SINGULARITIES OF ORDINARY DEFORMATION RINGS

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Abstract. Let R^{univ} be the universal deformation ring of a residual representation of a local Galois group. Kisin showed that many loci in $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/p])$ of interest are Zariski closed, and gave a way to study the generic fiber of the corresponding quotient of R^{univ}. However, his method gives little information about the quotient ring before inverting p. We give a method for studying this quotient in certain cases, and carry it out in the simplest non-trivial case. Precisely, suppose that V_0 is the trivial two dimensional representation and let R be the unique $\mathbf{Z}_p$-flat and reduced quotient of R^{univ} such that $\mathrm{MaxSpec}(R[1/p])$ consists of ordinary representations with Hodge–Tate weights 0 and 1. We describe the functor of points of (a slightly modified version of) R and show that the irreducile components of $\mathrm{Spec}(R)$ are normal and Cohen–Macaulay, but not Gorenstein. As a consequence, we find that certain global deformation rings are torsion-free and Cohen–Macaulay, but not Gorenstein.

Contents

1. Introduction

1.1. **Statement of results.** Let F be a finite extension of $\mathbf{Q}_p$, let k be a finite field of characteristic p, and let V_0 be a finite dimensional k-vector space carrying a continuous representation of the absolute Galois group G_F. Let $\mathcal{O}$ be a finite totally ramified extension of $W(k)$. There is then a universal ring R^{univ} parameterizing (framed) deformations of V_0 to artinnian $\mathcal{O}$-algebras. Consider the locus in $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/p])$ consisting of ordinary representations, i.e., those representations V such that

$$V|_{I_F} \cong \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where χ is the cyclotomic character. It is known that this set is Zariski closed, and therefore of the form $\mathrm{MaxSpec}(R[1/p])$ for a unique $\mathcal{O}$-flat reduced quotient R or R^{univ}. A natural problem (and one important for modularity lifting) is to understand the ring R: how many components does it have? what are its singularities? what is its functor of points?

Somewhat naively, define a deformation V of V_0 to an artinnian $\mathcal{O}$-algebra A to be *ordinary* if V admits a basis as an A-module such that (1) holds. One might then expect that R represents

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the deformation problem that assigns to A the set of ordinary framed deformations to A. If $\chi|_{I_F}$ is non-trivial modulo p then this is indeed the case, and one can use this natural moduli-theoretic description of R to study R. However, if $\chi|_{I_F}$ is trivial mod p then this description of R is incorrect: in fact, the stated deformation problem is not representable.

The failure of representability when $\chi|_{I_F}$ has trivial reduction comes from the fact that a deformation can be ordinary in multiple ways, that is, there can be multiple lines in V on which the Galois group acts by χ. This suggests that one should study the deformation problem assigning to A the set of pairs (V, L) where V is a framed deformation of V_0 to A and L is a rank one summand of V on which Galois acts through χ. Kisin [Ki2] showed that this deformation problem is represented by the formal completion of a projective morphism $\Theta\colon Z \to \mathrm{Spec}(R^{\mathrm{univ}})$, and that $\mathrm{Spec}(R)$ is the scheme-theoretic image of Θ. (In fact, this is how he showed X is Zariski closed.) It is easily seen that Θ is an isomorphism after inverting p, and this gives a way to analyze $R[1/p]$. Using this method, Kisin established some coarse results about R (such as its dimension and number of components) that he used to prove an $R[1/p] = \mathbf{T}[1/p]$ theorem.

Unfortunately, it is hard to prove finer results about R using the above method. Indeed, the above construction of R leaves its functor of points a complete mystery, since the points of a scheme-theoretic image are difficult to understand. In this paper, we give a method to describe the functor of points of R and to prove finer results about R. For instance, we show that (a slight modification of) R is normal and Cohen–Macaulay, but often not Gorenstein.

Our results have global applications. In §5 we give a simple example of how our results can be used to prove stronger $R = \mathbf{T}$ theorems. These ideas have already found application in the work of Calegari–Geraghty [CG], where modularity lifting is extended beyond the Taylor–Wiles method.

Remark 1.1.1. We concentrate on representations of the form (1) in this paper for simplicity. However, the method applies equally well to higher weight ordinary representations, or, even more generally, the locus in $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/p])$ consisting of extensions of two arbitrary characters. It is reasonable to hope that this method could be generalized to higher dimensional ordinary representations, though this would require additional work.

1.2. **Outline of the method in general.** Let X be an equidimensional Zariski closed subset of $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/p])$ and let R be the unique $\mathcal{O}$-flat reduced quotient of R^{univ} such that $\mathrm{MaxSpec}(R[1/p]) = X$. To analyze R we proceed along the following steps:

(a) Come up with a list of equations that make sense in arbitrary deformations of V_0 and which hold on the set X. Let $R^{\dagger}$ be the quotient of R^{univ} by these equations.
(b) Show that the map $R^{\mathrm{univ}} \to R$ factors through $R^{\dagger}$, and that the resulting map $R^{\dagger}[1/p]_{\mathrm{red}} \to R[1/p]$ is an isomorphism. This should not be difficult if one found enough equations in step (a). One should understand $R[1/p]$ from Kisin's method, and so one should now have some understanding of $R^{\dagger}[1/p]$.
(c) Show that $R^{\dagger} \otimes_{\mathcal{O}} k$ is reduced, equidimensional of the same dimension as $R^{\dagger}[1/p]$, and has the same number of minimal primes as $R^{\dagger}[1/p]$. As far as we know, there is no reason that this must be true in general; however, if it is true, it may be tractable to prove, as $R^{\dagger} \otimes_{\mathcal{O}} k$ represents an explicit deformation problem on k-algebras.
(d) Appeal to abstract commutative algebra facts to conclude that $R^{\dagger}$ is $\mathcal{O}$-flat and reduced (see Propositions 2.2.1 and 2.3.1).
(e) Conclude that $R^{\dagger} \to R$ is an isomorphism, as both are $\mathcal{O}$-flat and reduced and the map $R^{\dagger}[1/p]_{\mathrm{red}} \to R[1/p]$ is an isomorphism.

If the above steps can be carried out then one has a description of the functor of points of R, via the equations used to define $R^{\dagger}$. This can then be used to study R.

1.3. **Example of the method in an easy case.** Let us show how the method can be applied in an easy case. Let V_0 be the trivial two dimensional representation and let X be the set of representations that are conjugate to one of the form

$$\begin{pmatrix} \chi & * \\ & 1 \end{pmatrix}$$

on the full Galois group. We assume that $\chi = 1 \pmod p$, otherwise X is empty. There are two obvious families of equations that make sense in any deformation and that hold on X, namely:

(E1) $\mathrm{tr}(g) = \chi(g) + 1$ for $g \in G_F$.
(E2) $(g-1)(g'-1) = (\chi(g)-1)(g'-1)$ for $g, g' \in G_F$.

Let $R^\dagger$ be the quotient of R^{univ} by these equations. To be more precise, let $\rho^{\mathrm{univ}} \colon G_F \to \mathrm{GL}_2(R^{\mathrm{univ}})$ be the universal framed deformation. Let I be the ideal of R^{univ} generated by the elements

$$\mathrm{tr}(\rho^{\mathrm{univ}}(g)) - \chi(g) - 1,$$

as g varies over G_F, as well as the entries of the matrix

$$(\rho^{\mathrm{univ}}(g) - 1)(\rho^{\mathrm{univ}}(g') - 1) - (\chi(g) - 1)(\rho^{\mathrm{univ}}(g') - 1),$$

as g and g' vary over G_F. Then $R^\dagger = R^{\mathrm{univ}}/I$. Thus $R^\dagger$ is described as a quotient of R^{univ} by an explicit, though uncountable, set of relations.

The only really interesting step in the above procedure is (c), i.e., the analysis of the ring $R^\dagger \otimes_{\mathcal{O}} k$. To understand this ring, it suffices to understand maps $R^\dagger \to A$ where A is a k-algebra. From the equations defining $R^\dagger$, together with the fact that $\chi = 1 \pmod p$, we see that a map $R^{\mathrm{univ}} \to A$, corresponding to a deformation V, factors through $R^\dagger$ if and only if $\mathrm{tr}(g|_V) = 2$ for all $g \in G_F$ and $(g-1)(g'-1)V = 0$ for all $g, g' \in G_F$. It is easy to see that the action of G_F on such a deformation V factors through the abelianization of G_F. Using class field theory, it is therefore possible to give a very explicit description of $\mathrm{Spec}(R^\dagger \otimes_{\mathcal{O}} k)$ as a certain space of tuples of matrices. (In fact, it is (a formal completion of) the space $\mathcal{A}_{d+2}$ discussed in §3.3, with $d = [F : \mathbf{Q}_p]$.) This space can be analyzed by techniques of algebraic geometry and seen to be integral, normal and Cohen–Macaulay.

All the remaining points in the method carry through in this case, and so the map $R^\dagger \to R$ is an isomorphism. This gives a moduli-theoretic description of R: a map $R \to A$ corresponds to a framed deformation of V_0 to A where the equations (E1) and (E2) hold. Furthermore, since we know that $R^\dagger$ is $\mathcal{O}$-flat and $R^\dagger \otimes_{\mathcal{O}} k$ is Cohen–Macaulay, we find that R itself is Cohen–Macaulay. A similar argument can be used to show that R is normal.

1.4. **Plan of paper.** We begin in §2 by establishing some miscellaneous results we will need. In §3, we study certain moduli spaces of matrices. These spaces will show up as the special fibers of the rings $R^\dagger$ (as we saw in §1.3), and it is crucial to know that they are reduced and have the appropriate number of irreducible components. In §4 we carry out the analysis of ordinary deformation rings in the local case and prove the main results of the paper. Finally, in §5, we give the applications to global deformation rings.

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2. Preliminary results

In this section we establish some preliminary results that we will need later on. With the possible exception of Proposition 2.2.1, this material is well-known.

2.1. **Rings associated with vector bundles.** Let X be a geometrically integral scheme proper over k, let η be a vector bundle on X and let Z be the total space of the dual of η. Put

$$R = \Gamma(Z, \mathcal{O}_Z) = \Gamma(X, \mathrm{Sym}(\eta)).$$

In this section we prove two results about R, the first concerning its singularities and the second its minimal free resolution. We will only apply these results when $X = \mathbf{P}^1$, but the proofs are just as easy in the general case. The first result is the following (the notation is explained following the proposition):

Proposition 2.1.1. *Assume that X is normal and Cohen–Macaulay, that η is ample and generated by its global sections, and that*

$$\mathrm{H}^i(X, \mathrm{Sym}(\eta)) = \mathrm{H}^i(X, \mathrm{Sym}(\eta) \otimes \det(\eta) \otimes \omega_X) = 0$$

for $i > 0$. Then the map $Z \setminus Z_0 \to \mathrm{Spec}(R) \setminus \{0\}$ is an isomorphism and R is Cohen–Macaulay. Furthermore, if

$$\dim \mathrm{H}^0(X, \det(\eta) \otimes \omega_X) > 1$$

then the local ring of R at 0 is not Gorenstein.

Let us explain the notation and terminology used above. The ring R is naturally graded. Let R_+ be the ideal of positive degree elements. Then R/R_+ is identified with k. We write 0 for the point in $\mathrm{Spec}(R)$ corresponding to the ideal R_+. We write ω_Y for the dualizing complex of a variety Y over k, which is a coherent sheaf when Y is Cohen–Macaulay. A vector bundle η on X is said to be *ample* if the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(\eta^\vee)$ is ample. If η is a line bundle then $\mathbf{P}(\eta^\vee) = X$ and $\mathcal{O}(1)$ is just η, and so this notion of ample corresponds to the usual one. A direct sum of ample bundles is again ample. We write Z_0 for the image of the zero section of $\pi \colon Z \to X$.

