $x \in \mathrm{Frac}(\mathrm{Sym}(V))$ and $i_r(x)$ belongs to $\mathrm{Sym}(W)^{\otimes r}$ then x belongs to $\mathrm{Sym}(V)$.*
(c) *A series $f \in \mathrm{Sym}(V)[\![t]\!]$ is rational if and only if $i_r(f)$ is.*

Proof. The map $\mathrm{Sym}(V) \to \mathrm{Sym}(W)$ corresponds to an algebraic map $f : W^* \to V^*$ whose components, with respect to suitable bases, are homogeneous of positive degree. Since $V \to \mathrm{Sym}(W)$ is injective, the image of f is not contained in any linear subspace of V^*. It thus follows from Lemma 3.11 that for $r \gg 0$, any element of V^* is a sum of r elements of the image of f. Now, the map i_r corresponds to the map $i_r^* : (W^*)^r \to V^*$ given by $(w_1, \dots, w_n) \mapsto \sum f(w_i)$. We thus see that i_r^* is surjective for all $r \gg 0$.

Fix $r \gg 0$ and put $i = i_r$. We now show that (a) and (b) hold. The equation $i(x) = 0$ is equivalent to $x \circ i^* = 0$, where x is thought of as a function $V^* \to \mathbf{C}$. Since i^* is surjective, this equation implies $x = 0$. This shows that i is injective. Now say that x belongs to $\mathrm{Frac}(\mathrm{Sym}(V))$ and $i(x)$ is a polynomial. Then x defines a rational function on V^* such that $x \circ i^*$ is a regular function on $(W^*)^r$. Since i^* is surjective, x must be regular on V^* and thus it belongs to $\mathrm{Sym}(V)$.

We now prove (c). It is clear that if f is rational then $i(f)$ is as well. Thus let $f \in \mathrm{Sym}(V)[\![t]\!]$ be given and assume that $i(f)$ is rational. Write $f = \sum f_i t^i$ with $f_i \in \mathrm{Sym}(V)$. The rationality of $i(f)$ means that we can find a polynomial $q = 1 - \sum_{i=1}^m \alpha_i t^i$ with $\alpha_i \in \mathrm{Sym}(W)^{\otimes r}$ such that qf is a polynomial. Choose q with m minimal. We then have

$$f_k = \sum_{i=1}^m \alpha_i f_{k-i}$$

for all sufficiently large k. Thus for all large $k_1, \dots, k_m$ we have the equation $Ax = y$ where A is the $m \times m$ matrix (f_{k_i-j}), x is the column vector (α_i) and y is the column vector (f_{k_i}). By the minimality of m and Lemma 3.13 we can pick k_i so that $\det A$ is non-zero. We then find $x = A^{-1}y$, which shows that α_i belongs to $i(\mathrm{Frac}(\mathrm{Sym}(V)))$. Since α_i also belongs to $\mathrm{Sym}(W)^{\otimes r}$, statement (b) of the lemma implies that each α_i belongs to $i(\mathrm{Sym}(V))$. Thus $q = i(q')$ for a unique q' in $\mathrm{Sym}(V)[t]$ with the required properties to establish that f is rational. $\square$

We now prove the theorem:

Proof of Theorem 3.10. Let Γ, A and M be given. Let P be the set of partitions appearing in M. The set P is finite; indeed, only finitely many partitions appear in A (see the discussion preceding Theorem 2.6) and M is a quotient of $A \otimes F$ for some finite length object F of $\mathrm{Sym}(\mathcal{S})$. Let V be the subspace of $K(\mathcal{S})$ spanned by the s_λ with $\lambda \in P$. Thus $H^*_{M^\Gamma}$ belongs to $\mathrm{Sym}(V)[\![t]\!]$.

Let U be a finite dimensional vector space whose dimension exceeds the number of rows of any partition appearing in P and let $G = \mathrm{SL}(U)$. Then $K(G)$ is a polynomial ring. (We use $\mathrm{SL}(U)$ instead of $\mathrm{GL}(U)$ so that the Grothendieck group is a polynomial ring; it makes the argument a bit cleaner.) Evaluation on U gives a map $K(\mathcal{S}) \to K(G)$ which is injective when restricted to V. Let r be a large integer and let

$$\phi : \mathrm{Sym}(K(\mathcal{S})) \to K(G)^{\otimes r}$$

by the unique ring map which is given by

$$\phi([S]) = \sum_{i=1}^r \{[S(U)]\}_i$$

for $[S]$ in $K(\mathcal{S})$, where $\{-\}_i : K(G) \to K(G)^{\otimes r}$ is the inclusion in the ith factor. By Proposition 3.14(c), a power series $f \in \mathrm{Sym}(V)[\![t]\!]$ is a rational function if and only if $\phi(f)$ is. It thus suffices to show that $\phi(H^*_{M^\Gamma})$ is rational.

