PERIODICITY IN THE COHOMOLOGY OF SYMMETRIC GROUPS VIA DIVIDED POWERS

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ABSTRACT. A famous theorem of Nakaoka asserts that the cohomology of the symmetric group stabilizes. The first author generalized this theorem to non-trivial coefficient systems, in the form of **FI**-modules over a field, though one now obtains periodicity of the cohomology instead of stability. In this paper, we further refine these results. Our main theorem states that if M is a finitely generated **FI**-module over a noetherian ring $\mathbf{k}$ then $\bigoplus_{n \geq 0} \mathrm{H}^t(S_n, M_n)$ admits the structure of a **D**-module, where **D** is the divided power algebra over $\mathbf{k}$ in a single variable, and moreover, this **D**-module is "nearly" finitely presented. This immediately recovers the periodicity result when $\mathbf{k}$ is a field, but also shows, for example, how the torsion varies with n when $\mathbf{k} = \mathbf{Z}$. Using the theory of connections on **D**-modules, we establish sharp bounds on the period in the case where $\mathbf{k}$ is a field. We apply our theory to obtain results on the modular cohomology of Specht modules and the integral cohomology of unordered configuration spaces of manifolds.

Contents

1. Introduction

1.1. The main theorem. A famous theorem of Nakaoka asserts that the cohomology groups of the symmetric groups stabilize: for $n > 2t$, the restriction map

$$\mathrm{H}^t(S_n, \mathbf{k}) \to \mathrm{H}^t(S_{n-1}, \mathbf{k})$$

is an isomorphism, for any coefficient ring (or even abelian group) $\mathbf{k}$, with the symmetric groups acting trivially. It is natural to ask if there is some way to generalize this theorem to allow non-trivial coefficients. Thus suppose that for each $n \geq 0$ we have a representation M_n of S_n over a ring $\mathbf{k}$. We can then consider the groups $\mathrm{H}^t(S_n, M_n)$, for t fixed and n varying. Obviously, to hope for any sort of relationship between these groups we must assume that the M_n form a "coherent system" of representations, in some sense.

In recent years, several kinds of algebraic structures have been studied that can rightfully be called "coherent systems" of representations. In this paper, we focus on **FI**-modules,

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popularized by Church, Ellenberg, and Farb [CEF], though see §1.6 for additional discussion. An **FI**-module over $\mathbf{k}$ can be defined as a system of representations $\{M_n\}_{n\geq 0}$, as above, with transition maps $M_n \to M_{n+1}$ satisfying certain conditions; see §2.1 below. In a general **FI**-module, the transition maps could all be 0, and so the M_n's need not be related. However, in a *finitely generated* **FI**-module, the M_n's are very closely related. In this setting, the first author has found a generalization of Nakaoka's theorem:

Theorem 1.1 ([Nag]). *Suppose* $\mathbf{k}$ *is a field of characteristic* p. *Let* M *be a finitely generated* **FI***-module over* $\mathbf{k}$, *and fix* $t \geq 0$. *Then* $\dim_{\mathbf{k}} \mathrm{H}^t(S_n, M_n)$ *is periodic for* n *sufficiently large, with period a power of* p.

This is a fine theorem, but is somewhat deficient in that it is numerical rather than structural; that is, instead of simply having an equality of dimensions, one would like to have some sort of additional algebraic strucutre inducing isomorphisms between appropriate cohomology groups. The main purpose of this paper is to establish this structure.

Let M be a finitely generated **FI**-module. Define

$$\Gamma^t(M) = \bigoplus_{n\geq 0} \mathrm{H}^t(S_n, M_n),$$

regarded as a graded $\mathbf{k}$-module in the evident manner. We show that $\Gamma^t(M)$ canonically has the structure of a module over the divided power algebra $\mathbf{D}$ over $\mathbf{k}$ in a single variable (see §3 for the definition). The ring $\mathbf{D}$ is not noetherian in general, but it is coherent whenever $\mathbf{k}$ is noetherian [NS, Theorem 4.1], and so finitely presented $\mathbf{D}$-modules are reasonably well-behaved. Our main result is the following theorem.

Theorem 1.2 (Main theorem). *Assume that* $\mathbf{k}$ *is a commutative noetherian ring. Let* M *be a finitely generated* **FI***-module over* $\mathbf{k}$ *and let* $t \geq 0$. *Then there exists a finitely presented graded* $\mathbf{D}$*-module* K, *depending functorially on* M, *and a map of* $\mathbf{D}$*-modules* $\Gamma^t(M) \to K$ *that is an isomorphism in all sufficiently large degrees.*

It is easy to see that a finitely presented $\mathbf{D}$-module over a field of positive characteristic has eventually periodic dimensions (see §3.3), and so the above theorem recovers Theorem 1.1. However, it is stronger than Theorem 1.1 in two ways. For one, the $\mathbf{D}$-module structure provides more flexibility. For example, suppose that $M \to N$ is a map of finitely generated **FI**-modules over a field of positive characteristic. It follows from Theorem 1.2 that the image of the map $\Gamma^t(M) \to \Gamma^t(N)$ has eventually periodic dimension; Theorem 1.1 does not give this information. Theorem 1.2 also improves on Theorem 1.1 in that it does not require $\mathbf{k}$ to be a field. Thus, for example, Theorem 1.2 yields non-trivial information about how the torsion in $\mathrm{H}^t(S_n, M_n)$ varies with n in the case $\mathbf{k} = \mathbf{Z}$.

Remark 1.3. If M is a finitely generated **FI**-module then $\Gamma^t(M)$ need not be a finitely presented (or even finitely generated) $\mathbf{D}$-module, even for $t = 0$; see Example 4.14. Thus Theorem 1.2 is in some sense optimal. □

Remark 1.4. Theorem 1.1 also follows from a recent result of Harman ([Har, Proposition 3.3]). However, the arguments there are specific to fields. □

Remark 1.5. There are similar, but easier, results for the homology of **FI**-modules: for a finitely generated **FI**-module M, one can give $\bigoplus_n \mathrm{H}_t(S_n, M_n)$ the structure of a finitely generated $\mathbf{k}[t]$-module. It follows that $\dim_{\mathbf{k}} \mathrm{H}_t(S_n, M_n)$ stabilizes as $n \to \infty$. □

1.2. **Quantitative results.** Let M be a finitely generated **FI**-module over a field $\mathbf{k}$ of characteristic p. According to Theorem 1.1, $\Gamma^t(M)$ has eventually periodic dimensions. This raises two questions: what is the eventual period, and when does periodicity set in? We prove the following theorem that addresses these questions:

Theorem 1.6. *Suppose that M is generated in degrees $\leq g$ with relations in degrees $\leq r$ and has degree δ. Let q be the smallest power of p such that $\delta < q$. Then*

$$\dim_{\mathbf{k}} \mathrm{H}^t(S_n, M_n) = \dim_{\mathbf{k}} \mathrm{H}^t(S_{n+q}, M_{n+q})$$

holds for all $n \geq \max(g + r, 2t + \delta)$.

We recall that the degree δ of M is the degree of the polynomial $n \mapsto \dim_{\mathbf{k}}(M_n)$, and is bounded above by g. The above theorem greatly improves the bounds from [Nag], and we believe it is close to optimal.

1.3. **Application: configuration spaces.** Let $\mathrm{uConf}_n(\mathcal{M})$ be the unordered configuration space of n points on a manifold $\mathcal{M}$. In §5.2, we prove, under some mild assumptions on $\mathcal{M}$, that there is a finitely presented **D**-module K and **k**-module isomorphisms

$$\mathrm{H}^t(\mathrm{uConf}_n(\mathcal{M}), \mathbf{k})_n \cong K_n$$

for $n \gg 0$. When $\mathbf{k}$ is a field of positive characteristic, this recovers (without bounds) recent periodicity results on these cohomology groups (see [KM] for the sharpest result and summary). When $\mathbf{k} = \mathbf{Z}$, this provides new information about how the torsion in the cohomology of $\mathrm{uConf}_n(\mathcal{M})$ varies with n (see Example 5.4).

1.4. **Application: cohomology of Specht modules.** Let $\mathbf{k}$ be a field of characteristic p. For a partition λ, let $\mathbf{M}_\lambda$ be the associated Specht module over $\mathbf{k}$. Write $\lambda[n]$ for the partition $(n - |\lambda|, \lambda)$. In §5.1, we prove the following:

Theorem 1.7. *Let λ be a partition and let q be the smallest power of p greater than $|\lambda|$. Then*

$$\dim_{\mathbf{k}} \mathrm{H}^t(S_n, \mathbf{M}_{\lambda[n]}) = \dim_{\mathbf{k}} \mathrm{H}^t(S_n, \mathbf{M}_{\lambda[n+q]})$$

holds for $n \geq \max(2t + d, 2d + \lambda_1)$.

Note that a similar result can be obtained from [Har, Corollary 2.6] (but without bounds on the onset of periodicity) and that this is a specific case of a more general open problem [Hem, Problem 8.3.1].

1.5. **Overview of the proofs.** The main innovation of this paper is the systematic use of the **D**-module structure on $\Gamma^t(M)$. This not only leads to stronger results, but greatly leads to much proofs that are much cleaner than those in [Nag].

We deduce Theorem 1.2 from a generalization of a theorem of Dold and a structurual result on **FI**-modules due to the first author. The argument is roughly as follows. Suppose that V is a $\mathbf{k}[S_d]$-module, and let $\mathcal{I}(V)$ be the associated induced **FI**-module (see §2.4). We prove that $\Gamma^t(\mathcal{I}(V))$ is a finitely generated and relatively free **D**-module (that is, it has the form $M_0 \otimes_{\mathbf{k}} \mathbf{D}$ for some finitely generated **k**-module M_0.); this is our generalization of Dold's theorem. Now suppose that M is an arbitary finitely generated **FI**-module. The first author has shown that there is a sequence of **FI**-modules

$$0 \to M \to I^0 \to \cdots \to I^n \to 0$$

where each I^k is semi-induced (i.e., has a filtration with induced quotients) and whose homology is torsion. This gives a spectral sequence that computes $\Gamma^t(M)$ from the $\Gamma^i(I^j)$'s, ignoring finitely many degrees (due to the fact that the homology is possibly non-zero). Our generalization of Dold's theorem shows that each $\Gamma^i(I^j)$ is finitely generated and free. Thus, using the fact that $\mathbf{D}$ is coherent, it follows that $\Gamma^t(M)$ is isomorphic to a finitely presented $\mathbf{D}$-module outside of finitely many degrees.

To prove our generalization of Dold's theorem, and also to prove the quantitative bounds in Theorem 1.6, we appeal to the theory of connections. The ring $\mathbf{D}$ admits a derivation d defined by $d(x^{[n]}) = x^{[n-1]}$. A connection on a $\mathbf{D}$-module M is a $\mathbf{k}$-linear map $\nabla \colon M \to M$ satisfying a version of the Leibniz rule with respect to d. A fundamental result (Proposition 3.1) asserts that if M admits a connection then M is relatively free. Suppose that V is the trivial representation of S_0, so that $\mathcal{I}(V)_n = \mathbf{k}$ for all n. We show that the restriction map

$$\mathrm{H}^t(S_n, \mathbf{k}) \to \mathrm{H}^t(S_{n-1}, \mathbf{k})$$

induces a connection on the $\mathbf{D}$-module $\Gamma^t(\mathcal{I}(V))$. A similar construction applies to arbitrary representations V of S_d. This is how we prove our generalization of Dold's theorem. Similar ideas appear in Dold's original paper, but the use of connections greatly clarifies the arguments.