Before proving the proposition we give some lemmas. In these lemmas, we do not assume the hypotheses of the proposition.

Lemma 2.1.2. *If η is generated by global sections then $Z \to \mathrm{Spec}(R)$ is proper.*

Proof. Since η is generated by its global sections, it is a quotient of $V^* \otimes \mathcal{O}_X$ for some vector space V. Thus Z is a closed subset of $X \times V$. As the map $X \times V \to V$ is proper, we find that the map $Z \to V$ is proper. This map factors as $Z \to \mathrm{Spec}(R) \to V$, and so $Z \to \mathrm{Spec}(R)$ is proper. □

Lemma 2.1.3. *Assume that X is normal, that η is ample and that $f \colon Z \to \mathrm{Spec}(R)$ is proper. Then f induces an isomorphism $Z \setminus Z_0 \to (\mathrm{Spec}\, R) \setminus \{0\}$.*

Proof. Since η is ample, it follows that, for $z \in Z(\overline{k})$, we have $f(z) = 0$ if and only if $z \in Z_0(\overline{k})$. Furthermore, if $z \notin Z_0(\overline{k})$ then one can recover $\pi(z)$ from $f(z)$; since the restriction of f to each fiber of π is injective on $\overline{k}$-points, it follows that the restriction of f to $Z \setminus Z_0$ is injective on $\overline{k}$-points.

Let f' denote the map $Z \setminus Z_0 \to (\mathrm{Spec}\, R) \setminus \{0\}$ induced by f. As f is proper, so too is f'. We thus find that the image of f' is a closed subscheme of $(\mathrm{Spec}\, R) \setminus \{0\}$. Since R is integral and has dimension at most that of Z, and f' is injective on $\overline{k}$-points, it follows that f' is surjective. It follows that f' is finite (Zariski's main theorem) and birational (as it is generically flat), which implies (by normality) that f' is an isomorphism. □

Lemma 2.1.4. *Let $f \colon \widetilde{Y} \to Y$ be a proper birational map of schemes over k, with $\widetilde{Y}$ Cohen–Macaulay and Y affine. Suppose that $\mathrm{H}^i(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = \mathrm{H}^i(\widetilde{Y}, \omega_{\widetilde{Y}}) = 0$ for $i > 0$ and that the natural map $f^* \colon \mathrm{H}^0(Y, \mathcal{O}_Y) \to \mathrm{H}^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ is an isomorphism. Then Y is Cohen–Macaulay and ω_Y is isomorphic to $f_*(\omega_{\widetilde{Y}})$.*

Proof. Grothendieck duality for f states that

$$\mathrm{R}f_* \mathrm{R}\,\underline{\mathrm{Hom}}_{\widetilde{Y}}(F, f^! G) = \mathrm{R}\,\underline{\mathrm{Hom}}_Y(Rf_* F, G)$$

for $F \in \mathrm{D}^b(\widetilde{Y})$ and $G \in \mathrm{D}^b(Y)$. Take $F = \mathcal{O}_{\widetilde{Y}}$ and $G = \omega_Y$. The hypotheses of the lemma imply that $\mathrm{R}f_*F = \mathcal{O}_Y$; furthermore, $f^!G = \omega_{\widetilde{Y}}$. We thus find $\mathrm{R}f_*(\omega_{\widetilde{Y}}) = \omega_Y$. The hypotheses of the lemma imply that $\mathrm{R}f_*(\omega_{\widetilde{Y}}) = f_*(\omega_{\widetilde{Y}})$. Thus ω_Y is concentrated in a single degree, and so Y is Cohen–Macaulay. □

We now return to the proof of Proposition 2.1.1.

Proof of Proposition 2.1.1. The scheme Z is geometrically integral and Cohen–Macaulay, being a vector bundle over such a scheme. By hypothesis, $\mathrm{H}^i(Z, \mathcal{O}_Z) = \mathrm{H}^i(X, \mathrm{Sym}(\eta)) = 0$ for $i > 0$. We have $\omega_Z = \pi^*(\det(\eta) \otimes \omega_X)$; when X is smooth, this can be seen from taking the determinant of the exact sequence

$$0 \to \pi^*(\Omega^1_{X/k}) \to \Omega^1_{Z/k} \to \Omega^1_{Z/X} \to 0$$

after using the identification $\Omega^1_{Z/X} = \pi^*(\eta)$. Thus

$$\mathrm{H}^i(Z, \omega_Z) = \mathrm{H}^i(X, \mathrm{Sym}(\eta) \otimes \det(\eta) \otimes \omega_X)$$

vanishes for $i > 0$, by hypothesis. From Lemmas 2.1.2 and 2.1.3, we see that $Z \to \mathrm{Spec}(R)$ is proper and birational. Lemma 2.1.4 shows that R is Cohen–Macaulay and $\omega_R = \mathrm{H}^0(Z, \omega_Z)$. Now, ω_R is a graded R-module whose first graded piece is $\mathrm{H}^0(X, \det(\eta) \otimes \omega_X)$. This space injects into $\omega_R/R_+\omega_R$. Thus if $\mathrm{H}^0(X, \det(\eta) \otimes \omega_X)$ has dimension greater than 1 then the fiber of ω_R at 0 has dimension greater than 1 and R is not Gorenstein at 0. □

Suppose now that η is generated by its global sections, and write

$$0 \to \xi \to \epsilon \to \eta \to 0$$

with ϵ a globally free coherent $\mathcal{O}_X$-module. Let $V = \mathrm{Hom}(\epsilon, \mathcal{O}_X)$, so that we have canonical identifications $\epsilon = V^* \otimes \mathcal{O}_X$ and $\mathrm{Spec}(\mathrm{Sym}(\epsilon)) = X \times V$. Put $S = \mathrm{Sym}(V^*)$. Note that there is a natural map $S \to R$ respecting the grading. We regard k as an S-module via the identification $S/S_+ = k$.

Our second result is the following proposition. It is the basis of the "geometric method" for studying syzygies, as exposited in [We, Ch. 5]. We include a proof since it is short.

Proposition 2.1.5. *Assume $\mathrm{H}^i(X, \mathrm{Sym}(\eta)) = 0$ for $i > 0$. Then we have an isomorphism of graded vector spaces*

$$\mathrm{Tor}^n_S(R, k) = \bigoplus_{i \ge n} \mathrm{H}^{i-n}(X, \textstyle\bigwedge^i \xi)[i],$$

where $[\cdot]$ indicates the grading.

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "p'"'] & X \times V \arrow[d, "p"] \\ * \arrow[r, "i"] & V \end{tikzcd}$$

where $* = \mathrm{Spec}(k)$ and the horizontal maps are zero sections. We have isomorphisms (in the derived category of graded k-vector spaces)

$$\mathrm{Tor}^\bullet_S(R, k) = \mathrm{L}i^*\mathrm{R}p_*\mathcal{O}_Z = \mathrm{R}p'_*\mathrm{L}(i')^*\mathcal{O}_Z.$$

The first isomorphism comes from the fact that $\mathrm{R}p_*\mathcal{O}_Z = p_*\mathcal{O}_Z = R$, while the second is the base change map, which is an isomorphism in this case (as can be seen by applying the projection formula of [Ha, Prop. 5.6], taking $F = \mathcal{O}_Z$ and G the structure sheaf of the point). Now, the Koszul complex gives a resolution of $\mathrm{Sym}(\eta)$ as a $\mathrm{Sym}(\epsilon)$-module:

$$\cdots \to \mathrm{Sym}(\epsilon) \otimes \textstyle\bigwedge^2 \xi \to \mathrm{Sym}(\epsilon) \otimes \bigwedge^1 \xi \to \mathrm{Sym}(\epsilon) \to \mathrm{Sym}(\eta) \to 0.$$

The terms of this resolution are graded $\mathrm{Sym}(\epsilon)$-modules, with $\bigwedge^i \xi$ being in degree i. The differentials have degree one. Let $q\colon X \times V \to X$ be the projection. We can recast the above resolution as a quasi-isomorphism of complexes of coherent $\mathcal{O}_{X\times V}$ modules:

$$[\textstyle\bigwedge^\bullet (q^*\xi)] \to \mathcal{O}_Z,$$

As the sheaves in the Koszul complex are locally free $\mathcal{O}_{X\times V}$-modules, we can calculate $\mathrm{L}(i')^*$ by simply applying $(i')^*$. After doing so all differentials vanish, since the differentials in the Koszul complex have degree one. We thus have a quasi-isomorphism

$$[\textstyle\bigwedge^\bullet (\xi)] \to \mathrm{L}(i')^*\mathcal{O}_Z$$

where the complex on the left has zero differentials. Applying $\mathrm{R}p'_*$ yields the formula in the statement of the proposition. □

2.2. **A flatness criterion.** The main result of this section gives a fiberwise criterion for flatness over a discrete valuation ring. Before stating it, let us recall a few definitions. Let A be a noetherian ring. We say that A is *catenary* if for any two prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of A, any two maximal chains of prime ideals beginning at $\mathfrak{p}$ and ending at $\mathfrak{q}$ have the same length. This is a mild condition satisfied by most rings one encounters; for instance, every finitely generated algebra over a complete local noetherian ring is catenary. The *dimension* (resp. *codimension*) of a prime ideal $\mathfrak{p}$ of A is the length of the longest chain of primes beginning (resp. ending) at $\mathfrak{p}$, or, equivalently, the dimension of $A/\mathfrak{p}$ (resp. $A_{\mathfrak{p}}$). We say that A is *equidimensional* of dimension d if $\dim(\mathfrak{p}) + \mathrm{codim}(\mathfrak{p}) = d$ for all prime ideals $\mathfrak{p}$. If A is a noetherian catenary local ring then A is equidimensional if and only if its minimal primes all have the same dimension. We can now state the proposition:

Proposition 2.2.1. *Let A be a catenary noetherian local ring and let π be an element of the maximal ideal of A. Assume the following:*

- *The ring $A/\pi A$ is reduced.*
- *The rings $A[1/\pi]$ and $A/\pi A$ are equidimensional of the same dimension and have the same number of minimal primes.*

Then π is not a zero-divisor in A.

Remark 2.2.2. Let $\mathcal{O}$ be a discrete valuation ring with uniformizer π and let A be a catenary noetherian local $\mathcal{O}$-algebra. The above proposition gives a criterion for A to be flat over $\mathcal{O}$ in terms of conditions on the fibers of the map $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O})$.

Remark 2.2.3. The proof of the proposition will yield the following additional piece of information: extension gives a bijection between the minimal primes of A and those of $A[1/\pi]$, and similarly for $A/\pi A$ in place of $A[1/\pi]$.

We now prove the proposition. If π is nilpotent then the proposition is trivial, so assume this is not the case. Let d be the common dimension of $A[1/\pi]$ and $A/\pi A$, and let n be the common number of minimal primes in $A[1/\pi]$ and $A/\pi A$. Let I be the ideal of π-power torsion in A. We must show that $I = 0$. Put $B = A/I$. Note that B is still catenary, noetherian and local, and π is not a zero-divisor in B.