To understand the map ϕ we lift it to a functor Φ, as follows. First, identify $K(G)^{\otimes r}$ with $K(G^r)$. Let $U_1, \ldots, U_r$ be copies of U and put $G_i = \mathrm{SL}(U_i)$; we think of G^r as $G_1 \times \cdots \times G_r$. We regard ϕ as a map

$$\phi : \mathrm{Sym}(K(\mathcal{S})) \to K(G_1 \times \cdots \times G_r).$$

It can be described explicitly on $K(\mathcal{S})$ as follows:

$$\phi([S]) = \sum_{i=1}^{r} [S(U_i)].$$

Now define Φ to be the functor

$$\Phi : \mathrm{Sym}(\mathcal{S}) \to \mathrm{Sym}(\mathrm{Vec}), \qquad \Phi(F)(L) = \bigoplus_{L = L_1 \amalg \cdots \amalg L_r} F((U_1)_{L_1} \amalg \cdots \amalg (U_r)_{L_r}).$$

Here the sum is over all partitions of L into r parts and $(U_i)_{L_i}$ denotes the family (V, L_i) where $V_x = U_i$ for all $x \in L_i$. (A more conceptual description of Φ is as follows. Let $i : (\mathrm{fs})^r \to \mathrm{Vec}^f$ take $(L_1, \ldots, L_r)$ to $(U_1)_{L_1} \amalg \cdots \amalg (U_r)_{L_r}$ and let $j : (\mathrm{fs})^r \to (\mathrm{fs})$ be the addition map. Then $\Phi(F) = j_* i^* F$.) One readily verifies that Φ is a tensor functor. We now claim that Φ lifts ϕ, that is, we have

$$\phi([N]) = [\Phi(N)]$$

for all N in $\mathrm{Sym}(\mathcal{S})$. Both sides above are additive in N so it suffices to treat the case where N is a simple object of $\mathrm{Sym}(\mathcal{S})$. As we have previously stated (§2.2), the simple objects are of the form $\bigotimes \mathbf{S}_{\lambda_i}(S_i)$ where the λ_i are partitions and S_i are distinct simple objects of $\mathcal{S}$. Since Φ is a tensor functor and ϕ is a ring homomorphism, we are reduced to the case of considering $N = \mathbf{S}_\lambda(S)$. Put $k = |\lambda|$. Then N is the equivariant functor $\mathrm{Vec}^k \to \mathrm{Vec}$ given by

$$(V_1, \ldots, V_k) \mapsto \mathbf{M}_\lambda \otimes S(V_1) \otimes \cdots \otimes S(V_k).$$

We thus find that $\Phi(N) \in \mathrm{Sym}(\mathrm{Vec})$ is supported in order k and assigns to the set $\{1, \ldots, k\}$ the space

$$\mathbf{M}_\lambda \otimes \bigoplus_{i_1, \ldots, i_k} S(U_{i_1}) \otimes \cdots \otimes S(U_{i_k})$$

where the sum is over all $(i_1, \ldots, i_k)$ in $\{1, \ldots, r\}^k$. We therefore have

$$[\Phi(N)] = \frac{\dim \mathbf{M}_\lambda}{k!} \left(\sum_{i=1}^{r} [S(U_i)] \right)^k.$$

On the other hand,

$$[N] = \frac{\dim \mathbf{M}_\lambda}{k!} [S]^k.$$

Applying ϕ to the above gives exactly the previous formula for $[\Phi(N)]$. This proves the claim.

The above discussion, and the fact that Φ commutes with the formation of Γ invariants, shows that

$$\phi(H^*_{M^\Gamma}) = H^*_{\Phi(M)^\Gamma, G_1 \times \cdots \times G_r}.$$

As Φ is a tensor functor, $\Phi(A)$ is a twisted commutative algebra finitely generated in order 1 and $\Phi(M)$ is a finitely generated module over it. Thus $H^*_{\Phi(M)^\Gamma, G_1 \times \cdots \times G_r}$ is a rational function in $K(G_1 \times \cdots \times G_r)[\![t]\!]$ by Theorem 3.7. We thus find that $\phi(H^*_{M^\Gamma})$ is rational, which completes the proof. ☐

We note the following corollary of the proposition.

Corollary 3.15. *Let A be an algebra in $\mathrm{Sym}(\mathcal{S})$ finitely generated in order 1 on which a finite group Γ acts and let M be a finitely generated A^Γ-module. Then H^*_M is a rational function of the s_λ.*

Proof. Let $N = A \otimes_{A^\Gamma} M$. Then N is a finitely generated A-module and $N^\Gamma = M$, where Γ acts on N by acting trivially on M. We thus have that $H^*_M = H^*_{N^\Gamma}$ is rational by the proposition. ☐

3.4. **Hilbert series of Δ-modules.** Let M be a Δ-module. We define its Hilbert series to be the Hilbert series of the underlying object of $\mathrm{Sym}(\mathcal{S})$. Our main result on such Hilbert series is the following, which follows immediately from Corollary 3.15 and Proposition 2.11.

Theorem 3.16. *Let M be a small Δ-module. Then H_M^* is a rational function of the s_λ.*

Remark 3.17. We expect this result to hold for all finitely generated Δ-modules. We can prove it for a much larger class than the class of small Δ-modules, but we have not been able to prove it for all finitely generated Δ-modules.

4. Applications to syzygies

We now apply the theory we have developed to the study of syzygies of Segre embeddings.

4.1. **Set-up.** Let (V, L) be an object of Vec^f. Define rings

$$R(V,L) = \bigoplus_{k=0}^{\infty} R^{(k)}(V,L), \qquad R^{(k)}(V,L) = \bigotimes_{x \in L} \mathrm{Sym}^k(V_x)$$

and

$$P(V,L) = \bigoplus_{k=0}^{\infty} P^{(k)}(V,L), \qquad P^{(k)}(V,L) = \mathrm{Sym}^k\left(\bigotimes_{x \in L} V_x\right).$$

As $R^{(1)}(V,L) = P^{(1)}(V,L)$, there is a natural map $P(V,L) \to R(V,L)$, which is easily verified to be surjective. This map on rings is dual to the Segre embedding.