Suppose now that $I \to J$ is a map of induced $\mathbf{FI}$-modules over a field $\mathbf{k}$ of characteristic p. The induced map $\Gamma^t(I) \to \Gamma^t(J)$ need not respect the connections on these $\mathbf{D}$-modules. However, we show that if I and J are generated in degrees $< q$, where q is a power of p, then this map does respect the q-fold iterate of the connections. From this, we deduce that if M is a finitely generated $\mathbf{FI}$-module of degree $< q$ then $\Gamma^t(M)$ admits a connection with respect to the q-fold iterate of d. This is the key step in the proof of Theorem 1.6.

1.6. **Open problems.** As mentioned, $\mathbf{FI}$-modules provide one way to produce "coherent systems" of S_n-representations, but there are others. We mention two specific ones.

For a positive integer d, let $\mathbf{FI}_d$ be the category whose objects are finite sets, and where morphisms are injections together with a d-coloring of the complement of the image. Let M be a finitely generated $\mathbf{FI}_d$-module over a field $\mathbf{k}$. Then M_n is a representation of S_n. The first author has conjectured that $\dim_{\mathbf{k}} \mathrm{H}^t(S_n, M_n)$ is eventually a quasi-polynomial in n of degree at most $d-1$; note that this exactly reduces to Theorem 1.1 when $d = 1$. One can give these groups the structure of a module over the divided power algebra in d variables; ideally, this module structure would somehow be responsible for the quasi-polynomiality.

Let $\mathbf{FS}$ be the category whose objects are finite sets, and where morphisms are surjections. Let $\mathbf{FS}^{\mathrm{op}}$ be the opposite category, and let M be a finitely generated $\mathbf{FS}^{\mathrm{op}}$-module over $\mathbf{k}$. Once again, M_n is a representation of S_n. Given $n \le m$ there is a natural map $M_m \to M_n$ of S_n representations, and thus a restriction map

$$\mathrm{H}^t(S_m, M_m) \to \mathrm{H}^t(S_n, M_n).$$

When M is the trivial $\mathbf{FS}^{\mathrm{op}}$-module, these maps are isomorphisms for $n > 2t$: this is exactly Nakaoka's theorem. Are these maps always isomorphisms for $n \gg 0$? What can one say about $\mathrm{H}_t(S_n, M_n)$ in this setting?

1.7. **Outline of paper.** In §2 we review the theory of $\mathbf{FI}$-modules. The material here is mostly not new. In §3 we introduce the ring $\mathbf{D}$, and prove a number of basic facts about $\mathbf{D}$-modules. The main results of this paper are proved in §4.3. Finally, §5.1 and §5.2 contain applications of our results to the cohomology of Specht modules and configuration spaces respectively.

2. Background on FI-modules

2.1. **FI-modules.** Let **FI** be the category whose objects are finite sets and whose morphisms are injections. An **FI-module** with coefficients in $\mathbf{k}$, a commutative noetherian ring, is a functor from **FI** to the category $\mathrm{Mod}_{\mathbf{k}}$ of $\mathbf{k}$-modules. We write $\mathrm{Mod}_{\mathbf{FI}}$ for the category of **FI**-modules. Suppose that M is an **FI**-module. We write M_n for the value of M on the finite set $[n] = \{1, \ldots, n\}$. This is a representation of S_n over the ring $\mathbf{k}$. The inclusion $[n] \to [n+1]$ induces an S_n-equivariant map $M_n \to M_{n+1}$. In fact, one can think of an **FI**-module as a sequence $\{M_n\}_{n \geq 0}$ of representations equipped with transition maps $M_n \to M_{n+1}$ satisfying the following condition: the image of M_n in M_{n+r} lands in $M_{n+r}^{S_r}$ for all n and r, see [CEF, Remark 3.3.1].

2.2. **The twisted commutative algebra A.** A **representation of** S_* over $\mathbf{k}$ is a sequence $(V_n)_{n \geq 0}$ where V_n is a $\mathbf{k}[S_n]$-module. We write $\mathrm{Rep}_{\mathbf{k}}(S_*)$ for the category of such sequences. If V and W are two representations of S_*, we define their tensor product $V \otimes W$ to be the representation of S_* given by

$$(V \otimes W)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n}(V_i \otimes_{\mathbf{k}} W_j).$$

One can show that this tensor product is naturally symmetric, that is, there is a canonical isomorphism $V \otimes W \to W \otimes V$ that squares to the identity.

A **twisted commutative algebra** is a commutative algebra object in the tensor category $\mathrm{Rep}(S_*)$, that is, an S_*-representation A equipped with a multiplication map $A \otimes A \to A$ satisfying the appropriate axioms. (We note that, by Frobenius reciprocity, specifying such a multiplication map is the same as specifying $S_i \times S_j$ equivariant maps $A_i \otimes_{\mathbf{k}} A_j \to A_{i+j}$ for all i and j.) Given such an algebra A, an A-module is an S_*-representation M equipped with a multiplication map $A \otimes M \to M$ satisfying the appropriate axioms.

Let $\mathbf{A}$ be the S_*-representation given by $\mathbf{A}_n = \mathbf{k}$ with the trivial S_n action for all n. There is a canonical map $\mathbf{A}_i \otimes_{\mathbf{k}} \mathbf{A}_j \to \mathbf{A}_{i+j}$ for every i and j, and this defines a multiplication $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ that gives $\mathbf{A}$ the structure of a twisted commutative algebra. Suppose M is an $\mathbf{A}$-module. Then M is an S_*-representation, and the multiplication map $\mathbf{A}_1 \otimes_{\mathbf{k}} M_n \to M_{n+1}$ gives an S_n-equivariant map $M_n \to M_{n+1}$. These maps give $\{M_n\}_{n \geq 0}$ the structure of an **FI**-module. In this way, the categories $\mathrm{Mod}_{\mathbf{A}}$ and $\mathrm{Mod}_{\mathbf{FI}}$ are equivalent, see [SS2, Proposition 7.2.5]. We freely pass between the two perspectives.

2.3. **Generators and relations.** Since $\mathbf{A}$ is a commutative algebra object in a symmetric tensor category, there is a notion of tensor product of $\mathbf{A}$-modules and (since there are enough projectives) one can derive this. Regard $\mathbf{k}$ as an $\mathbf{A}$-module by letting the elements of positive degree act by 0. The $\mathbf{A}$-modules $\mathrm{Tor}_i^{\mathbf{A}}(M, \mathbf{k})$ are important quantities associated to the **FI**-module M. We let $g(M)$ be the maximal degree occurring in this module for $i = 0$, and $r(M)$ the maximal degree for $i = 1$. One can interpret $g(M)$ and $r(M)$ as the degrees of generators and relations for M.

2.4. **Induced and semi-induced FI-modules.** Let V be a representation of S_d. Define $\mathcal{I}(V)$ to be the S_*-representation given by

$$\mathcal{I}(V)_n = \mathrm{Ind}_{S_d \times S_{n-d}}^{S_n}(V \boxtimes \mathrm{triv}),$$

where here triv denotes the trivial representation of S_{n-d}. One easily verifies that $\mathcal{I}(V)$ has the structure of an **FI**-module; see [CEF, Definition 2.2.2] for details (note that there the notation $M(V)$ is used in place of $\mathcal{I}(V)$). We extend the construction of $\mathcal{I}(V)$ additively to objects V of $\mathrm{Rep}_{\mathbf{k}}(S_*)$. We call **FI**-modules of the form $\mathcal{I}(V)$ **induced FI-modules**. From the tca perspective, $\mathcal{I}(V)$ is the $\mathbf{A}$-module $\mathbf{A} \otimes V$. From this point of view, the following proposition is immediate:

Proposition 2.1. *Let $V \in \mathrm{Rep}_{\mathbf{k}}(S_*)$. Then for any* **FI***-module M we have*

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathbf{FI}}}(\mathcal{I}(V), M) = \mathrm{Hom}_{S_*}(V, M).$$

We say that an **FI**-module M is **semi-induced** if it has a finite length filtration where the graded pieces are induced. We have the following useful result:

Proposition 2.2 ([Dja, Proposition A.8, Theorem A.9])**.** *In a short exact sequence of* **FI**-*modules if any two of the objects are semi-induced then so is the third.*

Remark 2.3. In characteristic 0, induced **FI**-modules are projective, and so every semi-induced **FI**-module is induced. This is not true in general. □

2.5. **Finiteness properties.** An **FI**-module is **finitely generated** if it is a quotient of $\mathcal{I}(V)$ for some finitely generated object V of $\mathrm{Rep}_{\mathbf{k}}(S_*)$ (meaning each V_n is finitely generated as a $\mathbf{k}$-module and all but finitely many vanish). A fundamental result on **FI**-modules is the following noetherianity theorem, first proved in [CEFN, Theorem A] and later reproved in [SS2, Corollary 7.2.7]:

Theorem 2.4. *If M is a finitely generated* **FI***-module then M is noetherian, that is, any* **FI***-submodule of M is also finitely generated.*

2.6. **The shift functor.** Let V be an S_* representation. We define a new S_* representation $\Sigma(V)$ by $\Sigma(V)_n = V_{n+1}|_{S_n}$. We call this the **shift** of V. One readily verifies that if M is an **FI**-module then $\Sigma(M)$ is canonically an **FI**-module. Moreover, in this case, there is a canonical map of **FI**-modules $M \to \Sigma(M)$. If V is an S_*-representation then one has the important identity $\Sigma(\mathcal{I}(V)) = \mathcal{I}(V) \oplus \mathcal{I}(\Sigma(V))$. We make use of the projection map $\Sigma^n(\mathcal{I}(V)) \to \mathcal{I}(V)$ throughout without further mention of this identity.

2.7. **Torsion FI-modules.** Let M be an **FI**-module. We say that an element $x \in M_n$ is **torsion** if x maps to 0 under some transition map $M_n \to M_m$. We say that M itself is **torsion** if all of its elements are. We write $\mathrm{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$ for the category of torsion **FI**-modules. This is a Serre subcategory of $\mathrm{Mod}_{\mathbf{FI}}$ closed under direct sum, and so is a localizing subcategory.

2.8. **The degree of an FI-module.** We inductively define classes $\mathcal{C}_d$ of **FI**-modules as follows. First, $\mathcal{C}_{-1}$ is the class of torsion **FI**-modules. Having defined $\mathcal{C}_d$, we let $\mathcal{C}_{d+1}$ be the class of **FI**-modules M such that the cokernel of the natural map $M \to \Sigma(M)$ belongs to $\mathcal{C}_d$. We now define the **degree** of an **FI**-module M, denoted $\delta(M)$, to be the minimal d such that $M \in \mathcal{C}_d$, or ∞ if not such d exists. We have the following basic facts:

(a) $\delta(M) = -1$ if and only if M is torsion.
(b) If M is semi-induced then $\delta(M) = g(M)$.

(c) If N is a subquotient of M then $\delta(N) \le \delta(M)$. In particular, $\delta(M) \le g(M)$.
(d) If M is an extension of M' by M'' then $\delta(M) = \max(\delta(M'), \delta(M''))$.
(e) If M is non-torsion then $\delta(\mathrm{coker}(M \to \Sigma^k(M))) = \delta(M) - 1$, for any nonzero k.
(f) Suppose $\mathbf{k}$ is a field and M is finitely generated. There then exists a polynomial p such that $p(n) = \dim_{\mathbf{k}} M_n$ for all $n \gg 0$ [CEFN, Theorem B]. We have $\delta(M) = \deg(p)$, using the convention that the degree of the zero polynomial is -1.