Lemma 2.2.4. *No minimal prime of B contains π.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal primes of B, and suppose π belongs to $\mathfrak{p}_1$. We cannot have $n = 1$, as π is not nilpotent. Let x_i, for $2 \le i \le n$, be an element of $\mathfrak{p}_i$ which does not belong to $\mathfrak{p}_1$, and let x be the product of the x_i. Thus x belongs to $\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$, but is not nilpotent. As πx is nilpotent, we find that π^k kills x^k for some k, which contradicts the fact that π is not a zero-divisor of B. □

Lemma 2.2.5. *Any prime of B minimal over (π) has codimension 1.*

Proof. Krull's principal ideal theorem says the codimension is at most 1, while Lemma 2.2.4 says it cannot be 0. ☐

Lemma 2.2.6. *Let R be a noetherian local domain and let π be a non-zero element of the maximal ideal of R such that $R[1/\pi]$ is a field. Then R has exactly two prime ideals: its maximal ideal and the zero ideal.*

Proof. Since $R[1/\pi]$ is a field, given any non-zero $x \in R$, we can find a non-zero $y \in R$ such that $xy = \pi^n$ for some n. It follows that every non-zero principal ideal, and therefore every non-zero ideal, contains a power of π. Thus π belongs to every non-zero prime ideal. We thus see that the codimension one primes of R are in bijection with the minimal primes of $R/\pi R$, and are thus finite in number.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the codimension one primes of R. Every element of the maximal ideal $\mathfrak{m}$ of R belongs to one of these primes, by Krull's principal ideal theorem. Thus $\mathfrak{m} \subset \bigcup \mathfrak{p}_i$. However, this implies $\mathfrak{m} \subset \mathfrak{p}_i$ for some i, and so there is only one $\mathfrak{p}_i$ and it is maximal. This completes the proof. ☐

Remark 2.2.7. The above lemma is a special case of [EGA, 0_{IV}, Corollary 16.3.3], which is attributed to Artin–Tate. We thank Brian Conrad for this comment.

Lemma 2.2.8. *The ring B is equidimensional of dimension $d+1$ and contains n minimal primes.*

Proof. The primes of B not containing π correspond to the primes of $B[1/\pi] = A[1/\pi]$ via extension and contraction. Since no minimal prime of B contains π, it follows that the minimal primes of B and $B[1/\pi]$ are in bijection, and so B has n minimal primes. Let $\mathfrak{p}_0$ be a minimal prime of B, and let $\mathfrak{q}_0$ be its extension to $B[1/\pi]$. Let $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_d$ be a maximal chain of prime ideals in $B[1/\pi]$ and let $\mathfrak{p}_i$ be the contraction of $\mathfrak{q}_i$. Then $B/\mathfrak{p}_d$ is a local domain in which π is non-zero element of the maximal ideal, and $(B/\mathfrak{p}_d)[1/\pi] = B[1/\pi]/\mathfrak{q}_d$ is a field. It follows from Lemma 2.2.6 that $B/\mathfrak{p}_d$ is one dimensional, and so there is no prime between $\mathfrak{p}_d$ and the maximal ideal $\mathfrak{p}_{d+1}$ of B. Thus $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{d+1}$ is a maximal chain of length $d+1$. As B is catenary, every maximal chain between $\mathfrak{p}_0$ and $\mathfrak{p}_{d+1}$ has length $d+1$, and so B is equidimensional of dimension $d+1$. ☐

Lemma 2.2.9. *The ring $B/\pi B$ is equidimensional of dimension d.*

Proof. Let $\mathfrak{p}$ be a minimal prime of $B/\pi B$ and let $\widetilde{\mathfrak{p}}$ be its inverse image in B. Then $\widetilde{\mathfrak{p}}$ has codimension 1 by Lemma 2.2.5, and since B is equidimensional, dimension d. Of course, $\widetilde{\mathfrak{p}}$ and $\mathfrak{p}$ have the same dimension. This shows that all minimal primes of $B/\pi B$ have dimension d, and as $B/\pi B$ is catenary, noetherian and local, the proposition follows. ☐

Lemma 2.2.10. *Let $\mathfrak{p}$ be a minimal prime of B. Then there exists a prime $\mathfrak{q}$ of B minimal over (π) which contains $\mathfrak{p}$.*

Proof. The ring $B/\mathfrak{p}$ is a local domain of dimension $d+1$ in which π is non-zero element of the maximal ideal, and so $B/((\pi)+\mathfrak{p})$ has dimension d. Let $\mathfrak{q}$ be the inverse image in B of a minimal prime of $B/((\pi)+\mathfrak{p})$ of dimension d. Then $\mathfrak{q}$ has dimension d, and since B is equidimensional, codimension 1. It follows from Lemma 2.2.5 that $\mathfrak{q}$ is minimal over (π). ☐

Lemma 2.2.11. *The ring $B/\pi B$ is regular in codimension 0, that is, if $\mathfrak{q}$ is a minimal prime then $(B/\pi B)_{\mathfrak{q}}$ is a field.*

Proof. Let $\mathfrak{q}$ be a minimal prime of $B/\pi B$ and let $\mathfrak{p}$ be its inverse image in $A/\pi A$. Since $A/\pi A$ and $B/\pi B$ are equidimensional of the same dimension, $\mathfrak{p}$ is a minimal prime of $A/\pi A$. One readily verifies that the natural map $(A/\pi A)_{\mathfrak{p}} \to (B/\pi B)_{\mathfrak{q}}$ is a surjection of rings. As $A/\pi A$ is reduced, the former is a field, and so the latter is as well. ☐

Lemma 2.2.12. *The ring $B/\pi B$ contains at least n minimal primes.*

Proof. Let $\mathfrak{q}$ be a minimal prime over (π) in B. We claim that $\mathfrak{q}$ contains exactly one minimal prime of B. Since $(B/\pi B)_{\mathfrak{q}} = B_{\mathfrak{q}}/\pi B_{\mathfrak{q}}$ is a field (Lemma 2.2.11), we find that $\pi B_{\mathfrak{q}}$ is the maximal ideal of $B_{\mathfrak{q}}$. Since the maximal ideal of $B_{\mathfrak{q}}$ is principal, any other prime ideal of $B_{\mathfrak{q}}$ is the zero ideal, which establishes the claim. Since every minimal prime of B is contained in at least one prime minimal over (π) (Lemma 2.2.10), it follows that there are at least n primes of B minimal over (π). □

Lemma 2.2.13. *The map $A/\pi A \to B/\pi B$ is an isomorphism.*

Proof. It is a map of equidimensional noetherian rings of the same dimension such that the source is reduced and the target has at least the number of minimal primes as the source. □

Lemma 2.2.14. *We have $I = 0$.*

Proof. Since B has no π-torsion, the kernel of the map $A/\pi A \to B/\pi B$ is $I/\pi I$. Thus $I/\pi I = 0$, and so $I = 0$ by Nakayama's lemma. □

2.3. **Two more results from commutative algebra.** Let A be a ring and let π be an element of A. In this section we give a pair of results which allow us to transfer properties of $A/\pi A$ and $A[1/\pi]$ to A.

Proposition 2.3.1. *Suppose that π is not a zero-divisor, A is (π)-adically separated and $A/\pi A$ is reduced. Then A is reduced.*

Proof. Suppose x_1 is an element of A such that $x_1^n = 0$. Then $x_1 = 0$ in $A/\pi A$, and so we can write $x_1 = \pi x_2$. We thus find $\pi^n x_2^n = 0$, and so, since π is not a zero-divisor, $x_2^n = 0$. Reasoning as before, we can write $x_2 = \pi x_3$. Continuing in this manner, we find that x_1 belongs to (π^n) for all n, and is thus equal to 0. This completes the proof. □

Proposition 2.3.2. *Suppose that A is a domain, $A[1/\pi]$ is normal and $A/\pi A$ is reduced. Then A is normal.*

Proof. Let x be an element of the fraction field of A which is integral over A; we must show that x belongs to A. Since $A[1/\pi]$ is normal, x belongs to $A[1/\pi]$, and so we can write $x = y/\pi^n$ with $y \in A$ and $n \geq 0$ minimal. Assume $n > 0$. Let $\sum_{i=0}^d a_i x^i$ be a monic equation over A satisfied by x. We then have $\sum_{i=0}^d a_i y^i (\pi^n)^{d-i} = 0$. We find $y^d = 0$ in $A/\pi A$, which shows that y belongs to (π), contradicting the minimality of n. Thus $n = 0$ and x belongs to A. □

2.4. **A result from Galois theory.** Let $F/\mathbf{Q}_p$ be a finite extension of degree d and let F'/F be the maximal p-power Galois extension in which the inertia group is abelian and killed by p. Let $G = \mathrm{Gal}(F'/F)$ and let U be the inertia subgroup of G. We thus have a short exact sequence

$$1 \to U \to G \to \mathbf{Z}_p \to 0,$$

where $\mathbf{Z}_p$ has for a topological generator the arithmetic Frobenius element ϕ. We give U the structure of an $\mathbf{F}_p[\![T]\!]$-module by letting T act by $\phi - 1$. The following result determines the structure of U:

Proposition 2.4.1. *If F contains the pth roots of unity then $U \cong \mathbf{F}_p \oplus \mathbf{F}_p[\![T]\!]^{\oplus d}$; otherwise $U \cong \mathbf{F}_p[\![T]\!]^{\oplus d}$*

Proof. Fix an element $\widetilde{\phi}$ in G lifting ϕ. Let F_n be the unramified extension of F of degree p^n, and let F'_n be the maximal abelian extension of F_n of exponent p on which $\widetilde{\phi}^{p^n}$ acts trivially. Then F'_n is the fixed field of the normal closure of $\langle \widetilde{\phi} \rangle \subset G$ acting on F', and so $\mathrm{Gal}(F'_n/F_n)$ is identified with $U/T^{p^n}U$. By class field theory, the abelianized Galois group of F_n is identified with $(F_n^\times)^\wedge$, with the element $\widetilde{\phi}^{p^n}$ corresponding to some element x_n of valuation 1. It follows that we have isomorphism

$$\mathrm{Gal}(F'_n/F_n) = (F_n^\times)^\wedge/\langle x_n\rangle \otimes \mathbf{F}_p = U_{F_n} \otimes \mathbf{F}_p,$$

where U_{F_n} is the unit group of F_n. We thus find that $U/T^{p^n}U$ has dimension $\epsilon + p^n d$ over $\mathbf{F}_p$, where ϵ is 1 or 0 according to whether F contains the pth roots of unity or not. Now, since U is the inverse limit of the $U/T^{p^n}U$ and U/TU is finite dimensional, it follows that U is finitely generated over $\mathbf{F}_p[\![T]\!]$. Appealing to the structure theory of finitely generated $\mathbf{F}_p[\![T]\!]$-modules and our formula for the dimension of $U/T^{p^n}U$ gives the stated result. □

Remark 2.4.2. Suppose that F contains the pth roots of unity. By the above result, U contains a unique $\mathbf{F}_p$-line on which ϕ acts trivially. Let us now describe this line more explicitly. As F_n contains the pth roots of unity, there is some element y_n of $F_n^\times \otimes \mathbf{F}_p$ such that $F_{n+1} = F_n(y_n^{1/p})$. This element is unique up to scaling by elements of $\mathbf{F}_p^\times$. Alternatively, fixing a pth root of unity and letting $(,)$ be the $\mathbf{F}_p$-valued Hilbert symbol on $F_n^\times \otimes \mathbf{F}_p$, we can characterize y_n uniquely by $(y_n, x) = \mathrm{val}(x)$. The element y_n is a unit, invariant under ϕ and satisfies $N(y_{n+1}) = y_n$, where $N\colon F_{n+1}^\times \to F_n^\times$ is the norm map. The sequence (y_n) defines an element of the inverse limit of the $U_{F_n} \otimes \mathbf{F}_p$ (where the transition maps are the norm maps). By the above proof, this inverse limit is U. Since (y_n) is non-zero and ϕ-invariant, it spans the unique ϕ-invariant $\mathbf{F}_p$-line in U.