The associations $(V,L) \mapsto R^{(k)}(V,L)$ and $(V,L) \mapsto P^{(k)}(V,L)$ naturally define nice functors from Vec^Δ to Vec, as follows. Let $(V,L) \to (V',L')$ be a map in Vec^Δ. The map $P^{(k)}(V,L) \to P^{(k)}(V',L')$ is obtained by applying Sym^k to the map $\bigotimes_{x\in L} V_x \to \bigotimes_{y \in L'} V_y$. The map $R^{(k)}(V,L) \to R^{(k)}(V',L')$ is the tensor product over $x \in L$ of the maps

$$\mathrm{Sym}^k(V_x) \to \mathrm{Sym}^k\left(\bigotimes_{y \mapsto x} V'_y\right) \to \bigotimes_{y \mapsto x} \mathrm{Sym}^k(V'_y).$$

The first map comes from the given map $V_x \to \bigotimes_{y \mapsto x} V'_y$ and the functoriality of Sym^k, while the second map is the natural one. Note that the maps $P(V,L) \to P(V',L')$ and $R(V,L) \to R(V',L')$ (obtained by summing the previous maps over k) are both algebra homomorphisms.

As discussed in the introduction, to understand the syzygies of $R(V,L)$ as a $P(V,L)$-module, it suffices to understand the graded vector space F_p defined as follows:

$$F_p(V,L) = \mathrm{Tor}_{p-1}^{P(V,L)}(R(V,L), \mathbf{C}).$$

By the functorial properties of Tor, the association $(V,L) \mapsto F_p(V,L)$ defines a functor from Vec^Δ to Vec. It is easy to see that this is a nice functor (see the following section), and so F_p can be regarded as a Δ-module. Theorem A is the statement that F_p is finitely generated as a Δ-module, while Theorem B is the statement that $[F_p]^*$ is a rational function of the s_λ. These are proved in the following section.

4.2. **Proof of Theorems A and B.** As usual, put $W(V,L) = \bigotimes_{x \in L} V_x$. We can resolve $\mathbf{C}$ as a $P(V,L)$-module using the Koszul complex. The terms of this complex are $P(V,L) \otimes \bigwedge^i W(V,L)$. Tensoring this complex with $R(V,L)$ gives a complex which computes $F_\bullet(V,L)$. To be more precise, fix an integer k. For an integer i, put

$$M_i(V,L) = R^{(k-i)}(V,L) \otimes {\textstyle\bigwedge}^i W(V,L) = \left(\bigotimes_{x\in L} \mathrm{Sym}^{k-i}(V_x)\right) \otimes \bigwedge^i \left(\bigotimes_{x \in L} V_x\right).$$

Then the $M_i(V,L)$ form a complex, the homology of which is canonically identified with $F_p^{(k)}(V,L)$. Precisely, $F_p^{(k)}(V,L)$ is a subquotient of $M_{p-1}(V,L)$.

The association $(V,L) \mapsto M_i(V,L)$ naturally defines a functor $\mathrm{Vec}^\Delta \to \mathrm{Vec}$, and is clearly nice. Thus M_i is a Δ-module. The differentials in the Koszul complex are easily verified to be maps of Δ-modules, and so we can regard $M_\bullet$ as a complex in the abelian category Mod_Δ. The identification of $F_\bullet^{(k)}(V,L)$ with the homology of $M_\bullet(V,L)$ is an identification of functors $\mathrm{Vec}^\Delta \to \mathrm{Vec}$. In fact, this identification is essentially how the functoriality of $F_p^{(k)}$ is defined. Thus $F_p^{(k)}$ is a nice functor, and we have an identification $F_p^{(k)} = H_{p-1}(M_\bullet)$ in Mod_Δ.

We now claim that M_i is a small Δ-module. To see this, simply observe that if (V, L) is an object of Vec^f and $\mathscr{U}$ is a partition of L then the map $R^{(k-i)}(V, \mathscr{U}) \to R^{(k-i)}(V, L)$ is surjective, while the map $W(V, \mathscr{U}) \to W(V, L)$ is an isomorphism; thus the map $M_i(V, \mathscr{U}) \to M_i(V, L)$ is surjective. It follows that $\Psi(M_i)$ is equal to the order 1 piece of M_i, namely $\mathrm{Sym}^{k-i} \boxtimes \bigwedge^i$. This is a quotient of T_k, and so M_i is a quotient of $\Phi(T_k)$. This shows that M_i is small. It follows that $F_p^{(k)}$ is a small Δ-module as well, since smallness passes to subquotients. Theorem 2.10 now implies that $F_p^{(k)}$ is finitely generated, while Theorem 3.16 implies that $[F_p^{(k)}]^*$ is rational.

We have thus shown that each graded piece of F_p is finitely generated and has rational Hilbert series. Thus Theorems A and B follow from the following result, which states that F_p has only finitely many non-zero graded pieces. This result is well-known to the experts, so we only give a brief proof. We thank Aldo Conca for showing it to us.

Proposition 4.1. *For $p \geq 2$, the space F_p is supported in degrees $p, \ldots, 2p-2$.*

Proof. This follows from the fact that the ideal of the Segre variety is generated by a Gröbner basis of degree 2 [ERT, Prop. 17], together with general facts about Gröbner bases (such as the Taylor resolution, see [Eis, Exercise 17.11]). □

Remark 4.2. The bound in the proposition is not optimal. Indeed, it is known [Ru] that F_3 and F_4 are supported in degrees 3 and 4, while the upper bounds provided by the proposition are 4 and 6. The optimal upper bound is not known. However, our proof of Theorem A provides an algorithm for finding it.

