# STRENGTH AND HARTSHORNE'S CONJECTURE IN HIGH DEGREE

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## 1. Introduction

Hartshorne conjectured that every smooth, codimension $c$ subvariety of $\mathbf{P}^n$, with $c < \frac{1}{3}n$, is a complete intersection [Har74, p. 1017]. We prove this in the special case when $n \gg \deg X$.

**Theorem 1.1.** *There is a function $N(c, e)$ such that if $X \subseteq \mathbf{P}^n_{\mathbf{k}}$ is closed, equidimensional of codimension $c$, and has degree $e$, and if $X$ is nonsingular in codimension $\geq N(c, e)$, then $X$ is a complete intersection. In particular, $N(c, e)$ does not depend on $n$ or on the field $\mathbf{k}$.*

In characteristic zero, Hartshorne showed this in [Har74, Theorem 3.3]. In parallel, and also in characteristic zero, Barth and Van de Ven proved an effective version of this result, showing that $N = \frac{5}{2}e(e-7) + c$ works [Bar75]. Later improvements include: work of Ran [Ran83, Theorem], sharpening the bound and extending it to arbitrary characteristic, but only when $c = 2$; and Bertram-Ein-Lazarsfeld's [BEL91, Corollary 3], which sharpens the bound in arbitrary codimension, but only holds in characteristic zero. See also [BC83, HS85]. We believe our result is new in the case of positive characteristic and $c > 2$.

All of these previous results are proved by geometric means. For instance, the main ingredient in [Bar75] is an analysis of the variety of lines in $X$ through a point, and many of the proofs make use of Kodaira Vanishing and topological results like Lefschetz-type restriction theorems.

Our proof, on the other hand, is algebraic and employs quite different methods: we derive Theorem 1.1 as an elementary consequence of our result in [ESS] that the graded ultraproduct of polynomial rings is isomorphic to a polynomial ring (see Theorem 2.3 below). Although Theorem 1.1 is ineffective, it holds in arbitrary characteristic and is independent of the field. For $c > 2$, it is the first such result in arbitrary characteristic.

This approach connects Hartshorne's Conjecture with the circle of ideas related to strength (see §2 below for the definition) initiated by Ananyan and Hochster in [AH16]. Though we do not rely on the results of [AH16], those ideas certainly motivated our approach.

Our proof also has some overlap with the Babylonian tower theorems, like [BVdV74a, Theorems I and IV] and those in [Coa12, Fle85, Sat91] among others. From an algebraic perspective, the natural setting for such statements is an inverse limit of polynomial rings, and [ESS] shows such an inverse limit shares many properties with the ultraproduct ring.

## 2. Setup and Background

Each closed subscheme $X \subseteq \mathbf{P}^n_{\mathbf{k}}$ determines a homogeneous ideal $I_X \subseteq \mathbf{k}[x_0, \ldots, x_n]$. The scheme $X$, or the ideal $I_X$, is **equidimensional of codimension** $c$ if all associated primes

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of $I_X$ have codimension $c$, and $X$ is a **complete intersection** if $I_X$ is defined by a regular sequence. Since the minimal free resolution of $I_X$ is stable under extending the ground field $\mathbf{k}$, the property of being a complete intersection is also stable under field extension.

From here on, $\mathbf{k}$ and $\mathbf{k}_i$ will denote fields. If $R$ is a graded ring with $R_0 = \mathbf{k}$, then as in [AH16], we define the **strength** of a homogeneous element $f \in R$ to be the minimal integer $k \geq -1$ for which there is a decomposition $f = \sum_{i=1}^{k+1} g_i h_i$ with $g_i$ and $h_i$ homogeneous elements of $R$ of positive degree, or $\infty$ if no such decomposition exists. The **collective strength** of a set of homogeneous elements $f_1, \ldots, f_r \in R$ is the minimal strength of a non-trivial homogeneous $\mathbf{k}$-linear combination of the $f_i$.