3. Moduli spaces of matrices

In this section we study three moduli problems related to 2×2 nilpotent matrices. Certain local rings of these spaces will later be identified as the special fiber of certain Galois deformation rings. Throughout, k is a fixed field of characteristic not 2.

3.1. **Borel and nilpotent matrix algebras.** Let $\mathfrak{g} = M_2$ be the space of 2×2 matrices over k and let $\mathfrak{g}^\circ$ be the subspace of traceless matrices. Since 2 is invertible, the space $\mathfrak{g}$ is the direct sum of $\mathfrak{g}^\circ$ and the space of scalar matrices. We write $\underline{\mathfrak{g}}$ and $\underline{\mathfrak{g}}^\circ$ for the sheaves $\mathfrak{g} \otimes \mathcal{O}$ and $\mathfrak{g}^\circ \otimes \mathcal{O}$ on $\mathbf{P}^1$.

Let T be a k-scheme. A *nilpotent subalgebra* of $\mathfrak{g}_T$ is a line subbundle $\mathfrak{u}$ of $\mathfrak{g}_T$ such that $\ker(\mathfrak{u}) = \mathrm{im}(\mathfrak{u})$ is a line subbundle of $\mathcal{O}_T^2$. One easily sees that sending $\mathfrak{u}$ to $\ker(\mathfrak{u})$ defines a bijection between nilpotent subalgebras of $\mathfrak{g}_T$ and line subbundles of $\mathcal{O}_T^2$. One recovers $\mathfrak{u}$ from a line bundle $\mathcal{L}$ via the formula $\mathfrak{u} = \underline{\mathrm{Hom}}(\mathcal{O}_T^2/\mathcal{L}, \mathcal{L})$. Since the bundle $\mathcal{O}(-1)$ on $\mathbf{P}^1$ is the universal line subbundle of $\mathcal{O}^2$, it follows that the nilpotent subalgebra $\underline{\mathfrak{u}} = \underline{\mathrm{Hom}}(\mathcal{O}^2/\mathcal{O}(-1), \mathcal{O}(-1)) \cong \mathcal{O}(-2)$ of $\underline{\mathfrak{g}}$ on $\mathbf{P}^1$ is the universal nilpotent subalgebra of $\mathfrak{g}$.

A *Borel subalgebra* of $\mathfrak{g}_T$ is a rank three subbundle $\mathfrak{b}$ of $\mathfrak{g}_T$ for which there exists a line subbundle $\mathcal{L}$ of $\mathcal{O}_T^2$ such that $\mathfrak{b}\mathcal{L} \subset \mathcal{L}$. Again, one finds that the correspondence between $\mathfrak{b}$ and $\mathcal{L}$ is bijective, and so there is a universal Borel subalgebra $\underline{\mathfrak{b}}$ of $\underline{\mathfrak{g}}$ on $\mathbf{P}^1$. One can make similar definitions in the traceless case, and obtain a universal Borel subalgebra $\underline{\mathfrak{b}}^\circ$ of $\underline{\mathfrak{g}}^\circ$ on $\mathbf{P}^1$. We have $\underline{\mathfrak{b}} = \underline{\mathfrak{b}}^\circ \oplus \mathcal{O}$ since 2 is invertible. As every section of $\underline{\mathfrak{b}}^\circ$ induces an endomorphism of $\mathcal{O}^2/\mathcal{O}(-1)$, we get a canonical map $\underline{\mathfrak{b}}^\circ \to \mathcal{O}$, the kernel of which is $\underline{\mathfrak{u}}$. That is, we have an exact sequence

$$0 \to \underline{\mathfrak{u}} \to \underline{\mathfrak{b}}^\circ \to \mathcal{O} \to 0$$

of sheaves on $\mathbf{P}^1$. A global section of $\underline{\mathfrak{b}}^\circ$ is determined by the endomorphism of $\mathcal{O}^2$ it induces, and this endomorphism has image in $\mathcal{O}(-1)$. Since there are no non-zero maps $\mathcal{O}^2 \to \mathcal{O}(-1)$, we conclude that $\Gamma(\mathbf{P}^1, \underline{\mathfrak{b}}^\circ) = 0$. This, combined with the above exact sequence, implies that $\underline{\mathfrak{b}}^\circ \cong \mathcal{O}(-1)^{\oplus 2}$.

3.2. **Strongly nilpotent matrices.** Let T be a k-algebra. We say that a matrix m in $M_2(T)$ is *strongly nilpotent* if its trace and determinant are both 0. This implies $m^2 = 0$, but the converse does not hold in general (it does if T is a domain). If $\mathfrak{u}$ is a nilpotent subalgebra of $(M_2)_T$ then every element of $\mathfrak{u}$ is strongly nilpotent.

3.3. **The space $\mathcal{A}$.** Let $r \geq 1$ be an integer. Let $\mathcal{A} = \mathcal{A}_r$ be the functor which assigns to a k-algebra T the set $\mathcal{A}(T)$ of tuples $(m_1, \ldots, m_r)$ where each $m_i \in M_2(T)$ is a strongly nilpotent matrix such that $m_i m_j = 0$ for all i and j. We let $a \in \mathcal{A}(k)$ denote the tuple $(0, \ldots, 0)$. The main result of this section is the following theorem:

Theorem 3.3.1. *The functor $\mathcal{A}$ is (represented by) a geometrically integral normal Cohen–Macaulay affine scheme of dimension $r+1$. For $r > 1$ the local ring at a is not Gorenstein.*

It is clear that $\mathcal{A}$ is an affine scheme. In fact, write

$$m_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}.$$

Then $\mathcal{A} = \mathrm{Spec}(R)$, where R is the quotient of $k[a_i, b_i, c_i]_{1 \leq i \leq r}$ by the equations

$$a_i a_j = b_i c_j, \qquad a_i b_j = a_j b_i, \qquad a_i c_j = a_j c_i. \tag{2}$$

For $i \neq j$, these equations express the identity $m_i m_j = 0$. For $i = j$, the latter two equations are trivial, while the first expresses that m_i has determinant 0.

Let $\widetilde{\mathcal{A}}$ be the functor which attaches to a k-algebra T the set of tuples $(\mathfrak{u}; m_1, \ldots, m_r)$ where $\mathfrak{u}$ is a nilpotent subalgebra of $(M_2)_T$ and each m_i belongs to $\mathfrak{u}$. It is clear that $\widetilde{\mathcal{A}}$ is represented by the total space of the vector bundle $\underline{\mathfrak{u}}^{\oplus r}$ over $\mathbf{P}^1$, and is thus smooth and geometrically integral of dimension $r+1$. Let $\widetilde{R}$ be the ring of global functions on $\widetilde{\mathcal{A}}$. Then $\widetilde{R}$ is normal and geometrically integral.

Let $(\mathfrak{u}; m_1, \ldots, m_r)$ be an element of $\widetilde{\mathcal{A}}(T)$. Then each m_i is strongly nilpotent, and $m_i m_j = 0$ for all i and j. Thus $(m_1, \ldots, m_r)$ defines an element of $\mathcal{A}(T)$. We therefore have a map of schemes $\widetilde{\mathcal{A}} \to \mathcal{A}$, and thus a corresponding map of rings $R \to \widetilde{R}$.

Lemma 3.3.2. *The map $R \to \widetilde{R}$ is an isomorphism.*

Proof. Let $\eta^\vee$ be the vector bundle $\underline{\mathfrak{u}}^{\oplus r}$ on $\mathbf{P}^1$ and let $\epsilon^\vee$ be the bundle $(\underline{\mathfrak{g}}^\circ)^{\oplus r}$. Then $\eta^\vee$ is naturally a subbundle of $\epsilon^\vee$; let $\xi^\vee$ be the quotient bundle. We have isomorphisms $\eta = \mathcal{O}(2)^{\oplus r}$ and $\xi = \mathcal{O}(1)^{\oplus 2r}$. Note that $\widetilde{R} = \Gamma(\mathbf{P}^1, \mathrm{Sym}(\eta))$. Let S be the polynomial ring $k[a_i, b_i, c_i]$, which is identified with $\Gamma(\mathbf{P}^1, \mathrm{Sym}(\epsilon))$.

We now apply Proposition 2.1.5. We find

$$\widetilde{R}/S_+\widetilde{R} = \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}) \oplus \mathrm{H}^1(\mathbf{P}^1, \xi)[1] = k,$$

and so $S \to \widetilde{R}$ is surjective (by Nakayama's lemma). Let I be the kernel. Then $\mathrm{Tor}^1_S(R, k)$ is identified with I/S_+I, and so Proposition 2.1.5 gives

$$I/S_+I = \mathrm{H}^0(\mathbf{P}^1, \xi)[1] \oplus \mathrm{H}^1(\mathbf{P}^1, {\textstyle\bigwedge^2} \xi)[2].$$

The H^0 vanishes and the H^1 has dimension $\binom{r}{2}$. Thus I is generated in degree 2 and its degree 2 piece has dimension $\binom{r}{2}$. An elementary argument shows that the $\binom{r}{2}$ quadratic elements of S given in equation (2) are linearly independent, and so the kernel of $S \to R$ is equal to I. Thus the map $R \to \widetilde{R}$ is an isomorphism. $\square$

The theorem now follows immediately from Proposition 2.1.1. To be precise, the above lemma shows that R is normal and geometrically integral. Let η as above. The space Z in Proposition 2.1.1 is just $\widetilde{\mathcal{A}}$, and the point 0 of $\mathrm{Spec}(R)$ is just a. The proposition shows that $\widetilde{\mathcal{A}} \to \mathcal{A}$ is an isomorphism away from a, that $\widetilde{R} = R$ is Cohen–Macaulay and that the local ring of R at a is not Gorenstein when $r > 1$.

Remark 3.3.3. The ring R is isomorphic to the projective coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^r$ with respect to the bundle $\mathcal{O}(2, 1)$. The singularities of such Segre–Veronese rings have been well-studied, and more general results than Theorem 3.3.1 appear in the literature; see, for instance, [BM, §0.4].

3.4. **The space $\mathcal{B}$.** Let $\mathcal{B} = \mathcal{B}_r$ be the functor which assigns to a k-algebra T the set $\mathcal{B}(T)$ of tuples $(\phi, \alpha; m_1, \ldots, m_r)$ where ϕ is an element of $M_2(T)$ of determinant 1, α is an element of T which is a root of the characteristic polynomial of ϕ and the m_i are strongly nilpotent matrices in $M_2(T)$ such that $m_i m_j = 0$ and $m_i \phi = \alpha m_i$. Let $b \in \mathcal{B}(k)$ be the point $(1, 1; 0, \ldots, 0)$. The main result of this section is the following:

Theorem 3.4.1. *The functor $\mathcal{B}$ is (represented by) a geometrically integral normal Cohen–Macaulay affine scheme of dimension $r+3$. The local ring at the point b is not Gorenstein.*

It is clear that $\mathcal{B}$ is an affine scheme. In fact, write

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix}$$

and keep the notation for the m_i from the previous section. Then $\mathcal{B} = \operatorname{Spec}(R)$, where R is the quotient of $k[a_i, b_i, c_i, \phi_j, \alpha]$ (with $1 \le i \le r$ and $1 \le j \le 4$) by the equations (2), the equations

$$(3) \qquad a_i\phi_1 + b_i\phi_3 = \alpha a_i, \qquad a_i\phi_2 + b_i\phi_4 = \alpha b_i, \qquad c_i\phi_1 - a_i\phi_3 = \alpha c_i, \qquad c_i\phi_2 - a_i\phi_4 = -\alpha a_i,$$

the equation

$$(4) \qquad \alpha^2 - (\phi_1 + \phi_4)\alpha + (\phi_1\phi_4 - \phi_2\phi_3) = 0$$

and the equation

$$(5) \qquad \phi_1\phi_4 - \phi_2\phi_3 = 1.$$

Of course, the equaion (3) express the identity $m_i\phi = \alpha m_i$, while (4) expresses that α is a root of the characteristic polynomial of ϕ, and (5) expresses that ϕ has determinant 1.