4.3. **Computational aspects.** We now elaborate on the remark we have made that our proofs give algorithms to calculate the relevant objects. Say we would like to compute generators for the Δ-module $F_p^{(k)}$ of p-syzygies of degree k. (The algorithm for computing f_p^* proceeds along similar lines.) We proceed as follows. First, $F_p^{(k)}$ is the homology of the sequence

$$M_p \to M_{p-1} \to M_{p-2},$$

where the M's are as in §4.2. As shown there, each M_i is a quotient of the Δ-module $\Phi(T_k)$. This shows that each is canonically a $W_k^{S_k}$-module and the differentials respect this structure. Now, W_k only has partitions with at most k rows; the same is true for the M_i. Thus it suffices to see what happens when we evaluate on $\mathbf{C}^k$. Precisely, let A be the twisted commutative algebra $L \mapsto W_k^{S_k}((\mathbf{C}^k)_L)$ and let N_i be the A-module given by $L \mapsto M_i((\mathbf{C}^k)_L)$, where $(\mathbf{C}^k)_L$ denotes the constant family on $\mathbf{C}^k$. Let E be the homology of the complex $N_\bullet$ at $p-1$; thus $E_L = F_p^{(d)}((\mathbf{C}^k)_L)$. The row bounds then imply that generators for E as an A-module are generators for $F_p^{(d)}$ as a $W_k^{S_k}$-module. Proposition 2.9 thus implies that generators for E as an A-module are generators for $F_p^{(d)}$ as a Δ-module.

Now, A is equal to $\mathrm{Sym}(U\langle 1\rangle)^{S_k}$ where $U = (\mathbf{C}^k)^{\otimes k}$. It follows that any partition in A has at most $\dim U = k^k$ rows. Since M_i is a subquotient of W_k, it follows that N_i is a subquotient of $\mathrm{Sym}(U\langle 1\rangle)$. Thus any partition appearing in N_i has at most k^k rows as well. Thus we do not loose information by evaluating on $\mathbf{C}^{k^k}$ (regarding everything in the Schur model). That is, generators for $E(\mathbf{C}^{k^k})$ as an $A(\mathbf{C}^{k^k})$-module give generators for E.

Finally, $A(\mathbf{C}^{k^k})$ is the subring of the polynomial ring in k^{2k} variables which are S_k-invariant. Each of the modules $N_i(\mathbf{C}^{k^k})$ is a finite module over this ring. And $E(\mathbf{C}^{k^k})$ is the homology of the complex $N_\bullet$ at $i = p-1$. We have thus reduced the problem to a computation involving explicitly described finitely generated rings and modules. These computations can be done algorithmically.

As a corollary of the above discussion, we find that there is an algorithm that determines for a given p and k if $F_p^{(k)}$ is non-zero. Since F_p is known to be supported in at most degrees $p, \ldots, 2p-2$, we can use this algorithm to determine the exact set of degrees in which F_p is non-zero.

The above algorithm for computing a generating set of F_p runs in worse than p^p time, since it involves non-trivial linear algebra computations in modules over polynomial rings in k^{2k} variables, for $p \leq k \leq 2p-2$. This algorithm is therefore completely unsuitable for actual computations. We do not know if there are practical algorithms.

4.4. **The work of Lascoux.** Lascoux determined the entire minimal resolution of certain determinantal varieties (see [La], and also [PW], where a gap in [La] is resolved). The rank 1 case of his result exactly gives the leading term of our series f_p. We recall his results in our language.

First, we discuss some terminology. Let $m = s_{\lambda_1} \cdots s_{\lambda_n}$ be a monomial in the variables s_λ. We say that m has *order* n. We say that m has *degree* d if each λ_i is a partition of d. Every term in the series $[F_p^{(d)}]$ has degree d, while the orders of the terms are unbounded. The degree d, order n terms in f_p give information about the p-syzygies of degree d for the Segre embedding of an n-fold product of projective spaces.

Let $f_{p,2}$ be the order two two term of f_p. This is the leading order term of f_p. We consider its degree d piece $f_{p,2}^{(d)}$. Write $d = p - 1 + h$. Of course, if $h \le 0$ then $f_{p,2}^{(d)} = 0$. Proposition 4.1 implies that $f_{p,2}^{(d)} = 0$ for $h > p - 1$. Lascoux gives a much better bound: $f_{p,2}^{(d)} = 0$ for $h > \sqrt{p-1}$. Assume now $1 \le h \le \sqrt{p-1}$. Let S be the set of pairs of partitions (α, β) such that α has at most h columns, β has at most h rows and $|\alpha| + |\beta| = p - 1 - h^2$. Associate to (α, β) a new pair of partitions (μ, ν) as follows. Start with a rectangle with h columns and $h + 1$ rows. To get μ, append α to the bottom and β to the right. To get ν, append the dual of β to the bottom and the dual of α to the right. Lascoux's result is then

$$f_{p,2}^{(d)} = \tfrac{1}{2} \sum_{(\alpha,\beta) \in S} s_\mu s_\nu.$$

For example, say $p = 2$ and $d = 2$. Then $h = 1$. Since $p - 1 - h^2 = 0$ the set S consists of the single pair (α, β) where $\alpha = \beta$ is the zero partition. The partitions μ and ν are both $(1, 1)$ and so we find $f_{p,2}^{(2)} = \frac{1}{2} s_{(1,1)}^2$.