The above properties are all reasonably straightforward exercises. Some details can be found in [DV], where **FI**-modules of degree n are treated as degree n polynomial functors on the category $\Theta = \mathbf{FI}$.

Remark 2.5. We note that $\delta(M) = g(\Sigma^n M)$ for n large enough (see Theorem 2.6). In the case when $\mathbf{k}$ is a field we can take property (e) above as the definition of $\delta(M)$ and then the rest of the properties easily follow. This is the case that we use. □

2.9. **Resolutions by induced and semi-induced modules.** The following is a fundamental result on the structure of **FI**-modules due to the first author [Nag, Theorem A].

Theorem 2.6. *Let M be a finitely generated* **FI***-module. Then $\Sigma^n(M)$ is semi-induced for $n \gg 0$.*

As a consequence of this theorem, one obtains the following result [Nag, Theorem A], which is of paramount importance to this paper:

Theorem 2.7. *Let M be a finitely generated* **FI***-module of degree d. Then there exists a complex*

$$0 \to M \to I^0 \to \cdots \to I^d \to 0$$

with torsion cohomology, where I^k is finitely generated and semi-induced of degree $\le \delta(M) - k$.

It is easy to deduce Theorem 2.7 from Theorem 2.6, as follows. Given M, let $n \gg 0$ such that $\Sigma^n(M)$ is semi-induced. The map $M \to \Sigma^n(M)$ has torsion kernel and $\delta(\mathrm{coker}) \le \delta(M) - 1$. Thus, by induction on degree, there exists a complex as in the theorem for the cokernel, and this gives one for M as well, with $I^0 = \Sigma^n(M)$. The proof of the theorem applies equally well if M is a complex, rather than a module, and leads to the following result:

Theorem 2.8. *Let $M^\bullet$ be a finite length complex of finitely generated* **FI***-modules. Then, in the derived category, there exists an exact triangle*

$$T^\bullet \to M^\bullet \to I^\bullet \to$$

where $T^\bullet$ is a finite length complex of finitely generated torsion modules and $I^\bullet$ is a finite length complex of finitely generated semi-induced modules.

The following result is a refinement of [Dja, Theorem A.9(3)].

Proposition 2.9. *Let M be a semi-induced module with $\delta(M) \le d$. Then M admits a resolution $F_\bullet \to M \to 0$ of length at most $d+1$ such that each F_i is induced.*

Proof. Since $g(M) \le d$, $\mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, M)_r = 0$ for $r > d$. Let r be the least integer such that $\mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, M)_r$ is non-trivial (if no such r exists then $M = 0$ by the Nakayama lemma and there is nothing to prove). We prove by downward induction on r that there is a resolution $F_\bullet$ of M of length $d - r + 1$ where each F_i is a direct sum of induced modules. Let $F_0 = \bigoplus_{0 \le k \le d} \mathcal{I}(V_k)$

where $V_k = M_k$. We note that $\mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, M)_r = V_r = \mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, F_0)_r$ and $\mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, M)_k = 0 = \mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, F_0)_k$ for $k < r$. By construction, $\delta(F_0) \le d$ and there is a surjection $\psi \colon F_0 \to M$ (Proposition 2.1). We have $\mathrm{Tor}_0^{\mathbf{A}}(\mathbf{k}, \ker(\psi))_k = 0$ for $k \le r$ (semi-induced modules have no higher Tor; [Ram, Theorem B]). By Proposition 2.2, $\ker(\psi)$ is semi-induced. Clearly, we have $\delta(\ker(\psi)) \le d$. Thus by downward induction, $\ker(\psi)$ admits a resolution of the desired format. We can append F_0 to this resolution to get a resolution of M, completing the proof. □

In the proof above, the construction of F_0 (and thus F_i) is functorial in M, that is, if $f \colon M \to N$ is a map of semi-induced **FI**-modules with $\delta(M), \delta(N) \le d$ then there is a natural map $F_0(f) \colon F_0(M) \to F_0(N)$ making the obvious diagram commute. Thus the proof of the proposition applies equally well if M is a complex of semi-induced modules generated in degree $\le d$, rather than a module. Combining this observation with Theorem 2.8, we obtain:

Theorem 2.10. *Let $M^\bullet$ be a finite length complex of finitely generated* **FI***-modules with $\delta(M^i) \le d$ for each i. Then, in the derived category, there exists an exact triangle*

$$T^\bullet \to M^\bullet \to I^\bullet \to$$

where $T^\bullet$ is a finite length complex of finitely generated torsion modules and $I^\bullet$ is a finite length complex consisting of direct sums of induced modules with $\delta(I^i) \le d$ for each i.

2.10. **Derived saturation.** For our purposes, it will be useful to have a better understanding of the complex in Theorem 2.7. To state the relevant results, it will be convenient to introduce derived saturation. In characteristic 0, this theory was fully developed in [SS1]. We quickly sketch here how the ideas work over general rings $\mathbf{k}$.

Define the **generic category**, denoted $\mathrm{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$, to be the Serre quotient of $\mathrm{Mod}_{\mathbf{FI}}$ by the torsion subcategory $\mathrm{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$. We let $\mathrm{T} \colon \mathrm{Mod}_{\mathbf{FI}} \to \mathrm{Mod}_{\mathbf{FI}}^{\mathrm{gen}}$ be the localization functor, which is exact, and write $\mathrm{S} \colon \mathrm{Mod}_{\mathbf{FI}}^{\mathrm{gen}} \to \mathrm{Mod}_{\mathbf{FI}}$ for the right adjoint of T, the section functor. (This adjoint exists by the general theory of Grothendieck abelian categories: $\mathrm{Mod}_{\mathbf{FI}}^{\mathrm{tors}}$ is a localizing subcategory.) We let $\mathbf{S} = \mathrm{S} \circ \mathrm{T}$, the saturation functor. The saturation functor is left-exact, and we write $\mathrm{R}\mathbf{S}$ for its right-derived functor. We say that an **FI**-module M is **saturated** (resp. **derived saturated**) if the natural map $M \to \mathbf{S}(M)$ (resp. $M \to \mathrm{R}\mathbf{S}(M)$) is an isomorphism. By [Dja, Corollary A.4], M is saturated if and only if M is torsion free and the cokernel of the natural map $M \to \Sigma^k(M)$ is torsion free for each k.

Theorem 2.11 ([Dja, Theorem A.9])**.** *A finitely generated* **FI***-module is semi-induced if and only if it is derived saturated.*

Corollary 2.12. *Let M be a finitely generated* **FI***-module, and let $M \to I^\bullet$ be the complex of Theorem 2.7. Then $\mathrm{R}\mathbf{S}(M)$ is quasi-isomorphic to $I^\bullet$. In particular, $\mathrm{R}^i\mathbf{S}(M)$ is finite for all $i \ge 0$ and vanishes for $i \gg 0$.*

Proof. Clearly, $\mathrm{T}(I^\bullet)$ is a resolution of $\mathrm{T}(M)$ (if N is a torsion module then $\mathrm{T}(N) = 0$). By Theorem 2.11, $\mathrm{T}(I^k)$ is an S-acyclic for each k. Thus $\mathrm{R}\mathbf{S}(M) = \mathrm{S}(\mathrm{T}(I^\bullet)) = \mathrm{R}\mathbf{S}(I^\bullet)$. The remaining assertions now follow immediately from the properties of the complex $I^\bullet$. □

The above corollary shows that the complex $I^\bullet$ in Theorem 2.7 is well-defined up to quasi-isomorphism and depends functorially on M in the derived category. We denote the largest n such that $\mathrm{R}^i\mathbf{S}(M)_n$ is non-zero by $s^i(M)$.

Theorem 2.13. *The complex in Theorem 2.7 is exact in degrees* $\geq g(M)+r(M)$. *Moreover,* $s^i(M) \leq 2\delta(M) - 2i$ *for* $i \geq 1$ *and* $\delta(\mathbf{S}(M)) = \delta(M)$.

Proof. The first assertion follows from [Li, Theorem 1.3]. For the second assertion, we note that the cokernel of $f\colon M \to I^0$ is generated in degree $\leq \delta(M)$. Thus by [Li, Theorem 1.3] and Corollary 2.12, we have $s^i(M) = s^{i-1}(\mathrm{coker}(f)) \leq 2\delta(M) - 2(i-1) - 2 = 2\delta(M) - 2i$ for $i > 1$. The argument for $i = 1$ is similar and can be obtained from the bounds provided in the reference. We have $\mathbf{S}(M) \subset I^0$, and the kernel of the natural map $M \to \mathbf{S}(M)$ is torsion. It follows that $\delta(M) \leq \delta(\mathbf{S}(M)) \leq \delta(I^0) = \delta(M)$, completing the proof. $\square$

3. The divided power algebra and its modules

3.1. **Generalities.** Fix a commutative noetherian ring $\mathbf{k}$. Let $\mathbf{D}$ be the divided power algbera over $\mathbf{k}$ in a single variable x. Thus $\mathbf{D}$ has a $\mathbf{k}$-basis consisting of elements $x^{[n]}$ with $n \in \mathbf{N}$ in which multiplication is given by

$$x^{[n]}x^{[m]} = \tbinom{n+m}{n} x^{[n+m]}.$$

We regard $\mathbf{D}$ as graded, with $x^{[n]}$ having degree n. All $\mathbf{D}$-modules we consider are graded. The ring $\mathbf{D}$ is typically not noetherian: for example, if $\mathbf{k}$ has characteristic p then there is an isomorphism of $\mathbf{k}$-algebras

$$\mathbf{k}[y_0, y_1, y_2, \ldots]/(y_0^p, y_1^p, y_2^p, \ldots) \xrightarrow{\sim} \mathbf{D}$$

defined by $y_i \mapsto x^{[p^i]}$. Nonetheless, $\mathbf{D}$ is always coherent [NS, Theorem 4.1], and so finitely presented $\mathbf{D}$-modules form an abelian category.

For a $\mathbf{D}$-module M, put $\tau_{\geq n}(M) = \bigoplus_{i \geq n} M_i$; this is a $\mathbf{D}$-submodule of M. Following [NS, §9], we say that a $\mathbf{D}$-module M is **nearly finitely presented** if there exists a finitely presented $\mathbf{D}$-module N, called a **weak fp-envelope** of M, and an isomorphism of $\mathbf{D}$-modules $\tau_{\geq n}(M) \cong \tau_{\geq n}(N)$. We remark that if $\mathbf{k}$ is p-adically complete (e.g., if $\mathbf{k}$ has characteristic p) then weak fp-envelopes are unique up to canonical isomorphism [NS, Proposition 9.5].

3.2. **Connections.** Let $d\colon \mathbf{D} \to \mathbf{D}[1]$ be the $\mathbf{k}$-linear map defined by $x^{[k]} \mapsto x^{[k-1]}$, where we use the convention that $x^{[n]} = 0$ if $n < 0$. We leave to the reader the simple verification that d is a derivation. A **connection** on a $\mathbf{D}$-module M is a $\mathbf{k}$-linear map $\nabla\colon M \to M[1]$ that satisfies the Leibniz rule $\nabla(fm) = f\nabla(m) + d(f)m$ for all $f \in \mathbf{D}$ and $m \in M$. The following is a fundamental result about connections:

Proposition 3.1. *Let M be a $\mathbf{D}$-module equipped a connection ∇. Put $M_0 = M/\mathbf{D}_+M$, and write m_0 for the image of $m \in M$ in M_0. Define a map*

$$\Phi\colon M \to \mathbf{D} \otimes_{\mathbf{k}} M_0, \qquad \Phi(m) = \sum_{n \geq 0} x^{[n]}(\nabla^n m)_0.$$

Then:

(a) Φ is an isomorphism of $\mathbf{D}$-modules.
(b) Under Φ, the connection ∇ corresponds to $d \otimes 1$.
(c) The map Φ identifies $\ker(\nabla)$ with M_0.
(d) The natural map $\ker(\nabla) \otimes_{\mathbf{k}} \mathbf{D} \to M$ is an isomorphism of $\mathbf{D}$-modules.