Let $\mathcal{B}^\circ$ be defined like $\mathcal{B}$ except without any condition on the determinant of ϕ. Then $\mathcal{B}^\circ = \operatorname{Spec}(R^\circ)$, where R° is the quotient of $k[a_i, b_i, c_i, \phi_j, \alpha]$ by the equations (2), (3) and (4). Note that these equations are all homogeneous of degree two, and so R° is graded.

Let $\widetilde{\mathcal{B}}^\circ$ be the functor assigning to a k-algebra T the set of tuples $(\mathfrak{b}; \phi; m_1, \ldots, m_r)$ where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_T$, ϕ is an element of $\mathfrak{b}$ and the m_i are elements of $\mathfrak{u}$, the nilpotent radical of $\mathfrak{b}$. It is clear that $\widetilde{\mathcal{B}}^\circ$ is represented by the total space of the vector bundle $\underline{\mathfrak{b}} \oplus \underline{\mathfrak{u}}^{\oplus r}$ over $\mathbf{P}^1$. We write $\widetilde{R}^\circ$ for the ring of global functions on $\widetilde{B}^\circ$.

We let $\widetilde{\mathcal{B}}$ be defined like $\widetilde{\mathcal{B}}^\circ$ but with the condition $\det(\phi) = 1$ imposed. Note that $\widetilde{\mathcal{B}}$ is the fiber product of the universal Borel subgroup of $\mathrm{SL}(2)$ with a vector bundle over $\mathbf{P}^1$, and is therefore smooth and geometrically integral of dimension $r+3$. We let $\widetilde{R}$ be the ring of global functions on $\widetilde{\mathcal{B}}$.

Let $(\mathfrak{b}; \phi; m_1, \ldots, m_r)$ be an element of $\widetilde{\mathcal{B}}^\circ(T)$. Let α be the eigenvalue by which ϕ acts on the $[\mathfrak{b}, \mathfrak{b}]$ coinvariants of T^2 (e.g., the lower right entry of ϕ if $\mathfrak{b}$ is upper triangular). Then α satisfies the characteristic polynomial of ϕ, and $m_i\phi = \alpha m_i$. It follows that $(\phi, \alpha; m_1, \ldots, m_r)$ is an element of $\mathcal{B}^\circ$. We thus have a natural map $\widetilde{\mathcal{B}}^\circ \to \mathcal{B}^\circ$, and thus an induced map $R^\circ \to \widetilde{R}^\circ$. Of course, we also have $\widetilde{\mathcal{B}} \to \mathcal{B}$ and $R \to \widetilde{R}$.

We summarize the above definitions with two commutative diagrams

$$\begin{tikzcd} \widetilde{\mathcal{B}} \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{B}}^\circ \arrow[d] \\ \mathcal{B} \arrow[r, hook] & \mathcal{B}^\circ \end{tikzcd} \qquad\qquad \begin{tikzcd} \widetilde{R} & \widetilde{R}^\circ \arrow[l] \\ R \arrow[u] & R^\circ \arrow[l, two heads] \arrow[u] \end{tikzcd}$$

We now show that the vertical ring maps in the right diagram are isomorphisms.

Lemma 3.4.2. *The map $R^\circ \to \widetilde{R}^\circ$ is an isomorphism.*

Proof. Let $\eta^\vee$ be the vector bundle $\underline{\mathfrak{b}} \oplus \underline{\mathfrak{u}}^{\oplus r}$ on $\mathbf{P}^1$, which is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(-2)^{\oplus r}$. The bundle $\eta^\vee$ is naturally a subbundle of the constant bundle $\epsilon^\vee = \underline{\mathfrak{g}} \oplus (\underline{\mathfrak{g}}^\circ)^{\oplus r}$. The quotient bundle $\xi^\vee$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 2r}$. The ring $\widetilde{R}^\circ$ is $\Gamma(\mathbf{P}^1, \mathrm{Sym}(\eta))$. Let S be the ring $\Gamma(\mathbf{P}^1, \mathrm{Sym}(\epsilon))$, i.e., the polynomial ring $k[a_i, b_i, c_i, \phi_i]$. (Note: S does not contain α.) Then $\widetilde{R}^\circ$ is naturally an S-algebra, as is R°, and the map $R^\circ \to \widetilde{R}^\circ$ is S-linear.

It is clear that R°/S_+R° is isomorphic to $k \oplus k[1]$, and spanned by the images of 1 and α. Proposition 2.1.5 shows that

$$\widetilde{R}^\circ/S_+\widetilde{R}^\circ = \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}) \oplus \mathrm{H}^1(\mathbf{P}^1, \xi)[1] = k \oplus k[1].$$

Since α maps to a non-zero element of $\widetilde{R}^\circ$, we see that the map $R^\circ \to \widetilde{R}^\circ$ induces an isomorphism after quotienting by S_+, and is therefore surjective.

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & S \oplus S[1] \arrow[r] \arrow[d, equal] & R^\circ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r] & S \oplus S[1] \arrow[r] & \widetilde{R}^\circ \arrow[r] & 0 \end{tikzcd}$$

The map $S \oplus S[1] \to R^\circ$ sends the basis vectors to 1 and α. The map to $\widetilde{R}^\circ$ is defined similarly. These maps are surjective by the previous paragraph. We let M and N denote their kernels, so that $M \subset N$. Now, the equations for $\widetilde{R}^\circ$ are all of degree 2, which means that the natural map $M_2 \to M/S_+M$ is an isomorphism. On the other hand, N/S_+N is identified with $\mathrm{Tor}_1^S(\widetilde{R}^\circ, k)$, and so another application of Proposition 2.1.5 yields

$$N/S_+N = \mathrm{H}^0(\mathbf{P}^1, \xi)[1] \oplus \mathrm{H}^1(\mathbf{P}^1, {\textstyle\bigwedge^2} \xi)[2],$$

and so N/S_+N is concentrated in degree 2 and of dimension $4r + \binom{2r}{2}$. We thus have a diagram

$$\begin{tikzcd} M_2 \arrow[r] \arrow[d] & M/S_+M \arrow[d] \\ N_2 \arrow[r] & N/S_+N \end{tikzcd}$$

in which the horizontal maps are isomorphisms. Obviously, the left vertical map is injective. An elementary argument shows the $\binom{2r}{2}$ equations in (2) and the $4r$ equations in (3) are linearly independent, and so M_2 has dimension $4r + \binom{2r}{2}$. (Note: the equation (4) does not appear in M.) We therefore find that $M_2 \to N_2$ is surjective, and so $M/S_+M \to N/S_+N$ is surjective, and so $M \to N$ is surjective, i.e., $M = N$. This completes the proof. $\square$

Lemma 3.4.3. *The map $R \to \widetilde{R}$ is an isomorphism.*

Proof. Due to the previous lemma, it is enough to show that $\widetilde{R}^\circ \to \widetilde{R}$ is surjective with kernel generated by $\det(\phi) - 1$. Now, the space $\widetilde{\mathcal{B}}^\circ$ is affine over $\mathbf{P}^1$ and corresponds to the sheaf of algebras $\mathrm{Sym}(\eta)$. The space $\widetilde{\mathcal{B}}$ is also affine over $\mathbf{P}^1$; let $\mathcal{R}$ denote the corresponding sheaf of algebras. We have a short exact sequence of sheaves on $\mathbf{P}^1$

$$0 \to \mathrm{Sym}(\eta) \to \mathrm{Sym}(\eta) \to \mathcal{R} \to 0$$

where the first map is given by multiplication by $\det(\phi) - 1$, an element of $\Gamma(\mathbf{P}^1, \mathrm{Sym}(\eta))$. Since $\mathrm{H}^1(\mathbf{P}^1, \mathrm{Sym}(\eta)) = 0$, upon taking global sections we see that $\widetilde{R}^\circ \to \widetilde{R}$ is surjective and its kernel is generated by $\det(\phi) - 1$. $\square$

Let $\mathcal{B}_0^\circ(T)$ be the subset of $\mathcal{B}^\circ(T)$ consisting of tuples $(\phi, \alpha; m_1, \ldots, m_r)$ where $m_i = 0$ and $\phi = \alpha$ is a scalar matrix. Define $\widetilde{\mathcal{B}}^\circ$ similarly.

Lemma 3.4.4. *The map $\widetilde{\mathcal{B}}^\circ \setminus \widetilde{\mathcal{B}}_0^\circ \to \mathcal{B}^\circ \setminus \mathcal{B}_0^\circ$ is an isomorphism. The space $\mathcal{B}^\circ$ is Cohen–Macaulay; moreover, it is smooth away from $\mathcal{B}_0^\circ$ and not Gorenstein at any point in $\mathcal{B}_0^\circ$.*

Proof. Let η be as in Lemma 3.4.2. The $\mathcal{O}$ summand of η comes from the scalar matrices in $\mathfrak{b}$. Let η' be the complement of this, so that $\eta = \eta' \oplus \mathcal{O}$. Put $A = \Gamma(\mathbf{P}^1, \mathrm{Sym}(\eta'))$. Then $\widetilde{R}^\circ = R^\circ$ is the polynomial ring in one variable over A, and $\mathcal{B}^\circ = \mathrm{Spec}(A) \times \mathbf{A}^1$, with $\{0\} \times \mathbf{A}^1$ identified with $\mathcal{B}_0^\circ$. Proposition 2.1.1 applies to η'. We find that the map from the total space of η' to $\mathrm{Spec}(A)$ is an isomorphism away from 0, that A is Cohen–Macaulay and that the local ring of A at the point 0 is not Gorenstein. The lemma follows. $\square$

Lemma 3.4.5. *The map $\widetilde{\mathcal{B}} \to \mathcal{B}$ is birational.*

Proof. This follows immediately from the previous lemma and the fact that $\mathcal{B}_0^\circ \cap \mathcal{B}$ is a proper closed set in $\mathcal{B}$ (in fact, it consists of two closed points). $\square$

We now prove the theorem.

Proof of Theorem 3.4.1. Since $\widetilde{\mathcal{B}}$ is normal and geometrically integral, so is $R = \widetilde{R}$. As $\widetilde{\mathcal{B}} \to \mathcal{B}$ is birational, R has dimension $r+3$. Since R is the quotient of the Cohen–Macaulay ring R° by the non-zerodivisor $\det(\phi) - 1$, it is Cohen–Macaulay [BH, Thm. 2.1.3a]. Furthermore, since R° is not Gorenstein at the point b, and $\det(\phi) - 1$ belongs to the maximal ideal at b, we see that R is not Gorenstein at b [BH, Prop. 3.1.19b]. $\square$

3.5. **The space $\mathcal{C}$.** Let $\mathcal{C} = \mathcal{C}_r$ be the functor which assigns to a k-algebra T the set $\mathcal{C}(T)$ of tuples $(\phi, \alpha; m_1, \ldots, m_{r+1})$ where ϕ is an element of $M_2(T)$ of determinant 1, α is an element of T satisfying the characteristic polynomial of ϕ and the m_i are strongly nilpotent matrices in $M_2(T)$ such that the following equations hold:

$$m_i m_j = 0, \qquad m_i \phi = \alpha m_i, \qquad m_{r+1}\phi = \phi m_{r+1}.$$

The first two equations are for $1 \le i, j \le r+1$. The second equation is equivalent to $\phi m_i = \alpha^{-1} m_i$, as can be seen by taking the adjugate of each side. In particular, the final equation is equivalent to $(\alpha^2 - 1)m_{r+1} = 0$. We let $c \in \mathcal{C}(k)$ denote the point $(1, 1; 0, \ldots, 0)$. The main result of this section is the following theorem:

Theorem 3.5.1. *The functor $\mathcal{C}$ is (represented by) a geometrically reduced affine scheme which is equidimensional of dimension $r+3$. It has three irreducible components: two isomorphic to $\mathcal{A}_{r+2}$ (defined by the equations $\alpha = \pm 1$) and one isomorphic to $\mathcal{B}_r$ (defined by the equation $m_{r+1} = 0$).*

We require the following simple lemma.