4.5. **The polynomial g_p.** Let $G_p = \Psi(F_p)$ be the cokernel of the map $\Delta F_p \to F_p$. Since F_p is a finitely generated Δ-module, G_p is a finite length object of the category $\mathrm{Sym}(\mathcal{S})$. The object G_p is perhaps the most significant object in the theory of syzygies of Segre embeddings, at least from our point of view. It records precisely those syzygies that cannot be built out of syzygies on a product of fewer projective spaces. We let g_p be the *polynomial* $[G_p]$ (and $g_p^* = [G_p]^*$). The objects $L^i \Psi F_p$ for $i \ge 1$ are important as well — indeed, $[F_p]$ can be recovered from them — though they are bit less accessible.

We remark that our two main theorems can be rephrased using f_p and g_p so as to look more similar: Theorem A is exactly the statement that g_p^* is a polynomial, while Theorem B is exactly the statement that f_p^* is a rational function.

4.6. **Euler characteristics.** Define

$$\chi = \sum_{p \ge 1} (-1)^{p+1} f_p.$$

There are only finitely many terms of a given degree in the sum, and so it makes sense. We remark that $f_1 = 1$ — the first term in the resolution of R is always $P = P \otimes \mathbf{C}$ and so $F_1 = \mathbf{C}$ for any (V, L). The main result of this section is an explicit computation of χ. The notation used in the following proposition is defined below it.

Proposition 4.3. *We have*

$$\chi = \left[\sum_{k=0}^{\infty} \exp(s_{(k)}) \right] \boxtimes \left[\sum_{\lambda} \kappa_\lambda \exp(s'_\lambda) \right],$$

where κ_λ is the rational number $(-1)^{|\lambda|} \operatorname{sgn}(c_\lambda) z_\lambda^{-1}$. The second sum is taken over all partitions λ — including $\lambda = 0$, where the term is 1.

Extracting the degree k piece of the above formula yields:

Corollary 4.4. *For $k > 0$ we have*

$$\chi^{(k)} = \sum_{p=0}^{k} \left[\frac{(-1)^p}{p!} \sum_{\lambda \vdash p} (\# c_\lambda) \operatorname{sgn}(c_\lambda) \exp(s_{(k-p)} \boxtimes s'_\lambda) \right].$$

The $p = 0$ term of the above sum is $\exp(s_{(k)})$. We have $\chi^{(0)} = 1$.

We now define notation that will be in place for the rest of the section (and is used in the above proposition). Let λ be a partition of p. We let c_λ denote the conjugacy class of S_p corresponding to λ, normalized so that $\lambda=(1,\ldots,1)$ corresponds to the identity element, and we let $z_\lambda=p!/\#c_\lambda$ be the order of the centralizer of any element of c_λ. We let χ_λ denote the character of S_p corresponding to λ, normalized so that $\lambda=(1,\ldots,1)$ corresponds to the sign character sgn. The notation s_λ means what is has meant previously, namely the element $[\mathbf{S}_\lambda]$ of $K(\mathcal{S})$; in particular, $s_{(k)}=[\mathrm{Sym}^k]$. We define s'_λ to be the element of $K(\mathcal{S})$ of degree p which corresponds to the class function on S_p supported on c_λ and taking value z_λ there. Explicitly,

$$s'_\lambda=\sum_{\mu\vdash p}\chi_\mu(c_\lambda)s_\mu.$$

The symbol $\boxtimes$ is the point-wise tensor product: $s_\lambda\boxtimes s_\mu$ is computed using the Littlewood–Richardson rule. As usual, $W=W_1$ is the object of $\mathrm{Sym}(\mathcal{S})$ which assigns to (V,L) the tensor product of the V's. Throughout this section $\bigwedge^i W$ and $\mathbf{S}_\lambda(W)$ refer to point-wise operations in $\mathrm{Sym}(\mathcal{S})$. For instance, $\bigwedge^i W$ is the object of $\mathrm{Sym}(\mathcal{S})$ given by

$$(\textstyle\bigwedge^i W)(V,L)=\bigwedge^i(W(V,L))=\bigwedge^i\left(\bigotimes_{x\in L}V_x\right).$$

We now begin proving Proposition 4.3. We begin with the following.

Lemma 4.5. *We have*

$$\chi=[R]\boxtimes\left(\sum_{p=0}^{\infty}(-1)^p[\textstyle\bigwedge^p W]\right).$$

Proof. As discussed in §4.2, F_p is the homology of the complex $R\boxtimes\bigwedge^i W$ at $i=p-1$. The formula follows from standard facts about Euler characteristics. ☐

Lemma 4.6. *We have* $[R^{(d)}]=\exp(s_{(d)})$, *and so* $[R]=\sum_{d\ge 0}\exp(s_{(d)})$.