Proof. (a) Let $m \in M$ and $f \in \mathbf{D}$. We have the identity

$$\nabla^n(fm) = \sum_{i+j=n} \binom{n}{i} d^i(f)\nabla^j(m).$$

Thus

$$\begin{aligned}\Phi(fm) &= \sum_{n\ge 0}\sum_{i+j=n} \binom{n}{i} x^{[n]} d^i(f)_0 \nabla^j(m)_0 \\ &= \left(\sum_{i\ge 0} x^{[i]} d^i(f)_0\right)\left(\sum_{j\ge 0} x^{[j]}\nabla^j(m)_0\right) \\ &= f\Phi(m).\end{aligned}$$

In the last step we used Taylor's theorem: for any $f \in \mathbf{D}$ we have

$$f = \sum_{i\ge 0} x^{[i]} d^i(f)_0.$$

We have thus show that Φ is $\mathbf{D}$-linear. It is clear that Φ_0 is the identity map $M_0 \to M_0$, and so Φ is an isomorphism by the following lemma (basically Nakayama's).

(b) We have

$$(d\otimes 1)\Phi(m) = \sum_{n\ge 1} x^{[n-1]}(\nabla^n m)_0 = \sum_{n\ge 0} x^{[n]}(\nabla^{n+1} m)_0 = \Phi(\nabla m).$$

(c) By part (b), Φ induces an isomorphism $\ker(\nabla) \to \ker(d\otimes 1) = 1 \otimes M_0$.

(d) The map in question is just the inverse to Φ, under the identification $\ker(\nabla) = M_0$. $\square$

Lemma 3.2. *Let $f\colon M \to N$ be a map of $\mathbf{D}$-modules.*

(a) Suppose the map $f_0\colon M_0 \to N_0$ is surjective. Then f is surjective.

(b) Suppose f_0 is an isomorphism and N has the form $\mathbf{D} \otimes_{\mathbf{k}} N'$ for some $\mathbf{k}$-module N'. Then f is an isomorphism.

Proof. (a) We proceed by induction on degree. Thus let $n \in N$ have degree d, and suppose surjectivity holds in smaller degree. We can write $n_0 = f_0(m_0)$ for some $m \in M$, and thus $n - f(m) \in \mathbf{D}_+N$, and so $n - f(m) = \sum_{k\ge 1} x^{[k]} n_k$ for some $n_k \in N$. Since then n_k have strictly smaller degree than n, we have $n_k = f(m_k)$ for some $m_k \in M$, by induction. Thus $n - f(m) = f(m')$, where $m' = \sum_{k\ge 1} x^{[k]} m_k$, and so $n = f(m + m')$.

(b) Consider the exact sequence

$$0 \to K \to M \to N \to 0$$

Reducing mod $\mathbf{D}_+$, and using the fact that $\mathrm{Tor}_1^{\mathbf{D}}(N, \mathbf{k}) = 0$, gives

$$0 \to K_0 \to M_0 \to N_0 \to 0.$$

Since f_0 is an isomorphism, we find $K_0 = 0$, and so $K = 0$ (use part (a) with the map $0 \to K$). This proves the lemma. $\square$

Suppose now that $\mathbf{k}$ has characteristic p and let q be a power of p. Let $d_q\colon \mathbf{D} \to \mathbf{D}[q]$ be the q-fold iterate of d, i.e., $d_q(x^{[k]}) = x^{[k-q]}$. Then d_q is a derivation, since any q-fold iteration of a derivation in characteristic p is still a derivation. A **q-connection** on a $\mathbf{D}$-module M is a $\mathbf{k}$-linear map $\nabla\colon M \to M[q]$ satisfying the Leibniz rule with respect to d_q.

For example, if $\nabla \colon M \to M[1]$ is a connection then the q-fold iterate of ∇ is a q-connection. Let $\mathbf{D}^{(q)} = \bigoplus_{q \mid n} \mathbf{D}_n$, a subalgebra of $\mathbf{D}$. There is an algebra isomorphism $\mathbf{D}^{(q)} \to \mathbf{D}$ mapping $x^{[qn]}$ to x^n, under which d_q corresponds to d. Thus Proposition 3.1 yields:

Proposition 3.3. *Let M be a $\mathbf{D}$-module admitting a q-connection. Then there is an isomorphism of $\mathbf{D}^{(q)}$-modules $M \cong \mathbf{D}^{(q)} \otimes_{\mathbf{k}} M'$ for some graded $\mathbf{k}$-module M'.*

3.3. **Periodicity phenomena.** We assume in this section that $\mathbf{k}$ is a field of characteristic p. It is not difficult to see (and we will prove) that if M is a finitely presented $\mathbf{D}$-module then the dimension of M_n is eventually periodic in n with period a power of p. The main purpose of this section is to introduce invariants ϵ and λ that control the period and the onset of periodicity, and prove a few facts about them.

We first introduce some notation. Put $y_i = x^{[p^i]}$, and recall that $\mathbf{D}$ is isomorphic to $\mathbf{k}[y_i]_{i \geq 0}/(y_i^p)$. For a subset I of $\mathbf{N}$, let $\mathbf{D}_I$ be the subalgebra of $\mathbf{D}$ generated by the y_i with $i \in I$. We also put $\mathbf{D}_{<r} = \mathbf{D}_{[0,r)}$ and $\mathbf{D}_{\geq r} = \mathbf{D}_{[r,\infty)}$, where we use the usual interval notation for subsets of $\mathbf{N}$.

Proposition 3.4. *Let $f \colon M_1 \to M_2$ be a map of finitely generated free $\mathbf{D}_{\geq r}$-modules. Suppose that, in a suitable basis, the matrix entries of f only involve the variables $y_r, \ldots, y_{s-1}$, for some $s \geq r$. Then the kernel, cokernel, and image of f are free as $\mathbf{D}_{\geq s}$-modules.*

Proof. Let $\overline{M}_i$ be a free $\mathbf{D}_{[r,s)}$-module with the same basis as M_i, and define $\overline{f} \colon \overline{M}_1 \to \overline{M}_2$ using the same matrix that defines f. Then f is obtained from $\overline{f}$ by applying the exact functor $- \otimes_{\mathbf{k}} \mathbf{D}_{\geq s}$. Thus $\ker(f) = \ker(\overline{f}) \otimes_{\mathbf{k}} \mathbf{D}_{\geq s}$ is free over $\mathbf{D}_{\geq s}$, and similarly for the cokernel and image. □

Corollary 3.5. *Let M be a finitely presented $\mathbf{D}$-module. Then M is free as a $\mathbf{D}_{\geq r}$-module for some r.*

Proof. Apply the proposition to a presentation of M. □

Definition 3.6. Let M be a finitely presented $\mathbf{D}$-module. We define $\epsilon(M)$ to be the minimal non-negative integer r so that M is free as a $\mathbf{D}_{\geq r}$-module. □

Remark 3.7. If $s \geq r$ then $\mathbf{D}_{\geq r}$ is free as a $\mathbf{D}_{\geq s}$-module. It follows that if M is a finitely presented $\mathbf{D}$-module then M is free over $\mathbf{D}_{\geq r}$ for any $r \geq \epsilon(M)$. □

Let M be a finitely presented $\mathbf{D}$-module. For $r \geq \epsilon(M)$, let $g_r(M)$ be the maximal degree of a basis element of M as a $\mathbf{D}_{\geq r}$-module (this is independent of the choice of basis). Let $s \geq r$. Then $\mathbf{D}_{\geq r}$ is free as a $\mathbf{D}_{\geq s}$-module, with basis consisting of the monomials in variables $y_r, \ldots, y_{s-1}$. The monomial $m = y_r^{p-1} \cdots y_{s-1}^{p-1}$ is the one of maximal degree, and has degree $p^s - p^r$. Thus if e is a maximal degree basis element of M as a $\mathbf{D}_{\geq r}$-module then me is a maximal degree basis element of M as a $\mathbf{D}_{\geq s}$-module. It follows that we have the identity

$$g_s(M) = p^s - p^r + g_r(M).$$

This observation is the basis of the following definition.

Definition 3.8. Let M be a finitely presented $\mathbf{D}$-module. We define $\lambda(M)$ to be the common value of the expression $g_r(M) + 1 - p^r$, for $r \geq \epsilon(M)$. □

Example 3.9. We have $\epsilon(\mathbf{D}[d]) = 0$ and $\lambda(\mathbf{D}[d]) = d$. □

The importance of the λ and ϵ invariants comes from the following proposition.

Proposition 3.10. *Let M be a finitely presented* $\mathbf{D}$*-module. Then* $\dim_{\mathbf{k}}(M_n)$ *is periodic in n, for n sufficiently large, with period a power of p. More precisely, we have* $\dim(M_n) = \dim(M_m)$ *whenever* $n \equiv m \pmod{p^{\epsilon(M)}}$ *and* $n, m \geq \lambda(M)$.

Proof. Put $r = \epsilon(M)$ and $q = p^r$ and $a = \lambda(M)$. Write $M = V \otimes \mathbf{D}_{\geq r}$ where V is a finite dimensional graded vector space concentrated in degree $< q + a$. The degree n piece of $\mathbf{D}_{\geq r}$ is one dimensional if n is a non-negative multiple of q, and 0 otherwise. We thus see that

$$\dim(M_n) = \sum_{\substack{0 \leq k \leq n \\ k \equiv n \pmod{*} q}} \dim(V_k).$$

Thus if $n \equiv m \pmod{q}$ and $n \leq m$ then

$$\dim(M_m) - \dim(M_n) = \sum_{\substack{n < k \leq m \\ k \equiv n \pmod{*} q}} \dim(V_k).$$

Thus if $n \geq a$ then $k \geq q + a$ for every k appearing in the sum, and so $V_k = 0$ for such k. We thus find $\dim(M_m) = \dim(M_n)$. $\square$

We now explain how one can bound ϵ and λ in certain situations.

Proposition 3.11. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finitely presented $\mathbf{D}$*-modules. Then*

$$\epsilon(M_2) \leq \max(\epsilon(M_1), \epsilon(M_3)), \qquad \lambda(M_2) \leq \max(\lambda(M_1), \lambda(M_3)).$$

Proof. If M_1 and M_3 are free over $\mathbf{D}_{\geq r}$ then so is M_2; furthermore, a basis for M_2 can be obtained from bases of M_1 and M_3, and so $g_r(M_2)$ is equal to $\max(g_r(M_1), g_r(M_3))$. The proposition follows easily from these observations. $\square$

Proposition 3.12. *Let M be a finitely presented* $\mathbf{D}$*-module and let N be a finitely presented sub or quotient of M. Then* $\lambda(N) \leq \lambda(M)$.