Lemma 3.5.2. *Let R be a ring, let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{a}$ be a principal ideal of R not contained in $\mathfrak{p}$. Then $\mathfrak{a} \cap \mathfrak{p} = \mathfrak{a}\mathfrak{p}$.*

Proof. Clearly, $\mathfrak{a}\mathfrak{p}$ is contained in $\mathfrak{a} \cap \mathfrak{p}$. We now establish the reverse inclusion. Let $\mathfrak{a} = (a)$ and let x be an element of $\mathfrak{a} \cap \mathfrak{p}$. We can then write $x = ay$ for some $y \in R$. Since ay belongs to $\mathfrak{p}$ but a does not belong to $\mathfrak{p}$, we conclude that y belongs to $\mathfrak{p}$. Thus x belongs to $\mathfrak{a}\mathfrak{p}$. $\square$

We now prove the theorem.

Proof of Theorem 3.5.1. It is clear that $\mathcal{C}$ is represented by an affine scheme $\mathrm{Spec}(R)$; we do not write the equations, but keep our previous notation for elements of the ring R. The locus $\alpha = \pm 1$ in $\mathcal{C}$ is isomorphic to $\mathcal{A}_{r+2}$, the isomorphism taking a T-point $(\phi, \alpha; m_1, \ldots, m_{r+1})$ to $(\phi - \alpha, m_1, \ldots, m_{r+1})$. The locus $m_{r+1} = 0$ in $\mathcal{C}$ is isomorphic to $\mathcal{B}_r$ (obviously). It follows that $\mathfrak{p}_1 = (\alpha - 1)$, $\mathfrak{p}_2 = (\alpha + 1)$ and $\mathfrak{p}_3 = (a_{i+1}, b_{i+1}, c_{i+1})$ are prime ideals of R. We claim that their intersection is the zero ideal; this will prove the theorem. Since the rings $R/\mathfrak{p}_i$ all have the same dimension, there is no containment between the primes $\mathfrak{p}_i$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both prime principal ideals, the lemma gives $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2$, which is again a principal ideal. A second application of the lemma gives $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$. However, since $(\alpha^2 - 1)m_{r+1} = 0$, the product of the $\mathfrak{p}_i$ is zero. This completes the proof. □

Remark 3.5.3. The point c belongs to the two irreducible components of $\mathcal{C}$ corresponding to $\mathfrak{p}_1$ and $\mathfrak{p}_3$, and corresponds to the points a and b when these components are identified with $\mathcal{A}$ and $\mathcal{B}$.

4. Local deformation rings

4.1. **Set-up.** Let F be a finite extension of $\mathbf{Q}_p$ of degree d with absolute Galois group G_F. Let E be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer π and residue field k. Let V_0 be a two dimensional k-vector space equipped with a continuous action of G_F having cyclotomic determinant. Let $\mathscr{C}_{\mathcal{O}}$ denote the category of artinnian local $\mathcal{O}$-algebras with residue field k. Define a functor[1] D on $\mathscr{C}_{\mathcal{O}}$ by assigning to an algebra A the groupoid of all deformations of V_0 to A with cyclotomic determinant. That is, $D(A)$ is the category whose objects are free rank two A-modules V equipped with a continuous action of G_F (for the discrete topology) having cyclotomic determinant together with an isomorphism $V \otimes_A k \to V_0$; the morphisms in $D(A)$ are isomorphisms.

The category $D(A)$ is typically not discrete, and so D will typically not be representable. To circumvent this annoyance, we use framed deformations. For A as above, define $D^{\square}(A)$ to be the category of pairs $(V, \{e_1, e_2\})$, where V is an object of $D(A)$ and $\{e_1, e_2\}$ is a basis of V as an A-module. Morphisms in $D^{\square}(A)$ are required to respect the basis, and so there is at most one morphism between two objects. The functor $D^{\square}(A)$ is pro-representable by a complete local noetherian $\mathcal{O}$-algebra R^{univ} together with a free rank two R^{univ}-module V^{univ}. The action of G_F on V^{univ} is continuous when V^{univ} is given the $\mathfrak{m}^{\mathrm{univ}}$-adic topology ($\mathfrak{m}^{\mathrm{univ}}$ being the maximal ideal of R^{univ}).

Let $X \subset \mathrm{MaxSpec}(R^{\mathrm{univ}}[1/\pi])$ be the set of points where the corresponding representation $G_F \to \mathrm{GL}_2(E')$ is ordinary, in the sense that (1) holds. As mentioned in the introduction, this set is known to be Zariski closed, and therefore of the form $\mathrm{MaxSpec}(R[1/\pi])$ for a unique $\mathcal{O}$-flat reduced quotient R of R^{univ}. The purpose of §4 is to give an explicit description of R and prove certain results about its singularities. Generally speaking, the closer V_0 is to being irreducible, the nicer R should be. To simplify our disucssion, we only treat what should be the worst case, where V_0 is trivial.

We therefore assume for the rest of §4 that V_0 is trivial.

4.2. **Two sets of Galois representations.** Let x be a point in $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/\pi])$ with residue field E_x. Then E_x is a finite extension of E, and the action of G_F on $V_x = V^{\mathrm{univ}} \otimes_{R^{\mathrm{univ}}} E_x$ is continuous when V_x is given its p-adic topology.

As defined above, we let $X \subset \mathrm{MaxSpec}(R^{\mathrm{univ}}[1/\pi])$ denote the set of points x for which V_x is ordinary. We now define two subsets of X. We let X_1 be the subset consisting of points x where V_x is an extension of E_x by $E_x(\chi)$ on the full Galois group. And we let X_2 denote the subset of X where the representation is crystalline. We have the following basic result.

Lemma 4.2.1. *The set X is the union of the subsets X_1 and X_2.*

Proof. Let $x \in X$. Then $V_x|_{I_F}$ is an extension of E_x by $E_x(\chi)$, and so V_x is an extension of $E_x(\psi^{-1})$ by $E_x(\psi\chi)$ for some unramified character ψ. Thus V_x defines an element of $\mathrm{H}^1(G_F, E_x(\chi\psi^2))$. Now, we have isomorphisms

$$\mathrm{H}^1(G_F, E_x(\chi\psi^2)) \cong \mathrm{H}^1(I_F, E_x(\chi\psi^2))^{\mathrm{Gal}(F^{\mathrm{un}}/F)} = (((F^{\mathrm{un}})^\times)^\wedge \otimes E_x(\psi^2))^{\mathrm{Gal}(F^{\mathrm{un}}/F)}.$$

[1]The appropriate 2-categorical notions are to be understood throughout.

The first isomorphism is restriction, the second comes from Kummer theory. Here the $\wedge$ denotes p-adic completion. The valuation map on $(F^{\mathrm{un}})^\times$ defines a map

$$\mathrm{H}^1(G_F, E_x(\chi\psi^2)) \to E_x(\psi^2)^{\mathrm{Gal}(F^{\mathrm{un}}/F)}.$$

We thus see that either ψ^2 is trivial or the above map is zero. In the first case, ψ is also trivial, since ψ reduces to the trivial character as V_0 is trivial, and so x belongs to X_1. In the second case, x belongs to X_2. $\square$

4.3. **The ring R.** The following result is due to Kisin.

Proposition 4.3.1 (Kisin)**.** *Let $* \in \{1,2\}$. Then there exists a unique reduced $\mathcal{O}$-flat quotient R_* of R^{univ} such that $\mathrm{MaxSpec}(R_*[1/\pi])$ is equal to X_*. The ring R_* is a domain and equidimensional of dimension $d+4$. The ring $R_*[1/\pi]$ is regular.*

We recall the relevant pieces of the argument. For a complete proof, see §2.4 of Kisin's paper [Ki2] or Conrad's unpublished notes [Co]. We just deal with the $*=2$ case, the other case being similar. Define a functor D_2 on the category $\mathscr{C}_{\mathcal{O}}$ by assigning to an algebra A the groupoid $D_2(A)$ of pairs (V, L) where $V \in D(A)$ and L is a rank one A-module summand of V on which G_F acts through χ and such that the class in $H^1(G_F, L \otimes (V/L)^\vee)$ determined by the extension

$$0 \to L \to V_A^{\mathrm{univ}} \to V_A^{\mathrm{univ}}/L \to 0$$

belongs to $H^1_f(G_F, L \otimes (V_A^{\mathrm{univ}}/L)^\vee)$ (see [Ki2, §2.4.1] for the definition of H^1_f). There is a natural map $\Theta\colon D_2 \to D$ which forgets L. This map is relatively representable and projective; in fact, there is a closed immersion $D_2 \to \mathbf{P}^1_D$ lifting Θ. It follows that $D_2^\square$ is representable by a projective formal scheme $\widehat{Z}$ over $\mathrm{Spec}(R^{\mathrm{univ}})$. We let Z be the algebraization of $\widehat{Z}$, and still write Θ for the map $Z \to \mathrm{Spec}(R^{\mathrm{univ}})$.

The scheme Z is formally smooth over $\mathcal{O}$. The map $\Theta[1/\pi]$ is a closed immersion and induces a bijection between the closed points of $Z[1/\pi]$ and the set $X_2 \subset \mathrm{MaxSpec}(R^{\mathrm{univ}}[1/\pi])$. It follows that if we let R_2 be such that $\mathrm{Spec}(R_2)$ is the scheme-theoretic image of Θ, then R_2 is $\mathcal{O}$-flat and reduced and satisfies $\mathrm{MaxSpec}(R_2[1/\pi]) = X_2$. It is clear that R_2 is the unique quotient of R^{univ} with these properties. Since Z is formally smooth over $\mathcal{O}$ and $\Theta\colon Z[1/\pi] \to \mathrm{Spec}(R_2[1/\pi])$ is an isomorphism, it follows that $R_2[1/\pi]$ is formally smooth over E, and thus regular. One deduces that R_2 is a domain from the fact that the fiber of Θ over the closed point of R^{univ} is connected (it is $\mathbf{P}^1$ since V_0 is trivial); see [Ki, Cor. 2.4.6]. The dimension of R_2 can be calculated by looking at a tangent space.