Proof. We have

$$R_n^{(d)}(V_1,\ldots,V_n)=\mathrm{Sym}^d(V_1)\otimes\cdots\otimes\mathrm{Sym}^d(V_n).$$

Thus $[R_n^{(d)}]$ is equal to $\frac{1}{n!}[\mathrm{Sym}^d]^n$ in $\mathrm{Sym}^n(K(\mathcal{S}))$. The result follows. ☐

Lemma 4.7. *Let λ be a partition of $p>0$. Let $s_{\lambda,n}$ be the class in $\mathrm{Sym}^n(K(\mathcal{S}))$ of the functor $(V_1,\ldots,V_n)\mapsto\mathbf{S}_\lambda(V_1\otimes\cdots\otimes V_n)$. Then*

$$\sum_{n=0}^{\infty}s_{\lambda,n}=\frac{1}{p!}\sum_{\mu\vdash p}(\#c_\mu)\chi_\lambda(c_\mu)\exp(s'_\mu).$$

Note that the left side above is nothing other than $[\mathbf{S}_\lambda(W)]$.

Proof. A simple manipulation shows that for any vector spaces U and V we have

$$\mathbf{S}_\lambda(U\otimes V)=\bigoplus C_{\lambda\mu\nu}\mathbf{S}_\mu(U)\otimes\mathbf{S}_\nu(V),$$

where the sum is over all partitions μ and ν of p, and

$$C_{\lambda\mu\nu}=\dim(\mathbf{M}_\lambda\otimes\mathbf{M}_\mu\otimes\mathbf{M}_\nu)^{S_p}=\dim\mathrm{Hom}_{S_p}(\mathbf{M}_\lambda,\mathbf{M}_\mu\otimes\mathbf{M}_\nu).$$

(This appears as Exercise 6.11(b) in [FH].) We therefore have

$$\mathbf{S}_\lambda(V_1\otimes\cdots\otimes V_n)=\bigoplus_{\mu,\nu}C_{\lambda\mu\nu}\mathbf{S}_\mu(V_1)\otimes\mathbf{S}_\nu(V_2\otimes\cdots\otimes V_n).$$

We thus have a recurrence

$$s_{\lambda,n}=\frac{1}{n}\sum_{\mu,\nu}C_{\lambda\mu\nu}s_\mu s_{\nu,n-1}.$$

It will now be convenient to switch from working in the degree p piece of $K(\mathcal{S})$ to working in $K(S_p)$. The two are in isomorphism via $s_\lambda=[\mathbf{S}_\lambda]\leftrightarrow[\mathbf{M}_\lambda]$. Thus $s_{\lambda,n}$ can be regarded as an element of $\mathrm{Sym}^n(K(S_p))$. Note that the sum $\sum_\mu C_{\lambda\mu\nu}s_\mu$ is equal to $[\mathbf{M}_\lambda\otimes\mathbf{M}_\nu]$. We can thus rephrase our last expression as follows. Let v_n be the column vector $(s_{\lambda,n})_\lambda$ and let A be the matrix $([\mathbf{M}_\lambda\otimes\mathbf{M}_\mu])_{\lambda,\mu}$. Then

$$v_n=\tfrac{1}{n}Av_{n-1}.$$

We thus have

$$\sum v_n=\exp(A)v_0.$$

Note that v_0 has a 1 in the entry $\lambda = (p)$ and a 0 in all other entries. Indeed, an empty tensor product is equal to $\mathbf{C}$, so $s_{\lambda,0}$ is the class of $S_\lambda(\mathbf{C})$ in $\mathrm{Sym}^0(K(S_p)) = \mathbf{Q}$; in other words, $a_{\lambda,0} = \dim \mathbf{S}_\lambda(\mathbf{C})$. This is 1 if $\lambda = (p)$ and 0 otherwise. We therefore find that the initial vector v_0 in the above recurrence is quite simple. The problem is to determine the exponential of the matrix A. We will achieve this by diagonalizing A.

Let B be the matrix $(\chi_\lambda(c_\mu))_{\lambda,\mu}$. We index by rows first, then columns. Thus the rows of B are indexed by irreducible characters and the columns by conjugacy classes; B is the character table of S_p. Let D be the diagonal matrix given by $D_{\lambda\lambda} = z_\lambda \delta_\lambda$ where $z_\lambda = p!/\#c_\lambda$ is the cardinality of the centralizer of c_λ and δ_λ is the class function on S_p which assigns c_λ the value 1 and all other conjugacy classes 0. We then have the following fundamental identity

$$AB = BD. \tag{2}$$

We now explain this identity. First, we regard the entries of A as class functions, so $A_{\lambda\mu}$ is the character of $\mathbf{M}_\lambda \otimes \mathbf{M}_\mu$. The entry of the product AB at (λ, μ) evaluated at c_ζ is thus given by

$$\left(\sum_\nu A_{\lambda\nu} B_{\nu\mu} \right)(c_\zeta) = \sum_\nu \chi_\lambda(c_\zeta)\chi_\nu(c_\zeta)\chi_\nu(c_\mu) = \chi_\lambda(c_\zeta) \sum_\nu \chi_\nu(c_\zeta)\chi_\nu(c_\mu).$$

Now, for g and h in S_p the sum $\sum \chi_\nu(g)\chi_\nu(h)$ is the trace of (g, h) acting on the representation $\mathbf{C}[S_p]$ of $S_p \times S_p$. Since this is a permutation representation, the trace is given by the number of fixed points. An element $x \in S_p$ is a fixed point if $gxh^{-1} = x$, or equivalently, if $g = xhx^{-1}$. Thus the number of fixed points is 0 if g and h are not conjugate, and is otherwise the size of the centralizer of g. We therefore find

$$\chi_\lambda(c_\zeta) \sum_\nu \chi_\nu(c_\zeta)\chi_\nu(c_\mu) = \chi_\lambda(c_\mu) z_\mu \delta_\mu(c_\zeta) = B_{\lambda\mu} D_{\mu\mu}(c_\zeta).$$

This proves (2).