Proof. Let I_r be the maximal ideal of $\mathbf{D}_{\geq r}$, i.e., $(y_r, y_{r+1}, \ldots)$. Note that if M is free as a $\mathbf{D}_{\geq r}$-module then $g_r(M)$ is equal to $\operatorname{maxdeg}(M/I_rM)$, where we write $\operatorname{maxdeg}(V)$ for the maximal degree occurring in a graded vector space V. Suppose that N is a quotient of M. Then for any $r \geq 0$, we have a surjection $M/I_rM \to N/I_rN$, and so

$$\operatorname{maxdeg}(N/I_rN) \leq \operatorname{maxdeg}(M/I_rM).$$

Thus $g_r(N) \leq g_r(M)$ whenever the two sides are defined, and the result follows. Now suppose N is a sub of M. Let r be sufficiently large so that M/N is free as a $\mathbf{D}_{\geq r}$-module. Then the natural map $N/I_rN \to M/I_rM$ is injective, and so again one has the above inequality, and the result follows. $\square$

Corollary 3.13. *Let $f \colon M \to N$ be a map of finitely presented* $\mathbf{D}$*-modules. Then*

$$\lambda(\ker(f)) \leq \lambda(M), \qquad \lambda(\operatorname{im}(f)) \leq \lambda(M), \qquad \lambda(\operatorname{coker}(f)) \leq \lambda(N).$$

For an integer n, we let $\ell(n)$ be the smallest non-negative integer r such that $n \leq p^r$. Note that $\ell(a+b) \leq \max(\ell(a), \ell(b)) + 1$.

Proposition 3.14. *Let $f\colon M \to N$ be a map of finitely presented* $\mathbf{D}$*-modules. Then*

$$\epsilon(*) \le \max(\epsilon(M), \epsilon(N), \ell(\lambda(M))) + 1,$$

where $*$ is the kernel, cokernel, or image of f.

Proof. Let $r = \max(\epsilon(M), \epsilon(N))$, and let s be the right side of the inequality in the statement of the proposition. Let $\{v_i\}$ and $\{w_j\}$ be bases for M and N as $\mathbf{D}_{\ge r}$-modules. Then $f(v_i) = \sum_j a_{i,j} w_j$ for some $a_{i,j} \in \mathbf{D}_{\ge r}$. Since the w_j have non-negative degree, we find

$$\deg(a_{i,j}) \le \deg(v_i) < p^r + \lambda(M) \le p^s.$$

Thus $a_{i,j}$ can only involve the variables $y_r, \ldots, y_{s-1}$, and so the kernel, cokernel, and image of f are free over $\mathbf{D}_{\ge s}$ by Proposition 3.4. □

Corollary 3.15. *Let $I \subset \mathbf{D}$ be a homogeneous ideal generated in degrees $\le d$. Then*

$$\deg \operatorname{Tor}_1^{\mathbf{D}}(\mathbf{k}, I) \le d - 1 + p^{\ell(d)+1}.$$

Proof. Let $F \to I$ be a surjection where F is a free $\mathbf{D}$-module generated in degrees $\le d$. Let K be its kernel. Then we have $\lambda(K) \le d$ and $\epsilon(K) \le \ell(d) + 1$. This shows that

$$\deg \operatorname{Tor}_1^{\mathbf{D}}(\mathbf{k}, I) \le g_{\epsilon(K)} \le d - 1 + p^{\ell(d)+1},$$

completing the proof. □

4. Cohomology of FI-modules

4.1. **The functor Γ.** Write $\mathrm{Mod}_{\mathbf{k}}^{\mathbf{N}}$ for the category of graded $\mathbf{k}$-modules supported in non-negative degrees. We define a functor

$$\Gamma\colon \operatorname{Rep}_{\mathbf{k}}(S_*) \to \mathrm{Mod}_{\mathbf{k}}^{\mathbf{N}}, \qquad \Gamma(V)_n = V_n^{S_n}.$$

This functor is left-exact, and so we can consider its right derived functor $\mathrm{R}\Gamma$. We write Γ^t in place of $\mathrm{R}^t\Gamma$. Since the functor Γ is simply computed pointwise, so is Γ^t, and thus

$$\Gamma^t(V)_n = \mathrm{H}^t(S_n, V_n).$$

We now investigate some of further properties of this functor.

Proposition 4.1. *If V and W are S_*-representations then there is a canonical isomorphism $\Gamma(V \otimes W) = \Gamma(V) \otimes \Gamma(W)$, that is, Γ is a tensor functor.*

Proof. Recall that if $H \subset G$ are finite groups and V is a representation of H then we have a canonical identification $(\operatorname{Ind}_H^G(V))^G = V^H$. This follows, for instance, from Frobenius reciprocity. The proposition follows from this:

$$\begin{aligned}\Gamma(V \otimes W)_n = (V \otimes W)_n^{S_n} &= \bigoplus_{i+j=n} (\operatorname{Ind}_{S_i \times S_j}^{S_n}(V_i \otimes V_j))^{S_n} \\ &= \bigoplus_{i+j=n} V_i^{S_i} \otimes W_j^{S_j} = \bigoplus_{i+j=n} \Gamma(V)_i \otimes \Gamma(W)_j = (\Gamma(V) \otimes \Gamma(W))_n\end{aligned}$$ □

In fact, Γ is a symmetric tensor functor, as the isomorphism of $\Gamma(V \otimes W)$ with $\Gamma(V) \otimes \Gamma(W)$ just constructed clearly respects the symmetries on both tensor products. It follows from this that Γ takes commutative algebras in $\operatorname{Rep}_{\mathbf{k}}(S_*)$, that is, twisted commutative $\mathbf{k}$-algebras, to commutative algebras in $\mathrm{Mod}_{\mathbf{k}}^{\mathbf{N}}$, that is, commutative graded $\mathbf{k}$-algebras. In particular, $\Gamma(\mathbf{A})$ is a graded commutative $\mathbf{k}$-algebra. The following proposition identifies it:

Proposition 4.2. *The algebra $\Gamma(\mathbf{A})$ is the divided power algbera $\mathbf{D}$.*

Proof. Since Γ is a tensor functor, the multiplication map $\Gamma(\mathbf{A})\otimes\Gamma(\mathbf{A}) \to \Gamma(\mathbf{A})$ is the image of the multiplication map $\mathbf{A}\otimes\mathbf{A} \to \mathbf{A}$ under Γ. The latter map in degree n is $\oplus_{i\leq n}\Psi_n^i$ where $\Psi_n^i \colon \operatorname{Ind}_{S_i\times S_{n-i}}^{S_n} \mathbf{A}_i \otimes_{\mathbf{k}} \mathbf{A}_{n-i} \to \mathbf{A}_n$ is the map induced by the multiplication map $\mathbf{A}_i \otimes_{\mathbf{k}} \mathbf{A}_{n-i} \to \operatorname{Res}_{S_i\times S_{n-i}}^{S_n} \mathbf{A}_n$ taking $a\otimes b$ to ab. Clearly, the images of the source and the target of Ψ_n under Γ can be naturally identified with $\mathbf{k}$ and $\Gamma(\Psi_n)$ is the multiplication by the binomial coefficient $\binom{n}{i}$ under this identification. This shows that $\Gamma(\mathbf{A})$ is the divided power algebra $\mathbf{D}$. □

Proposition 4.3. *Let M be an $\mathbf{FI}$-module, and let $\Gamma^t(M) = \bigoplus_{n\geq 0}\mathrm{H}^t(S_n, M_n)$ be the corresponding $\Gamma(\mathbf{A})$-module. Then the action of $x^{[m-n]} \in \Gamma(\mathbf{A})_{m-n}$ on $\Gamma^t(M)$ is given by the composition $\psi_{n,m}$ of the following maps:*

$$\mathrm{H}^t(S_n, M_n) \to \mathrm{H}^t(S_n\times S_{m-n}, M_n) \to \mathrm{H}^t(S_n\times S_{m-n}, M_m) \to \mathrm{H}^t(S_m, M_m).$$

The first map is pull-back along the group homomorphism $S_n\times S_{m-n} \to S_n$. The second map is induced by the $\mathbf{FI}$-module transition map $M_n \to M_m$. Finally, the last map is corestriction.

Proof. Since all the maps in the composite are functorial in M, we may assume that $t=0$. Since Γ is a tensor functor, the multiplication $\Gamma(\mathbf{A})_{m-n}\otimes\Gamma(M)_n \to \Gamma(M)_m$ is the image of the multiplication map $\mathbf{A}_{m-n}\otimes M_n \to M_m$ under Γ. The latter map $\Psi\colon \operatorname{Ind}_{S_{m-n}\times S_n}^{S_m}\mathbf{k}\otimes_{\mathbf{k}} M_n \to M_m$ is the natural map induced by the $\mathbf{FI}$ structure on M. Clearly, $(\operatorname{Ind}_{S_{m-n}\times S_n}^{S_m}\mathbf{k}\otimes_{\mathbf{k}} M_n)^{S_m}$ and $(M_m)^{S_m}$ are naturally isomorphic to $\Gamma(M)_m$ and $\Gamma(M)_n$ respectively, and $(\Psi)^{S_n}$ is the composite map in the assertion under this isomorphism. This completes the proof. □

4.2. **A connection on the cohomology of the symmetric group.** In this section we reformulate a result due to Dold in terms of connections and generalize it to induced **FI**-modules. More precisely, we prove the following:

Proposition 4.4. *Let V be an object of $\mathrm{Rep}_{\mathbf{k}}(S_*)$ and denote the $\mathbf{D}$-module $\Gamma^t(\mathcal{I}(V))$ by M. Then M admits a connection $\nabla\colon M \to M[1]$. Moreover, if $V_n = 0$ for $n > d$ then we have the following:*

- *(a)* $\ker\nabla_n = 0$ *for* $n > 2t+d$.
- *(b)* $M \cong M_0\otimes_{\mathbf{k}}\mathbf{D}$ *for a graded* $\mathbf{k}$*-module* M_0 *supported in degrees* $\leq 2t+d$.

The rest of §4.2 is devoted to proving this proposition. For this, we introduce some additional objects. Let $\mathrm{Rep}_{\mathbf{k}}(S_*)^G$ be the category of G-equivariant objects of $\mathrm{Rep}_{\mathbf{k}}(S_*)$: an object is a sequence $(M_n)_{n\geq 0}$ where M_n is a $\mathbf{k}[S_n\times G]$-module. For $M\in\mathrm{Rep}_{\mathbf{k}}(S_*)^G$, the module $\Gamma(M)$ has an action of G, and we put $\Gamma_G(M) = \Gamma(M)^G$. We let Γ_G^t be the tth right derived functor of Γ^t. Thus for $M\in\mathrm{Rep}_{\mathbf{k}}(S_*)^G$ we have

$$\Gamma_G^t(M)_n = \mathrm{H}^t(S_n\times G, M_n).$$

Let $\mathrm{Mod}_{\mathbf{A}}^G$ denote the category of G-equivariant $\mathbf{A}$-modules, using the trivial action of G on $\mathbf{A}$. For $M\in\mathrm{Mod}_{\mathbf{A}}^G$, the $\mathbf{D}$-module $\Gamma^t(M)$ has an action of G, and so $\Gamma_G^t(M)$ is also a $\mathbf{D}$-module. The following result is a direct analog of Proposition 4.3.

Proposition 4.5. *Let M be an G-equivariant $\mathbf{A}$-module. Then the action of $x^{[m-n]}\in\mathbf{D}$ on $\Gamma_G^t(M)_n$ is given by the composition $\varphi_{n,m}$ of the following maps:*

$$\mathrm{H}^t(S_n\times G, M_n) \to \mathrm{H}^t(S_n\times S_{m-n}\times G, M_n) \to \mathrm{H}^t(S_n\times S_{m-n}\times G, M_m) \to \mathrm{H}^t(S_m\times G, M_m).$$

The first map is pull-back along the group homomorphism $S_n \times S_{m-n} \times G \to S_n \times G$. The second map is induced by the $\mathbf{FI}$-module transition map $M_n \to M_m$. Finally, the last map is corestriction.