From Proposition 4.3.1, we immediately obtain the following theorem:

Proposition 4.3.2. *There exists a unique reduced $\mathcal{O}$-flat quotient R of R^{univ} with the property that $\mathrm{MaxSpec}(R[1/\pi])$ is equal to X. The ring R is equidimensional of dimension $d+4$ and has two minimal primes, namely the kernels of the surjections $R \to R_*$.*

4.4. **The ring $\widetilde{R}$.** Fix a Frobenius element ϕ of G_F. We assume $\chi(\phi) = 1$ for convenience. Let $\widetilde{D}$ be the functor on $\mathscr{C}_{\mathcal{O}}$ assigning to A the set of pairs (V, α), where V belongs to $D(A)$ and $\alpha \in A$ is a root of the characteristic polynomial of ϕ on V. The framed version $\widetilde{D}^\square = \widetilde{D} \times_D D^\square$ is pro-representable by a complete local noetherian $\mathcal{O}$-algebra $\widetilde{R}^{\mathrm{univ}}$. The ring $\widetilde{R}^{\mathrm{univ}}$ is the quotient of $R^{\mathrm{univ}}[\alpha]$ by a monic degree two polynomial (the characteristic polynomial of ϕ on V^{univ}).

We now define a map

$$\Phi\colon X \to \mathrm{MaxSpec}(\widetilde{R}^{\mathrm{univ}}[1/\pi]).$$

Thus let x be a point in X. The space V_x contains a unique line L_x on which I_F acts through the cyclotomic character. This line is stable by G_F since I_F is a normal subgroup. Let α_x be the scalar through which ϕ acts on V_x/L_x. Then $\widetilde{x} = (V_x, \alpha_x)$ defines a point of $\mathrm{MaxSpec}(\widetilde{R}^{\mathrm{univ}})$. We put $\Phi(x) = \widetilde{x}$. Put $\widetilde{X} = \Phi(X)$ and $\widetilde{X}_* = \Phi(X_*)$.

The main result of this section is the following:

Proposition 4.4.1. *Let $* \in \{1,2\}$. Then there is a unique reduced $\mathcal{O}$-flat quotient $\widetilde{R}_*$ of $\widetilde{R}^{\mathrm{univ}}$ such that* $\mathrm{MaxSpec}(\widetilde{R}_*[1/\pi])$ *is equal to $\widetilde{X}_*$. The ring $\widetilde{R}_*$ is a domain and equidimensional of dimension $d+4$. The natural map $R_* \to \widetilde{R}_*$ is an isomorphism after inverting π.*

This proposition can be proved in the same manner as Proposition 4.3.1. Instead of doing this, we deduce it from Proposition 4.3.1, as this is a bit shorter. We only treat the $*=2$ case, as the other is similar.

Let Z be the scheme constructed in the previous section. Let $\widetilde{\Theta}\colon Z \to \mathrm{Spec}(\widetilde{R}^{\mathrm{univ}})$ be the map defined by taking (V,L) to (V,α), where α is the scalar through which ϕ acts on V/L. We have a commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "\widetilde{\Theta}"'] \arrow[dr, "\Theta"] & \\ \mathrm{Spec}(\widetilde{R}^{\mathrm{univ}}) \arrow[rr] & & \mathrm{Spec}(R^{\mathrm{univ}}) \end{tikzcd}$$

where the bottom horizontal map is the natural one (forget α), and

$$\begin{tikzcd} & Z[1/\pi]' \arrow[dl, "\widetilde{\Theta}"'] \arrow[dr, "\Theta"] & \\ \widetilde{X}_2 & & X_2 \arrow[ll, "\Phi"'] \end{tikzcd}$$

where the prime denotes the set of closed points. Since $\Theta[1/p]$ is a closed immersion, it follows from the first diagram that $\widetilde{\Theta}[1/p]$ is as well. We thus see that $\widetilde{\Theta}$ is injective in the second diagram; since we know that Θ and Φ and bijections, it follows that $\widetilde{\Theta}$ is as well.

Let $\widetilde{R}_2$ be such that $\mathrm{Spec}(\widetilde{R}_2)$ is the scheme-theoretic image of $\widetilde{\Theta}$. Since Z is formally smooth over $\mathcal{O}$, the ring $\widetilde{R}_2$ is $\mathcal{O}$-flat and reduced; furthermore, by the above comments, $\mathrm{MaxSpec}(\widetilde{R}_2[1/\pi]) = \widetilde{X}_2$. It is clear that $\widetilde{R}_2$ is the unique quotient of $\widetilde{R}^{\mathrm{univ}}$ with these properties. Since $\Theta[1/\pi]\colon Z[1/\pi] \to \mathrm{Spec}(R_2[1/\pi])$ and $\widetilde{\Theta}[1/\pi]\colon Z[1/\pi] \to \mathrm{Spec}(\widetilde{R}_2[1/\pi])$ are both isomorphisms, it follows from the first diagram above that $R_2[1/\pi] \to \widetilde{R}_2[1/\pi]$ is an isomorphism. We thus conclude that $\widetilde{R}_2$ is a domain and equidimensional of dimension $d+4$ from the corresponding results for R_2.

Remark 4.4.2. The map $R_1 \to \widetilde{R}_1$ is an isomorphism.

The above proposition immediately implies the following one.

Proposition 4.4.3. *There is a unique reduced $\mathcal{O}$-flat quotient $\widetilde{R}$ of $\widetilde{R}^{\mathrm{univ}}$ such that* $\mathrm{MaxSpec}(\widetilde{R}[1/\pi])$ *is equal to $\widetilde{X}$. The ring $\widetilde{R}$ is equidimensional of dimension $d+4$ and has two minimal primes, the kernels of the surjections $\widetilde{R} \to \widetilde{R}_*$. The natural map $R \to \widetilde{R}$ is an isomorphism after inverting π.*

4.5. **The ring $\widetilde{R}^\dagger$.** For $A \in \mathscr{C}_{\mathcal{O}}$, let $\widetilde{D}^\dagger(A)$ denote the subset of $\widetilde{D}(A)$ consisting of those pairs (V,α) such that the following conditions are satisfied:

- $\mathrm{tr}\, g = \chi(g) + 1$ for all $g \in I_F$.
- $(g-1)(g'-1) = (\chi(g)-1)(g'-1)$ for $g, g' \in I_F$.
- $(g-1)(\phi-\alpha) = (\chi(g)-1)(\phi-\alpha)$ for $g \in I_F$.
- $(\phi-\alpha)(g-1) = (\alpha^{-1}-\alpha)(g-1)$ for $g \in I_F$.

The functor $\widetilde{D}^{\dagger,\square}$ is clearly prorepresentable by a ring $\widetilde{R}^\dagger$; to obtain $\widetilde{R}^\dagger$, simply form the quotient of $\widetilde{R}^{\mathrm{univ}}$ by the above equations.

Lemma 4.5.1. *The natural map $\widetilde{R}^{\mathrm{univ}} \to \widetilde{R}$ factors through $\widetilde{R}^\dagger$.*

Proof. The map $\widetilde{\Theta} : Z \to \mathrm{Spec}(\widetilde{R}^{\mathrm{univ}})$ clearly factors through the closed immersion $\mathrm{Spec}(\widetilde{R}^\dagger) \to \mathrm{Spec}(\widetilde{R}^{\mathrm{univ}})$, which proves the lemma. $\square$

4.6. **The main theorems.** We prove two main theorems. The first is the following:

Theorem 4.6.1. *The natural map $\widetilde{R}^\dagger \to \widetilde{R}$ is an isomorphism.*

This theorem gives a description of the points of $\widetilde{R}$. Our second theorem is the following:

Theorem 4.6.2. *Let $* \in \{1, 2\}$. The ring $\widetilde{R}_*$ is normal and Cohen–Macaulay but not Gorenstein.*

The rest of this section is devoted to proving these two theorems. We begin with some lemmas.

Lemma 4.6.3. *The natural map $\widetilde{R}^\dagger[1/\pi]_{\mathrm{red}} \to \widetilde{R}[1/\pi]$ is an isomorphism.*

Proof. Let $\widetilde{x}$ be a point in $\mathrm{MaxSpec}(\widetilde{R}^\dagger[1/\pi])$ and let (V, α) be the corresponding representation and eigenvalue of ϕ. Let x be the image of $\widetilde{x}$ in $\mathrm{MaxSpec}(R^{\mathrm{univ}}[1/\pi])$, so that $V = V_x$. The equations of §4.5 hold on V. The first of these, namely $\mathrm{tr}(g) = \chi(g) + 1$ for $g \in I_F$, shows that the semi-simplification of $V|_{I_F}$ is $\chi \oplus 1$.

Suppose that $V|_{I_F}$ is an extension of χ by 1, so that with respect to a suitable basis the action of I_F is given by

$$g \mapsto \begin{pmatrix} 1 & f(g) \\ & \chi(g) \end{pmatrix}.$$

The second equation of §4.5 shows that

$$(\chi(g) - 1)f(g') = (\chi(g') - 1)f(g)$$

for all $g, g' \in I_F$, which shows that the cocycle f is a coboundary (fix g' with $\chi(g') \neq 1$). Thus the extension is split.

The previous paragraph shows that we can regard $V|_{I_F}$ as an extension of 1 by χ. Thus x belongs to X. Let β be the eigenvalue of ϕ on the inertial coinvariants of V. Then the fourth equation in §4.5 shows that $(\beta^{-1} - \alpha)(g - 1) = (\alpha^{-1} - \alpha)(g - 1)$ holds on V_x for all $g \in I_F$. This implies $\beta = \alpha$ (consider $g \in I_F$ with $\chi(g) \neq 1$), and so $\widetilde{x} = \Phi(x)$. This shows that $\widetilde{x}$ belongs to $\widetilde{X}$.

We have just shown that the inclusion $\widetilde{X} = \mathrm{MaxSpec}(\widetilde{R}[1/\pi]) \subset \mathrm{MaxSpec}(\widetilde{R}^\dagger[1/\pi])$ induced by the surjection $\widetilde{R}^\dagger \to \widetilde{R}$ is in fact an equality. The lemma follows. $\square$

Lemma 4.6.4. *The ring $\widetilde{R}^\dagger/\pi\widetilde{R}^\dagger$ is isomorphic to the complete local ring of the scheme $\mathcal{C}_d$ at the point c (see §3.5 for the definition of $\mathcal{C}_d$ and c, and recall $d = [F : \mathbf{Q}_p]$).*

Proof. Let $\widehat{\mathcal{C}}$ denote the formal completion of $\mathcal{C}_d$ at the point c. Let $\mathscr{C}_k$ denote the category of complete local noetherian k-algebras with residue field k. For $A \in \mathscr{C}_k$, the set $\widehat{\mathcal{C}}(A)$ consists of those elements of $\mathcal{C}(A)$ whose image in $\mathcal{C}(k)$ is the point c. We will show that the functors $\widetilde{D}^{\dagger,\square}$ and $\widehat{\mathcal{C}}$ are isomorphic on the category $\mathscr{C}_k$. As $\widetilde{R}^{\dagger,\square}/\pi\widetilde{R}^\dagger$ represents the former functor, this will prove the lemma.

Let $A \in \mathscr{C}_k$. We regard elements of $\widetilde{D}^\square(A)$ as pairs (ρ, α) where $\rho\colon G_F \to \mathrm{GL}_2(A)$ is a homomorphism reducing to the trivial homomorphism modulo the maximal ideal of A and α is an element of A satisfying the characteristic polynomial of $\rho(\phi)$. An element (ρ, α) of $\widetilde{D}^\square(A)$ belongs to $\widetilde{D}^{\dagger,\square}(A)$ if and only if the following equations hold:

- $\mathrm{tr}\,\rho(g) = 2$ for all $g \in I_F$.
- $(\rho(g) - 1)(\rho(g') - 1) = 0$ for $g, g' \in I_F$.
- $(\rho(g) - 1)(\rho(\phi) - \alpha) = 0$ for $g \in I_F$.
- $(\rho(\phi) - \alpha)(\rho(g) - 1) = (\alpha - \alpha^{-1})(\rho(g) - 1)$ for $g \in I_F$.