The equation (2) diagonalizes A. However, for it to be useful we need to compute B^{-1}. This is straightforward. Let C be the diagonal matrix given by $C_{\lambda\lambda} = z_\lambda^{-1}$. Then the orthonormality of characters is precisely the identity

$$BCB^t = 1$$

and so $B^{-1} = CB^t$. (This again uses the fact that all representations of symmetric groups are self-dual, which is equivalent to their characters being real valued.)

We now find

$$\exp(A)v_0 = B \exp(B^{-1}AB)B^{-1}v_0 = B \exp(D)CB^t v_0.$$

Simple matrix multiplication now gives the stated formula. □

Lemma 4.8. *Let x and y belong to $K(\mathcal{S})$. Then* $\exp(x) \boxtimes \exp(y) = \exp(x \boxtimes y)$.

Proof. For an object F of $\mathcal{S}$ let F' be the object of $\mathrm{Sym}(\mathcal{S})$ given by $(V, L) \mapsto \bigotimes_{x \in L} F(V_x)$. Then $\exp([F]) = [F']$. Now, let x and y in $K(\mathcal{S})$ be given. Since $\boxtimes$ is additive and exp is multiplicative, it suffices to treat the case where $x = [F]$ and $y = [G]$, with F and G in $\mathcal{S}$. We then have

$$\exp(x) \boxtimes \exp(y) = [F'] \boxtimes [G'] = [F' \boxtimes G'] = [(F \boxtimes G)'] = \exp([F \boxtimes G]) = \exp(x \boxtimes y).$$

The key fact is the obvious formula $F' \boxtimes G' = (F \boxtimes G)'$. □

The proposition and corollary follow easily from the above lemmas.

4.7. **Examples for small** p**.** The main result of [Ru] states that Segre embeddings satisfy the Green-Lazarsfeld property N_3 but not N_4. This implies that F_2, F_3 and F_4 are only supported in degrees 2, 3 and 4 respectively. From this, we deduce the following equalities:

$$f_2 = -\chi^{(2)}, \qquad f_3 = \chi^{(3)}, \qquad f_4 = -\chi^{(4)}, \qquad f_5^{(5)} = \chi^{(5)}.$$

These values have been computed in Proposition 4.3. They are listed explicitly, and in simplified form, in Figure 1 (other than $f_5^{(5)}$). We explain how the value for f_2 given in the figure was derived, the values of f_3 and f_4 being gotten in a similar fashion. Proposition 4.3 gives

$$f_2 = -\exp(s_{(2)}) + \exp(s_{(1)} \boxtimes s'_{(1)}) - \tfrac{1}{2} \left(\exp(s'_{(1,1)}) - \exp(s'_{(2)}) \right).$$

$s = [\mathrm{Sym}^2], \quad w = [\bigwedge^2]$
$f_2 = \frac{1}{2}e^{s+w} + \frac{1}{2}e^{s-w} - e^s$
$g_2 = \frac{1}{2}w^2$
$s = [\mathrm{Sym}^3], \quad w = [\bigwedge^3], \quad t = [\mathbf{S}_{(2,1)}]$
$f_3 = \frac{1}{3}e^{s+w+2t} - \frac{1}{3}e^{s+w-t} - e^{s+t} + e^s$
$g_3 = wt$
$s = [\mathrm{Sym}^4], \quad w = [\bigwedge^4], \quad a = [\mathbf{S}_{(3,1)}], \quad b = [\mathbf{S}_{(2,2)}], \quad c = [\mathbf{S}_{(2,1,1)}]$
$f_4 = \frac{1}{8}e^{s+w+3a+2b+3c} - \frac{1}{8}e^{s+w-a+2b-c} + \frac{1}{4}e^{s-w-a+c} - \frac{1}{4}e^{s-w+a-c} + \frac{1}{2}e^{s+b-c} - \frac{1}{2}e^{s+2a+b+c} + e^{s+a} - e^s.$
$g_4 = aw + \frac{1}{2}c^2 \quad (+?)$

FIGURE 1. Values of f_p and g_p for p small.

We have $s'_{(1)} = s_{(1)}$, while $s'_{(1,1)} = s_{(2)} + s_{(1,1)}$ and $s'_{(2)} = s_{(2)} - s_{(1,1)}$. Now, the product $s_{(1)} \boxtimes s_{(1)}$ is just the usual product in $K(\mathcal{S})$, i.e., it is the class of the functor $V \mapsto \mathrm{Sym}^1(V) \otimes \mathrm{Sym}^1(V)$. This, of course, is equal to $s_{(2)} + s_{(1,1)}$. We thus find

$$f_2 = \tfrac{1}{2}\exp(s_{(2)} + s_{(1,1)}) + \tfrac{1}{2}\exp(s_{(2)} - s_{(1,1)}) - \exp(s_{(2)}).$$

This is the value given in the figure.

We have previously stated (without proof) that the equation for $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$ generates F_2 as a Δ-functor. This implies that g_2 is the order two piece of f_2. Similarly, we have stated (without proof) that any non-zero syzygy for $\mathbf{P}^1 \times \mathbf{P}^2$ in $\mathbf{P}^5$ generates F_3. This implies that g_3 is the order two piece of f_3. Finally, the order two piece of g_4 is the same as that of f_4; we are unaware if g_4 has any terms of higher order. These remarks explain the values of g_2, g_3 and g_4 given in the figure.