The following lemma of Dold is crucial for our argument.

Lemma 4.6. *Let M be a $\mathbf{k}[G]$-module with S_{n+m} acting trivially. Then in the below diagram, the map defined by the middle path is the sum of the ones defined by the top and the bottom paths.*

$$\begin{tikzcd}
\mathrm{H}^t(S_n \times S_{m-1} \times S_1 \times G, M) \arrow[rrd, bend left, "\mathrm{cor}"] & & \\
\mathrm{H}^t(S_n \times S_m \times G, M) \arrow[u, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] \arrow[r, "\mathrm{cor}"] & \mathrm{H}^t(S_{n+m} \times G, M) \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_{n+m-1} \times S_1 \times G, M) \\
\mathrm{H}^t(S_{n-1} \times S_1 \times S_m \times G, M) \arrow[rru, bend right, "\mathrm{cor} \circ \zeta^\star"'] & &
\end{tikzcd}$$

Here ζ is the permutation defined by

$$(\zeta(1), \ldots, \zeta(n+m)) = (1, 2, \ldots, n-1, n+m, n, n+1, \ldots, n+m-1),$$

and res *and* cor *are restriction and corestriction respectively.*

Proof. Let P be a $\mathbf{k}[S_{n+m} \times G]$-free resolution of $\mathbf{k}$. Then P is an H-free resolution for every subgroup H of $S_{n+m} \times G$. To keep Dold's ([Dol]) notation, we denote the cochain complex $\mathrm{Hom}_{\mathbf{k}[G]}(P, M)$ by A. There is an action of S_{n+m} on A given by $(\sigma f)(p) = \sigma(f(\sigma^{-1}p))$, and for any subgroup H of S_{n+m}, we have $\mathrm{H}^t(H \times G, M) = \mathrm{H}^t(A^H)$. With this observation in mind, the assertion follows by applying [Dol, Lemma 1, dual version] to this A. $\square$

Proposition 4.7. *Let V be a $\mathbf{k}[G]$-module, regarded as an object of $\mathrm{Rep}_{\mathbf{k}}(S_*)^G$ concentrated in degree 0. The restriction map*

$$\mathrm{H}^t(S_n \times G, V) \to \mathrm{H}^t(S_{n-1} \times G, V)$$

defines a connection ∇ on the $\mathbf{D}$-module $\Gamma^t_G(\mathcal{I}(V))$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^t(S_n \times G, V) \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_n \times S_{m-1} \times S_1 \times G, V) \arrow[rrd, bend left, "\mathrm{cor}"] & & \\
\mathrm{H}^t(S_n \times G, V) \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{res}"'] \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_n \times S_m \times G, V) \arrow[u, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \arrow[r, "\mathrm{cor}"] & \mathrm{H}^t(S_{n+m} \times G, V) \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_{n+m-1} \times S_1 \times G, V) \\
\mathrm{H}^t(S_{n-1} \times S_1 \times G, V) \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_{n-1} \times S_1 \times S_m \times G, V) \arrow[rru, bend right, "\mathrm{cor} \circ \zeta^\star"'] & &
\end{tikzcd}$$

By Proposition 4.5, the composite maps defined by the top, the middle and the bottom paths are $x^{[m-1]}$, $\nabla x^{[m]}$ and $x^{[m]}\nabla$ respectively. Since restriction is functorial, we see that the top left and bottom left squares commute. Thus, by the previous lemma, we have $\nabla x^{[m]} = x^{[m]}\nabla + d(x^{[m]})$. This completes the proof. $\square$

We shall need a group theoretic analog of the binomial identity

$$\binom{n+m}{n}\binom{n}{d} = \binom{n+m}{d}\binom{n+m-d}{n-d}.$$

We introduce a notation first. Let $B \subset A \subset U$ be finite sets, and let $S \in \binom{A}{B}$ (that is, S is a subset of A of size $|B|$). We denote, by γ_S, the element of $\mathrm{Aut}_{\mathbf{FI}}(U)$ that fixes the complement of A pointwise, takes B to S, and is order preserving on both S and $A \setminus S$. The following lemma is easy to verify and provides a group theoretic analog of the identity above.

Lemma 4.8. *Let $A_1 \sqcup A_2 \sqcup A_3$ is a partition of U. Then the following identity holds in $\mathbf{k}[\mathrm{Aut}_{\mathbf{FI}}(U)]$*

$$\sum_{S' \in \binom{U}{A_1 \sqcup A_2}} \sum_{S \in \binom{A_1 \sqcup A_2}{A_1}} \gamma_{S'}\gamma_S = \sum_{T' \in \binom{U}{A_2 \sqcup A_3}} \sum_{T \in \binom{A_2 \sqcup A_3}{A_2}} \gamma_{T'}\gamma_T.$$

Moreover, all the elements of $\mathrm{Aut}_{\mathbf{FI}}(U)$ appearing in the left (right) hand side are distinct.

Proposition 4.9. *Let V be an object of $\mathrm{Rep}_{\mathbf{k}}(S_*)$ concentrated in degree d, and let $V' = V_d$, regarded as an object of $\mathrm{Rep}_{\mathbf{k}}(S_*)^{S_d}$ concentrated in degree 0. Then Shapiro isomorphism $\Gamma^t(\mathcal{I}(V)) \to \Gamma^t_{S_d}(\mathcal{I}(V'))[d]$ is a map of $\mathbf{D}$-modules.*

Proof. Let $f\colon [n] \to [n+m]$ be the natural inclusion. We need to verify the commutative of the following diagram.

$$\begin{tikzcd}
\mathrm{H}^t(S_n, \mathcal{I}(V)_n) \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_n \times S_m, \mathcal{I}(V)_n) \arrow[r, "f_\star"] & \mathrm{H}^t(S_n \times S_m, \mathcal{I}(V)_{n+m}) \arrow[r, "\mathrm{cor}"] & \mathrm{H}^t(S_{n+m}, \mathcal{I}(V)_{n+m}) \\
\mathrm{H}^t(S_{n-d} \times S_d, V) \arrow[u, "\mathrm{Shapiro}"] \arrow[r, "\mathrm{res}"] & \mathrm{H}^t(S_{n-d} \times S_m \times S_d, V) \arrow[r, "\mathrm{id}"] & \mathrm{H}^t(S_{n-d} \times S_m \times S_d, V) \arrow[r, "\mathrm{cor}"] & \mathrm{H}^t(S_{n+m-d} \times S_d, V) \arrow[u, "\mathrm{Shapiro}"']
\end{tikzcd}$$

Each of the cohomology groups in the diagram above can be calculated by using a common $\mathbf{k}[S_{n+m}]$-free resolution of $\mathbf{k}$. Thus is suffices to verify the commutativity of the following diagram for a given $\mathbf{k}[S_{n+m}]$-module P.

$$\begin{tikzcd}
\mathrm{Hom}_{S_n}(P, \mathcal{I}(V)_n) \arrow[r, "\mathrm{res}"] & \mathrm{Hom}_{S_n \times S_m}(P, \mathcal{I}(V)_n) \arrow[r, "f_\star"] & \mathrm{Hom}_{S_n \times S_m}(P, \mathcal{I}(V)_{n+m}) \arrow[r, "\mathrm{cor}"] & \mathrm{Hom}_{S_{n+m}}(P, \mathcal{I}(V)_{n+m}) \\
\mathrm{Hom}_{S_{n-d} \times S_d}(P, V) \arrow[u, "Shapiro"] \arrow[r, "\mathrm{res}"] & \mathrm{Hom}_{S_{n-d} \times S_m \times S_d}(P, V) \arrow[r, "\mathrm{id}"] & \mathrm{Hom}_{S_{n-d} \times S_m \times S_d}(P, V) \arrow[r, "\mathrm{cor}"] & \mathrm{Hom}_{S_{n+m-d} \times S_d}(P, V) \arrow[u, "Shapiro"']
\end{tikzcd}$$

Set $A_1 = [d]$, $A_2 = [n] \setminus [d]$ and $A_3 = [n+m] \setminus [n]$. Then we have

$$\mathcal{I}(V)_{n+m} = \mathbf{k}[\mathrm{Aut}_{\mathbf{FI}}(U)] \otimes_{\mathbf{k}[\mathrm{Aut}_{\mathbf{FI}}(A_2 \sqcup A_3) \times \mathrm{Aut}_{\mathbf{FI}}(A_3)]} V.$$

Fix an element $a \in \mathrm{Hom}_{S_{n-d} \times S_d}(P, V)$. The images, say φ and ψ, of a in $\mathrm{Hom}_{S_{n+m}}(P, \mathcal{I}(V)_{n+m})$ along the top and the bottom paths are given by

$$\begin{aligned}
\varphi(p) &= \sum_{S' \in \binom{U}{A_1 \sqcup A_2}} \sum_{S \in \binom{A_1 \sqcup A_2}{A_1}} \gamma_{S'}\gamma_S \otimes a(\gamma_S^{-1}\gamma_{S'}^{-1} p) \\
\psi(p) &= \sum_{T' \in \binom{U}{A_2 \sqcup A_3}} \sum_{T \in \binom{A_2 \sqcup A_3}{A_2}} \gamma_{T'} \otimes \gamma_T a(\gamma_T^{-1}\gamma_{T'}^{-1} p) \\
&= \sum_{T' \in \binom{U}{A_2 \sqcup A_3}} \sum_{T \in \binom{A_2 \sqcup A_3}{A_2}} \gamma_{T'}\gamma_T \otimes a(\gamma_T^{-1}\gamma_{T'}^{-1} p)
\end{aligned}$$

Thus by the previous lemma, we see that $\varphi = \psi$. This completes the proof. $\square$

Lemma 4.10. *Suppose $n > 2t \geq 0$. Then the restriction map $\mathrm{H}^{t+1}(S_n, \mathbf{Z}) \to \mathrm{H}^{t+1}(S_{n-1}, \mathbf{Z})$ is an isomorphism.*

Proof. By the universal coefficient theorem for relative cohomology we have:

$$\mathrm{H}^t((S_n, S_{n-1}), G) = \left[\mathrm{H}^t((S_n, S_{n-1}), \mathbf{Z}) \otimes_{\mathbf{Z}} G\right] \oplus \mathrm{Tor}_1^{\mathbf{Z}}(\mathrm{H}^{t+1}((S_n, S_{n-1}), \mathbf{Z}), G).$$

Nakaoka's stability theorem ([Nak]) states that $\mathrm{H}^t(S_n, G) = \mathrm{H}^t(S_{n-1}, G)$ for $n > 2t$ where G is an arbitrary abelian group with trivial action of S_n. Thus we must have

$$\mathrm{Tor}_1^{\mathbf{Z}}(\mathrm{H}^{t+1}((S_n, S_{n-1}), \mathbf{Z}), G) = 0$$

for all abelian groups G. This implies that $\mathrm{H}^{t+1}((S_{2t+1}, S_{2t}), \mathbf{Z})$ is flat as a $\mathbf{Z}$-module. Since $t+1 > 0$, $\mathrm{H}^{t+1}((S_{2t+1}, S_{2t}), \mathbf{Z})$ is a torsion $\mathbf{Z}$-module. A flat torsion $\mathbf{Z}$ module must be trivial. The assertion now follows. □

Proof of Proposition 4.4. It suffices to treat the case where V is concentrated in degree d. Let $V' = V_d$, regarded as an object of $\mathrm{Rep}_{\mathbf{k}}(S_*)^{S_d}$ in degree 0. Set $M = \Gamma^t_{S_d}(\mathcal{I}(V))$. By Proposition 4.9, $\Gamma^t(\mathcal{I}(V))$ is isomorphic to $M[d]$ as $\mathbf{D}$-modules, and by Proposition 4.7 we have a connection ∇ on M.