These equations come from combining the defining equations of $\widetilde{D}^\dagger$ with the assumption that χ reduces to 1 modulo p. Of course, we also have $\det \rho(g) = 1$ for any such deformation. These conditions imply that $\rho(g) - 1$ is strongly nilpotent for any $g \in I_F$, and thus $\rho(g)^p = 1$ for any such g. We thus see that if (ρ, α) belongs to $\widetilde{D}^{\dagger,\square}(A)$ then $\rho|_{I_F}$ factors through the maximal abelian quotient of I_F of exponent p. Of course, ρ factors through the maximal p-power quotient of G_F since its reduction modulo the maximal ideal of A is trivial.

Let G be the maximal p-power quotient of G_F in which inertia is abelian and of exponent p. Let U be the inertia group in G. We have a short exact sequence

$$0 \to U \to G \to \mathbf{Z}_p \to 0.$$

The image of ϕ is a topological generator of $\mathbf{Z}_p$. We give U the structure of an $\mathbf{F}_p[\![T]\!]$-module by letting T act by $\phi - 1$. As computed in Proposition 2.4.1, U is isomorphic to $\mathbf{F}_p \oplus \mathbf{F}_p[\![T]\!]^{\oplus d}$. Let $g_1, \ldots, g_d$ be an $\mathbf{F}_p[\![T]\!]$-basis for the free part of U and let g_{d+1} be a generator of the T-torsion of U. Note that to give a continuous map from G to some discrete group Γ is the same as giving elements $\overline{\phi}$ and $\overline{g}_1, \ldots, \overline{g}_{d+1}$ of Γ such that the $\overline{g}_i$ commute with each other, $\overline{g}_i^p = 1$ for each i, $\overline{\phi}$ has finite order and $\overline{\phi}$ and $\overline{g}_{d+1}$ commute.

For $A \in \mathscr{C}_k$, we define a map $\widetilde{D}^{\dagger,\square}(A) \to \widehat{\mathcal{C}}(A)$ by taking (ρ, α) to the tuple $(\phi, \alpha; m_1, \ldots, m_{d+1})$ where $\phi = \rho(\phi)$ (apologies for the bad notation) and $m_i = \rho(g_i) - 1$. The defining equations for $\widetilde{D}^{\dagger,\square}$ given above show that this map is a bijection. $\square$

We now prove the first theorem.

Proof of Theorem 4.6.1. We follow the plan laid out in §1.2. We have already completed step (a) by guessing the equations for $\widetilde{R}$ and defining the ring $\widetilde{R}^\dagger$. We now complete the process.

(b) By Lemma 4.6.3, $\widetilde{R}^\dagger[1/\pi]_{\mathrm{red}} \to \widetilde{R}[1/\pi]$ is an isomorphism. In particular, by Proposition 4.4.3, $\widetilde{R}^\dagger[1/\pi]$ is equidimensional of dimension $d+3$ and has two minimal primes.
(c) By Lemma 4.6.4 and Theorem 3.5.1, $\widetilde{R}^\dagger/\pi\widetilde{R}^\dagger$ is reduced, equidimensional of dimension $d+3$ and has two minimal primes.
(d) By Proposition 2.2.1, the ring $\widetilde{R}^\dagger$ is flat over $\mathcal{O}$. By Proposition 2.3.1 it is reduced.
(e) Since $\widetilde{R}^\dagger$ is $\mathcal{O}$-flat and reduced and the map $\widetilde{R}^\dagger[1/\pi]_{\mathrm{red}} \to \widetilde{R}[1/\pi]$ is an isomorphism, it follows that $\widetilde{R}^\dagger \to \widetilde{R}$ is an isomorphism.

This completes the proof. $\square$

We now turn to the second theorem.

Proof of Theorem 4.6.2. As shown in the proof of Proposition 2.2.1, the ring $\widetilde{R}$ has two minimal primes, and these minimal primes are naturally in correspondence with those of $\widetilde{R}/\pi\widetilde{R}$ and $\widetilde{R}[1/\pi]$. It is clear that one of these minimal primes is defined by the equation $\alpha = 1$. The quotient by this minimal prime is the rings $\widetilde{R}_1$. We thus see that $\widetilde{R}_1/\pi\widetilde{R}_1$ is isomorphic to the complete local ring of $\mathcal{A}_{d+2}$ at a. Finally, the second minimal prime gives $\widetilde{R}_2$. It is clear that $\widetilde{R}_2/\pi\widetilde{R}_2$ is isomorphic to the complete local ring of $\mathcal{B}_d$ at b, since this is the only thing left over.

Appealing to Theorem 3.3.1 and Theorem 3.4.1, we see that $\widetilde{R}_*/\pi\widetilde{R}_*$ is integral, normal, Cohen–Macaulay and not Gorenstein. The ring R_* is a domain and $R_*[1/\pi]$ is normal (as it is regular). It follows that R_* is normal (by Proposition 2.3.2), Cohen–Macaulay (by [BH, Thm. 2.1.3a]) and not Gorenstein (by [BH, Prop. 3.1.19b]). $\square$

Remark 4.6.5. We can analyze the ring $\widetilde{R}_1$ directly, without using the ring $\widetilde{R}$. Indeed, we can define $\widetilde{R}_1$ as the quotient of $\widetilde{R}$ be the equation $\alpha = 1$, which realizes it directly as a quotient of R^{univ}. We can then go through the above arguments, but specifically for $\widetilde{R}_1$. However, we have not found a way to analyze $\widetilde{R}_2$ directly: we do not know the equations that cut it out from $\widetilde{R}$. Note, however, that we do know how to cut out $\widetilde{R}_2/\pi\widetilde{R}_2$ from $\widetilde{R}/\pi\widetilde{R}$: it is defined by the equation $\rho(g_{d+1}) = 1$, where g_{d+1} is as in the proof of Lemma 4.6.4. We have therefore studied $\widetilde{R}_2$ indirectly by studying the entire ring $\widetilde{R}$.

5. Global deformation rings

We now give some global applications of our results. These applications are really just some superficial comments about how our result can be combined with other known results, so we do not bother going into many details.

Let E, $\mathcal{O}$ and k be as in the previous section. Let F be a totally real field, let Σ be a finite set of finite places of F, including all those above p, and let $\overline{\rho} : G_{F,\Sigma} \to \mathrm{GL}_2(k)$ be a totally odd continuous representation of the absolute Galois group of F unramified away from Σ. We assume that $\overline{\rho}$ is absolutely irreducible and has determinant χ, the cyclotomic character. We define several deformation rings (all with fixed determinant χ):

- For $v \in \Sigma$, let $R_v^{\square,\mathrm{univ}}$ denote the universal framed deformation ring of $\overline{\rho}|_{G_{F_v}}$.
- For $v \in \Sigma$, choose a finite $R_v^{\square,\mathrm{univ}}$-algebra $R_v^{\square}$ which $\mathcal{O}$-flat and equidimensional of dimension $[F_v : \mathbf{Q}_p] + 4$ if $p \mid v$ or 4 if $p \nmid v$.
- Let $R_{\mathrm{loc}}^{\square,\mathrm{univ}}$ be the completed tensor product of the $R_v^{\square,\mathrm{univ}}$ over $\mathcal{O}$ and let $R_{\mathrm{loc}}^{\square}$ be the completed tensor product of the $R_v^{\square}$ over $\mathcal{O}$.
- Let $R^{\square,\mathrm{univ}}$ be the universal deformation ring of $\overline{\rho}$ with framings at each $v \in \Sigma$ and let $R^{\square}$ be the completed tensor product of $R^{\square,\mathrm{univ}}$ with $R_{\mathrm{loc}}^{\square}$ over $R_{\mathrm{loc}}^{\square,\mathrm{univ}}$.
- Let R^{univ} be the universal (unframed) deformation ring of $\overline{\rho}$ and let R be the descent of $R^{\square}$ from $R^{\square,\mathrm{univ}}$ to R^{univ}; it is a finite R^{univ}-algebra.

We then have the following result, taken from the discussion in [KW] before Corollary 4.7.

Proposition 5.0.6. *Assume R is finite over $\mathcal{O}$ and each $R_v^{\square}$ is Cohen–Macaulay. Then R is flat over $\mathcal{O}$ and Cohen–Macaulay. Furthermore, R is Gorenstein if and only if each $R_v^{\square}$ is.*

Proof. By [Ki3, Prop. 4.1.5], we have a presentation

$$R^{\square,\mathrm{univ}} = R_{\mathrm{loc}}^{\square,\mathrm{univ}}[\![x_1, \ldots, x_{r+n-1}]\!]/(f_1, \ldots, f_{r+s}),$$

where $n = \#\Sigma$, $s = [F : \mathbf{Q}]$ and r is a non-negative integer. Tensoring over $R_{\mathrm{loc}}^{\square,\mathrm{univ}}$ with $R_{\mathrm{loc}}^{\square}$ gives a presentation

$$R^{\square} = R_{\mathrm{loc}}^{\square}[\![x_1, \ldots, x_{r+n-1}]\!]/(f_1, \ldots, f_{r+s}).$$

This shows that $R^{\square}$ has dimension at least $4n$. Since $R^{\square}$ is a power series ring over R in $4n-1$ variables and R is finite over $\mathcal{O}$, we see that R has dimension 1 and $R^{\square}$ has dimension $4n$. Furthermore, writing $R^{\square} = R[\![T_1, \ldots, T_{4n-1}]\!]$, we see that $T_1, \ldots, T_{4n-1}, f_1, \ldots, f_{r+s}, p$ is a system of parameters for $R_{\mathrm{loc}}^{\square}[\![x_1, \ldots, x_{r+n-1}]\!]$. Since each $R_v^{\square}$ is Cohen–Macaulay, so too is $R_{\mathrm{loc}}^{\square}[\![x_1, \ldots, x_{r+n-1}]\!]$, and it follows that $T_1, \ldots, T_{4n-1}, f_1, \ldots, f_{r+s}, p$ is a regular sequence. This shows that $R^{\square}$ is $\mathcal{O}$-flat and Cohen–Macaulay, and furthermore that $R^{\square}$ is Gorenstein if and only if each $R_v^{\square}$ is. Finally, note that $R^{\square}$ is a power series ring over R, and so all these properties can be transferred to R. $\square$

Remark 5.0.7. Finiteness of R is known in many cases. When $\overline{\rho}$ is modular, one can often obtain finiteness of R using the Taylor–Wiles argument as modified by Kisin. Even without modularity it is often still possible to obtain finiteness by using potential modularity.

The following proposition is a very special case, showing how the above proposition can be combined with the main results of this paper.

Proposition 5.0.8. *Suppose Σ consists exactly of the primes over p and that for each $v \in \Sigma$ the local representation $\overline{\rho}|_{G_{F_v}}$ is trivial. For $v \in \Sigma$, let $R_v^{\square}$ be one of the rings $\widetilde{R}_*$ constructed in §4.4. Then, assuming R is finite over $\mathcal{O}$, it is $\mathcal{O}$-flat, Cohen–Macaulay and not Gorenstein.*

We make one final comment. If $\overline{\rho}$ is assumed to be modular, then one has a surjection $R \to \mathbf{T}$, for a certain Hecke algebra $\mathbf{T}$. Kisin's version of the Taylor–Wiles argument often allows one to show that this map is an isomorphism after inverting p. We simply remark that if one knows a priori that R is torsion-free, then Kisin's result implies that $R \to \mathbf{T}$ is an isomorphism without inverting p. Thus in situations like the above proposition, it should be possible to obtain modularity lifting theorems for artinnian deformations.

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