4.8. **Syzygies of Δ-schemes.** We now give a few more examples to which our theory applies. A Δ-*scheme* is a functor X assigning to each object (V, L) of Vec^{Δ} a $\mathbf{G}_m$-stable closed subscheme $X(V, L)$ of $\bigotimes_{x \in L} V_x^*$, and to each morphism $(V, L) \to (V', L')$ a map $X(V', L') \to X(V, L)$, making the obvious diagrams commute. A Δ-scheme can be viewed as a functor from Vec^{Δ} to the category of morphisms of schemes. More concretely, a Δ-scheme can be thought of as a rule assigning to each tuple of vector spaces $(V_1, \ldots, V_n)$ a $\mathbf{G}_m$-stable closed subscheme $X_n(V_1, \ldots, V_n)$ of $\bigotimes V_i^*$ such that: X_n is functorial in the V_i; X_n is S_n-equivariant; and $X_{n+1}(V_1, \ldots, V_{n+1})$ is contained in $X_n(V_1, \ldots, V_{n-1}, V_n \otimes V_{n+1})$.

Before proceeding, we mention some examples of Δ-schemes. The functor $\mathbf{V}$ defined by $\mathbf{V}(V, L) = \bigotimes_{x \in L} V_x^*$ is a trivial example; it is the final object in the category of Δ-schemes. A more interesting example is the family of Segre varieties: take $X(V, L)$ to be the set of pure tensors in $\bigotimes_{x \in L} V_x^*$. If X and Y are Δ-schemes, then so too are $X \cap Y$, $X \cup Y$ and $X + Y$ (all computed inside of $\mathbf{V}$). If X is a Δ-schemes then so too are its tangent variety and secant varieties. Applying these constructions to the Segre embeddings give important examples.

Let X be a Δ-scheme. Let $R(V, L)$ be the coordinate ring of $X(V, L)$ and let $P(V, L)$ be the coordinate ring of $\bigotimes_{x \in L} V_x^*$, i.e., $\mathrm{Sym}(\bigotimes_{x \in L} V_x)$. The rings $R(V, L)$ and $P(V, L)$ are graded (from the $\mathbf{G}_m$-action), and we have a surjection $P(V, L) \to R(V, L)$. As before, put $F_p(V, L) = \mathrm{Tor}_{p-1}^{P(V,L)}(R(V, L), \mathbf{C})$. Then F_p forms a Δ-module; in fact, each graded piece $F_p^{(d)}$ of F_p is a Δ-module. The following result is proved like Theorems A and B.

Proposition 4.9. *For any Δ-scheme X, the Δ-module $F_p^{(d)}$ is finitely generated and has rational Hilbert series.*

In particular, we see that F_p itself is finitely generated and has rational Hilbert series if (and only if) F_p is supported in finitely many degrees. We know of only a small number of examples where these degree bounds have been established. The GSS conjecture, recently established by Raicu [Ra], implies that the ideal for the secant variety to the Segres are generated by cubics, and so F_2 is finitely generated. In fact, the GSS conjecture is exactly the statement that F_2 is generated in order 2 and degree 3. The ideal for the tangent variety to the Segre is generated in degrees ≤ 6 [LW], and so F_2 is finitely generated in this case. In [LW], certain generators for the ideal are conjectured. If one could determine an explicit finite generating set for F_2 then this conjecture would be reduced to a finite check.

One might hope that there are general results about Δ-schemes which guarantee (under reasonable hypotheses) uniform degree bounds on syzygies. We hope to return to this question in the future.

5. Questions and problems

(**1**) Are finitely generated twisted commutative algebras noetherian? We proved this for algebras generated in order 1, and can also prove it for certain algebras in order 2. However, we do not know what to expect in general. We note that if all finitely generated twisted commutative algebras were noetherian, then our proof of Theorem 2.6 would show that all finitely generated algebras in $\mathrm{Sym}(\mathcal{S})$ are noetherian as well.

(**2**) Let M be a finitely generated module over a twisted commutative algebra finitely generated in order 1. We have shown that $H_M(t)$ is a polynomial of t and e^t. How does the form of this polynomial relate to the structure of M? For example, is the maximal power of e^t related to the number of rows in M?

(**3**) Our series f_p forgets the S_n-equivariance of $F_{p,n}$. Is there an object (such as a rational function) that can be described with a finite amount of data and determines the $F_{p,n}$ with their equivariant structure?

(**4**) Is the series f_p a polynomial in the s_λ and the $e^{\pm s_\lambda}$? This does not follow from what we have proved, but one might hope that it is true based on some of our results. In fact, based on the computations of f_2, f_3 and f_4, one might hope for a stronger statement: f_p is a polynomial in only the $e^{\pm s_\lambda}$. We suspect this is false, but do not know.

(**5**) Compute f_p and g_p for more values of p. We have given an algorithm to do this, but it is too inefficient to use. It would be particularly interesting to compute f_5 since this is the first place where the Green-Lazarsfeld property fails and the value is not given by the Euler characteristic formula. This computation may even resolve the previous question.

(**6**) Is the series $\sum_{i\geq 0}(-1)^i[L^i\Psi F_p]^* q^i$ a rational function of the s_λ? This series contains more information than f_p and g_p, since one can recover f_p^* by applying Φ and letting q go to 1, and one can recover g_p^* by setting q to 0.

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