We claim that ∇_n is an isomorphism for $n > 2t$. To see this, observe that

$$M_n = \mathrm{H}^t(S_d \times S_n, V \boxtimes \mathbf{Z}) = \bigoplus_{a+b=t+1} \mathrm{Tor}_1^{\mathbf{Z}}(\mathrm{H}^a(S_d, V), \mathrm{H}^b(S_n, \mathbf{Z})) \oplus \bigoplus_{a+b=t} \mathrm{H}^a(S_d, V) \otimes \mathrm{H}^b(S_n, \mathbf{Z})$$

By Lemma 4.10, the restriction map $\mathrm{H}^b(S_n, \mathbf{Z}) \to \mathrm{H}^b(S_{n-1}, \mathbf{Z})$ is an isomorphism for $n > 2t$. Since the Künneth formula commutes with restriction, the claim follows.

The above claim implies $\ker(\nabla_n) = 0$ for $n > 2t$, which proves statement (a). By Proposition 3.1, we have $M/\mathbf{D}_+M = \ker \nabla$ and an isomorphism $M \cong \mathbf{D} \otimes_{\mathbf{k}} \ker(\nabla)$, and so (b) follows. □

4.3. **Proof of main theorem.** For an **FI**-module M, define $\underline{\Gamma}(M) = \Gamma(\mathbf{S}(M))$, where $\mathbf{S}$ is the saturation functor. The functor $\underline{\Gamma}$ is left-exact, so we can consider its right derived functors $\mathrm{R}\underline{\Gamma}$. We put $\underline{\Gamma}^t = \mathrm{R}^t\underline{\Gamma}$. We note that the map $M \to \mathbf{S}(M)$ induces a map $\Gamma(M) \to \underline{\Gamma}(M)$, and thus maps $\Gamma^t(M) \to \underline{\Gamma}^t(M)$ for all $t \geq 0$.

Theorem 4.11. *Let M be a finitely generated* **FI***-module. Then:*

(a) $\underline{\Gamma}^t(M)$ is a finitely presented **D***-module.*
(b) The map $\Gamma^t(M) \to \underline{\Gamma}^t(M)$ is an isomorphism in degrees $\geq g(M) + r(M)$.
(c) If **k** *has characteristic p, then $\lambda(\underline{\Gamma}^t(M)) \leq 2t + \delta(M)$.*

Proof. (a) If M is induced then $\Gamma^t(M)$ is finitely presented by Proposition 4.4. If M is semi-induced, then $\Gamma^t(M)$ is finitely presented by dévissage to the induced case ($\mathbf{D}$ is coherent). Finally, for a general module M, consider a complex $M \to I^\bullet$ as in Theorem 2.7, so that $\mathrm{R}\mathbf{S}(M) = I^\bullet$. Then $\mathrm{R}\underline{\Gamma}(M) = \mathrm{R}\Gamma(I^\bullet)$, and so there is a spectral sequence

$$\Gamma^i(I^j) \implies \underline{\Gamma}^{i+j}(M),$$

from which it follows that $\underline{\Gamma}^t(M)$ is finitely presented.

(b) The complex $M \to I^\bullet$ of Theorem 2.7 is exact in degrees $\geq g(M) + r(M)$ (Theorem 2.13), and so it follows that the map $\Gamma^t(M) \to \Gamma^t(I^\bullet) = \underline{\Gamma}^t(M)$ is an isomorphism in degrees $\geq g(M) + r(M)$, since Γ^t is computed degree-wise.

(c) First, suppose M is induced from degree $\leq d$. Then Propositions 4.4 shows that $\Gamma^t(M)$ is a free $\mathbf{D}$-module finitely generated in degrees $\leq 2t + d$, and so $\lambda(\Gamma^t(M)) \leq 2t + d$.

Now suppose that M is semi-induced of degree $\leq d$. Then there is a short exact sequence

$$0 \to N \to M \to I \to 0$$

where I is induced of degree d and N is semi-induced of degree $< d$. We obtain an exact sequence

$$\Gamma^t(N) \to \Gamma^t(M) \to \Gamma^t(I)$$

We have seen that $\lambda(\Gamma^t(I)) \leq 2t+d$, and by induction we can assume $\lambda(\Gamma^t(N)) \leq 2t+d-1$. It follows from Propositions 3.11 and 3.12 that $\lambda(\Gamma^t(M)) \leq 2t+d$.

Next suppose that $I^\bullet$ is a finite length complex of semi-induced modules with I^i of degree $\leq d-i$. We will show $\Gamma^t(I^\bullet) \leq 2t+d$. Let $J^\bullet$ be the complex $I^1 \to I^2 \to \cdots$. We have a short exact sequence of complexes

$$0 \to J^\bullet[1] \to I^\bullet \to I^0 \to 0.$$

We thus obtain an exact sequence

$$\Gamma^{t-1}(J^\bullet) \to \Gamma^t(I^\bullet) \to \Gamma^t(I^0).$$

By induction on d, we can assume $\lambda(\Gamma^{t-1}(J^\bullet)) \leq 2(t-1)+(d-1)$, and the previous paragraph shows $\lambda(\Gamma^t(I^0)) \leq 2t+d$. Once again, we conclude $\Gamma^t(I^\bullet) \leq 2t+d$.

Finally, let M be a finitely generated **FI**-module of degree $\leq d$. Let $M \to I^\bullet$ be as in Theorem 2.7. Then $\underline{\Gamma}^t(M) = \Gamma^t(I^\bullet)$, and so $\lambda(\underline{\Gamma}^t(M)) \leq 2t+\delta(M)$ by the previous paragraph. □

Corollary 4.12. *If M is a finitely generated* **FI***-module then $\Gamma^t(M)$ is nearly finitely presented.*

Corollary 4.13. *Let M be a finitely generated* **FI***-module over a field* $\mathbf{k}$ *of characteristic p, and let $t \geq 0$ be given. Then there exists a power q of p such that*

$$\dim_{\mathbf{k}} \mathrm{H}^t(S_n, M_n) = \dim_{\mathbf{k}} \mathrm{H}^t(S_{n+q}, M_{n+q})$$

holds for all $n \geq \max(g(M)+r(M), 2t+\delta(M))$.

Example 4.14. $\Gamma^t(M)$ may not be finitely presented: assume that $t=0$ and M be the **FI**-module over $\mathbf{F}_p$ satisfying $M_0 = \mathbf{F}_p$ and $M_n = 0$ for $n > 0$. Then we have $\Gamma^0(M) = \mathbf{D}/\mathbf{D}_+$ which is clearly not finitely presented ($\mathbf{D}_+$ is not finitely generated). □

4.4. **Bounds on the period.** In this section, $\mathbf{k}$ is a field of characteristic p. Our goal is to prove the following result:

Theorem 4.15. *Let M be a finitely generated* **FI***-module and suppose $\delta(M) < q$, with $q = p^r$. Then the* **D***-module $\underline{\Gamma}^t(M)$ admits a q-connection. In particular, $\epsilon(\underline{\Gamma}^t(M)) \leq r$.*

Corollary 4.16. *In Corollary 4.13, one can take q to be any power of p such that $\delta(M) < q$.*

We require a number of preparatory results. In this section, $\mathrm{R}^\bullet\Gamma(V)$ denotes $\bigoplus_{t\geq 0} \Gamma^t(V)$. Note that this space is bigraded, since each $\Gamma^t(V)$ is itself a graded vector space. We call t the cohomological index; writing $\Gamma(V) = \bigoplus_{n\geq 0} \Gamma(V_n)$, we call n the **FI**-index.

Proposition 4.17. *The functor $\mathrm{R}^\bullet\Gamma$ respects tensor products, that is, if M and N are representations of S_* then there is a natural isomorphism $\mathrm{R}^\bullet\Gamma(M \otimes N) = \mathrm{R}^\bullet\Gamma(M) \otimes \mathrm{R}^\bullet\Gamma(N)$. Furthermore, this isomorphism is compatible with the symmetries of the tensor products (using the appropriate sign rule on the cohomological index).*

Proof. Since $\mathrm{R}\Gamma$ and $\otimes$ commute with arbitrary direct sums, it suffices to treat the case where M is a concentrated in degree m and N is concentrated in degree n. The result then follows from Shapiro's lemma and the Künneth formula:

$$\begin{aligned}\mathrm{R}^\bullet\Gamma(M\otimes N) &= \mathrm{H}^\bullet(S_{m+n}, \mathrm{Ind}_{S_m\times S_n}^{S_{n+m}}(M_m\otimes N_n)) = \mathrm{H}^\bullet(S_m\times S_n, M_m\otimes N_n)\\ &= \mathrm{H}^\bullet(S_m, M_m)\otimes \mathrm{H}^\bullet(S_n, N_n) = \mathrm{R}^\bullet\Gamma(M)\otimes \mathrm{R}^\bullet\Gamma(N)\end{aligned}$$

The symmetry property is standard. ∎

In particular, we see that $\mathbf{E} = \mathrm{R}^\bullet\Gamma(\mathbf{A})$ naturally has the structure of an algebra. Note that $\mathbf{E}_n^t = \mathrm{H}^t(S_n, \mathbf{k})$. Let $\mathbf{d}\colon \mathbf{E}_n^t \to \mathbf{E}_{n-1}^t$ be the map induced by restricting cohomology classes from S_n to S_{n-1}.

Proposition 4.18. *The map $\mathbf{d}\colon \mathbf{E}\to\mathbf{E}$ is a derivation.*

Proof. We need to show that in the diagram below the, middle row is the sum of the top and the bottom rows (where $\cup$ is the cup product and rest of the notation are as in Lemma 4.6):

$$\begin{tikzcd}
\mathrm{H}^a(S_n,\mathbf{k})\otimes \mathrm{H}^b(S_{m-1},\mathbf{k}) \arrow[r, "\cup"] & \mathrm{H}^{a+b}(S_n\times S_{m-1}\times S_1,\mathbf{k}) \arrow[drrr, dashed, bend left, "\mathrm{cor}"] & & \\
\mathrm{H}^a(S_n,\mathbf{k})\otimes \mathrm{H}^b(S_m,\mathbf{k}) \arrow[u, "\mathrm{id}\otimes\mathrm{res}"] \arrow[d, "\mathrm{res}\otimes\mathrm{id}"'] \arrow[r, "\cup"] & \mathrm{H}^{a+b}(S_n\times S_m,\mathbf{k}) \arrow[u, dashed, "\mathrm{res}"'] \arrow[d, dashed, "\mathrm{res}"'] \arrow[r, dashed, "\mathrm{cor}"] & \mathrm{H}^{a+b}(S_{n+m},\mathbf{k}) \arrow[r, dashed, "\mathrm{res}"] & \mathrm{H}^{a+b}(S_{n+m-1}\times S_1,\mathbf{k}) \\
\mathrm{H}^a(S_{n-1},\mathbf{k})\otimes \mathrm{H}^b(S_m,\mathbf{k}) \arrow[r, "\cup"] & \mathrm{H}^{a+b}(S_{n-1}\times S_1\times S_m,\mathbf{k}) \arrow[urrr, dashed, bend right, "\mathrm{cor}\circ\zeta^\star"'] & &
\end{tikzcd}$$

By Lemma 4.6, the dashed part has the property that the middle row is the sum of the top and the bottom rows. Also, the cup product commutes with restriction. Thus the top-left and the bottom-left squares commute. The assertion now follows. ∎

Let V be a representation of S_*. We constructed a connection ∇ on the $\mathbf{D}$-module $\Gamma^t(\mathfrak{I}(V))$. Write ∇ still for the induced map on $\mathrm{R}^\bullet\Gamma(\mathfrak{I}(V))$.

Proposition 4.19. *Under the identification $\mathrm{R}^\bullet\Gamma(\mathfrak{I}(V)) = \mathrm{R}^\bullet\Gamma(V)\otimes\mathbf{E}$, the map ∇ corresponds to $1\otimes\mathbf{d}$.*

Proof. It suffices to consider the case when V is concentrated in degree d. With the notation of Proposition 4.9, the identification is given by the composite

$$\mathrm{R}^\bullet\Gamma(\mathfrak{I}(V)) \to \mathrm{R}^\bullet\Gamma_{S_d}(\mathfrak{I}(V'))[d] \to \mathrm{R}^\bullet\Gamma(V)\otimes\mathbf{E}$$

where the first map is the Shapiro isomorphism and the second map is the Künneth isomorphism. Recall that ∇ is defined to correspond (via the Shapiro isomorphism) to the connection on $\mathrm{R}^\bullet\Gamma_{S_d}(\mathfrak{I}(V'))[d]$, and the connection on $\mathrm{R}^\bullet\Gamma_{S_d}(\mathfrak{I}(V'))[d]$ is given in degree n by the restriction map (Proposition 4.7)

$$\mathrm{H}^t(S_{n-d}\times S_d, V)\to \mathrm{H}^t(S_{n-d-1}\times S_d, V).$$

Since the Künneth isomorphism commutes with restriction, we see that ∇ corresponds to $1\otimes\mathbf{d}$. ∎

Proposition 4.20. *Let V and W be representations of S_* and let $f\colon \mathfrak{I}(V)\to\mathfrak{I}(W)$ be a map of $\mathbf{A}$-modules. Suppose V is supported in degrees $< q$, where q is a power of p. Then the map $\Gamma^t(\mathfrak{I}(V))\to\Gamma^t(\mathfrak{I}(W))$ induced by f is compatible with the connection ∇^q, for all $t\geq 0$.*

Proof. The map $\mathrm{R}^\bullet\Gamma(\mathcal{I}(V)) \to \mathrm{R}^\bullet\Gamma(\mathcal{I}(W))$ induced by f is one of $\mathbf{E}$-modules. Since $\mathrm{R}^\bullet\Gamma(\mathcal{I}(V))$ is isomorphic to $\mathrm{R}^\bullet\Gamma(V) \otimes \mathbf{E}$, it suffices to show that f carries $\mathrm{R}^\bullet\Gamma(V)$ into the kernel of ∇^q. But this is automatic: every element of $\mathrm{R}^\bullet\Gamma(V)$ has $\mathbf{FI}$-degree $< q$, and so the same is true for the elements of the image, and they are thus annihilated by ∇^q for degree reasons. □

Remark 4.21. Note that the statement of the proposition is about a single cohomology group, but the proof (which is the simplest one we know) makes use of the $\mathbf{E}$-module structure on all cohomology groups. □

Proof of Theorem 4.15. By Corollary 2.12, $\mathrm{R}\mathbf{S}(M)$ is quasi-isomorphic to a bounded complex $I^\bullet$ of semi-induced modules generated in degree $\leq \delta(M)$. By Proposition 2.9 (and the following discussion), $I^\bullet$ is quasi-isomorphic to a bounded complex of induced modules generated in degree $\leq \delta(M)$. The result now follows from Proposition 4.20 and Proposition 3.3. □

5. Applications

5.1. **Periodicity in the cohomology of Specht modules.** Assume that $\mathbf{k}$ is a field of characteristic p. Recall that $\mathbf{M}_\lambda$ is the Specht module corresponding to a partition λ and that $\lambda[n]$ is the partition $(n - |\lambda|, \lambda)$. We have the following:

Theorem 5.1. *Let λ be a partition of d and q be a power of p strictly larger than d. If $n \geq \max(2d + \lambda_1, 2t + d)$, then we have*

$$\mathrm{H}^t(S_n, \mathbf{M}_{\lambda[n]}) = \mathrm{H}^t(S_{n+q}, \mathbf{M}_{\lambda[n+q]}).$$

We start with a proposition.

Proposition 5.2. *Suppose λ is a partition of d. There is an $\mathbf{FI}$-module L_λ such that the following hold:*

- *(a)* $(L_\lambda)_n = \mathbf{M}_{\lambda[n]}$
- *(b)* $\delta(\lambda_\lambda) = d$
- *(c)* $g(L_\lambda) = d + \lambda_1$
- *(d)* $r(L_{\lambda_1}) \leq 2d + \lambda_1 + 1$
- *(e) There exists a finite length complex $0 \to L_\lambda \to I^\bullet$ exact in degrees $\geq 2d + \lambda_1$ where, for each i, I^i is a semi-induced module with $\delta(I^i) \leq d$.*

Proof. We first construct L_λ. Let V_λ be the permutation module of S_d corresponding to the partition λ. Now note that for $n \geq d + \lambda_1$, $\mathcal{I}(V_\lambda)_n$ is naturally a permutation module for the partition $\lambda[n]$. For each $n \geq d + \lambda_1$, inductively pick a Young tableau T_n of shape $\lambda[n]$ with entries in $[n]$ such that T_n and T_m agree on $\lambda[d + \lambda_1]$. We can think of $\mathcal{I}(V_\lambda)_n$ as the $\mathbf{k}[S_n]$-module generated by T_n. Let $C_{\lambda[n]}$ be the column subgroup corresponding to T_n. By construction, we have $C_n = i_{d+\lambda_1,n} C_{d+\lambda_1}$ where $i_{m,n} \colon [m] \to [n]$ is the natural inclusion. Thus if $e = \sum_{\sigma \in C_{d+\lambda_1}} \mathrm{sgn}(\sigma)\sigma$ then we have

$$\mathbf{M}_{\lambda[n]} = (i_{d+\lambda_1,n} e) T_n = i_{d+\lambda_1,n}(e T_{d+\lambda_1}) = i_{d+\lambda_1,n} \mathbf{M}_{d+\lambda_1}.$$

This shows that the sub-$\mathbf{FI}$-module of $\mathcal{I}(V_\lambda)$ generated in degree $d+\lambda_1$ by $eT_{d+\lambda_1}$ is isomorphic to $\mathbf{M}_{\lambda[n]}$ in every degree $n \geq d+\lambda_1$. Define L_λ to be this sub-$\mathbf{FI}$-module. This already proves (a) and (c). It is easy to check via the hook length formula that $\dim_{\mathbf{k}} \mathbf{M}_{\lambda[n]}$ is eventually a polynomial in n of degree d. This proves (b). Now consider the exact sequence

$$0 \to L_\lambda \to \mathcal{I}(V_\lambda) \to K \to 0.$$

We have $g(K) \le d$ and $r(K) \le g(L_\lambda) = d + \lambda_1$. So by Theorem 2.13, K admits a complex as in Theorem 2.7 exact in degrees $\ge 2d + \lambda_1$. (e) now follows from the exact sequence above. In [CE, Theorem A], it is proven that Castelnuovo-regularity of an **FI**-modules M is at most $g(M) + r(M) - 1$. It follows that $r(L_\lambda) = \mathrm{Tor}^1_{\mathbf{A}}(L_\lambda, \mathbf{k}) = \mathrm{Tor}^2_{\mathbf{A}}(K, \mathbf{k}) \le g(K) + r(K) + 1$. Thus $r(L_\lambda) \le 2d + \lambda_1 + 1$, completing the proof. □

Proof of Theorem 5.1. Let $I^\bullet$ be the complex in Proposition 5.2(e). Then we have $\underline{\Gamma}^t(L_\lambda) = \Gamma^t(I^\bullet)$, and the natural map $\Gamma^t(L_\lambda) \to \underline{\Gamma}^t(L_\lambda)$ is an isomorphism in degrees $\ge 2d + \lambda_1$. Now by Proposition 4.20, $\Gamma^t(I^\bullet)$ admits a q-connection. We also have $\lambda(\Gamma^t(I^\bullet)) \le 2t + d$ by Theorem 4.11(c). Thus by Proposition 3.10, we see that $\Gamma^t(I^\bullet)_n = \Gamma^t(I^\bullet)_{n+q}$ for $n \ge 2t + d$. The result now follows because $\Gamma^t(I^\bullet)_n = \Gamma^t(L_\lambda)_n = \mathrm{H}^t(S_n, \mathbf{M}_{\lambda[n]})$ for $n \ge 2d + \lambda_1$. □

5.2. **Integral cohomology of unordered configuration spaces.** Let $\mathcal{M}$ be a manifold. The unordered configuration space $\mathrm{uConf}_n(\mathcal{M})$ is given by

$$\mathrm{uConf}_n(\mathcal{M}) \coloneqq \{(P_1, P_2, \dots, P_n) \in \mathcal{M}^n \colon P_i \ne P_j\}/S_n.$$

Some recent results have shown that under certain mild assumption on $\mathcal{M}$, the cohomology groups $\mathrm{H}^t(\mathrm{uConf}_n(M), \mathbf{F}_p)$ are periodic (see [KM] for the latest result). We have an integral version in this direction:

Theorem 5.3. *Suppose* $\mathbf{k}$ *is a commutative noetherian ring and fix a* $t \ge 0$*. Let* $\mathcal{M}$ *be a connected orientable manifold of dimension* ≥ 2 *with the homotopy type of a finite CW complex. Then there exists a finitely presented* $\mathbf{D}$*-module* M *such that*

$$\mathrm{H}^t(\mathrm{uConf}_n(\mathcal{M}), \mathbf{k}) = M_n$$

for $n \gg 0$*.*

Proof. In [Nag, §4], it is shown that there is a bounded below complex $\mathcal{U}^\bullet$ of finitely generated **FI**-modules such that $\Gamma^s(\mathcal{U}^\bullet)_n = \mathrm{H}^s(\mathrm{uConf}_n(\mathcal{M}), \mathbf{k})$ for all s (the proof in the reference is only given when $\mathbf{k}$ is a field of positive characteristic but it works without change for any noetherian ring). Since t is fixed, we may assume without loss of generality that $\mathcal{U}^\bullet$ is bounded and that $\Gamma^t(\mathcal{U}^\bullet)_n = \mathrm{H}^t(\mathrm{uConf}_n(\mathcal{M}), \mathbf{k})$. By Corollary 2.12, $\mathrm{R}\mathbf{S}(\mathcal{U}^\bullet)$ is quasi-isomorphic to a bounded complex $I^\bullet$ of induced **FI**-modules. Since $\mathbf{D}$ is coherent and $\Gamma^x(I^y)$ is a finitely presented $\mathbf{D}$-module (Proposition 4.4), we conclude that $\Gamma^t(\mathrm{R}\mathbf{S}(\mathcal{U}^\bullet))$ is finitely presented. The result now follows because the cone of the map $\mathcal{U}^\bullet \to \mathrm{R}\mathbf{S}(\mathcal{U}^\bullet)$ is quasi-isomorphic to a bounded complex of finitely generated torsion modules (Theorem 2.7 and Corollary 2.12) and hence is supported in finitely many degrees. □

Example 5.4. It is known that $\mathrm{H}^2(\mathrm{uConf}_n(\mathcal{S}^2), \mathbf{Z}) = \mathbf{Z}/(2n-2)$ for $n \ge 2$ (see [FB]). Thus M can be taken to be the cokernel of the map $\mathbf{D}[2] \to \mathbf{D}[1]$ given by $x^{[0]} \mapsto 2x^{[1]}$. □

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