**Lemma 2.1.** *Let $R$ be a graded ring with $R_0 = \mathbf{k}$. If $I \subseteq R$ is homogeneous and finitely generated, then $I$ has a generating set $g_1, \ldots, g_r$ where the strength of $g_k$ equals the collective strength of $g_1, \ldots, g_k$ for each $1 \leq k \leq r$.*

*Proof.* Choose any homogeneous generators $f_1, \ldots, f_r$ of $I$. We prove the statement by induction on $r$. For $r = 1$ the statement is tautological. Now let $r > 1$. By definition of collective strength, we have a $\mathbf{k}$-linear combination $g_r = \sum_{i=1}^{r} a_i f_i$ such that the strength of $g_r$ equals the collective strength of $f_1, \ldots, f_r$. After relabeling, we can assume that $a_r \neq 0$ and it follows that $f_1, \ldots, f_{r-1}, g_r$ generate $I$. Applying the induction hypothesis to the ideal $(f_1, \ldots, f_{r-1})$ yields the desired result. $\square$

Let $Q = (f_1, \ldots, f_r) \subseteq \mathbf{k}[x_1, x_2, \ldots]$. The ideal of $c \times c$ minors of the Jacobian matrix of $(\frac{\partial f_i}{\partial x_j})$ does not depend on the choice of generators of $Q$. We denote this ideal by $J_c(Q)$.

**Lemma 2.2.** *Let $Q = (f_1, \ldots, f_r)$ be a homogeneous ideal in $\mathbf{k}[x_1, x_2, \ldots]$. If the strength of $f_i$ is at most $s$ for $c \leq i \leq r$, then* $\operatorname{codim} J_c(Q) \leq (r - c + 1)(2s + 2)$.

*Proof.* For each $c \leq i \leq r$, we write $f_i = \sum_{j=0}^{s} a_{i,j} g_{i,j}$ where $a_{i,j}$ and $g_{i,j}$ have positive degree for all $i, j$. Write $L_i$ for the ideal $(a_{i,j}, g_{i,j} \mid 0 \leq j \leq s)$ and let $L = L_c + L_{c+1} + \cdots + L_r$. The $i$th row of the Jacobian matrix has entries $\frac{\partial f_i}{\partial x_k}$; thus by the product rule, every entry in this row is in $L_i$. Since every $c \times c$ minor of the Jacobian matrix will involve row $i$ for some $c \leq i \leq r$, it follows that $J_c(Q) \subseteq L$. Thus $\operatorname{codim} J_c(Q) \leq \operatorname{codim} L$, which by the Principal Ideal Theorem is at most $(r - c + 1)(2s + 2)$, as this is the number of generators of $L$. $\square$

We briefly recall the definition of the ultraproduct ring, referring to [ESS, §4.1] for a more detailed discussion. Let $\mathcal{I}$ be an infinite set and let $\mathcal{F}$ be a non-principal ultrafilter on $\mathcal{I}$. We refer to subsets of $\mathcal{F}$ as **neighborhoods of** $*$, where $*$ is an imaginary point of $\mathcal{I}$. For each $i \in \mathcal{I}$, let $\mathbf{k}_i$ be an infinite perfect field. Let $\mathbf{S}$ denote the graded ultraproduct of $\{\mathbf{k}_i[x_1, x_2, \ldots]\}$, where each polynomial ring is given the standard grading. An element $g \in \mathbf{S}$ of degree $d$ corresponds to a collection $(g_i)_{i \in \mathcal{I}}$ of degree $d$ elements $g_i \in \mathbf{k}_i[x_1, x_2, \ldots]$, modulo the relation that $g = 0$ if and only if $g_i = 0$ for all $i$ in some neighborhood of $*$. For a homogeneous $g \in \mathbf{S}$ we write $g_i$ for the corresponding element in $\mathbf{k}_i[x_1, x_2, \ldots]$, keeping in mind that this is only well-defined for $i$ in some neighborhood of $*$. The following comes from [ESS, Theorems 1.2 and 4.6]:

**Theorem 2.3.** *Let $K$ be the ultraproduct of perfect fields $\{\mathbf{k}_i\}$ and fix $f_1, \ldots, f_r \in \mathbf{S}$ of infinite collective strength. There is a set $\mathcal{U}$, containing the $f_i$, such that $\mathbf{S}$ is isomorphic to the polynomial ring $K[\mathcal{U}]$.*

The following result was first proven in [CMPV, Theorem 4.2] (though that result is more general than this lemma). We provide an alternate proof, to illustrate how it also follows quickly from Theorem 2.3.

**Lemma 2.4.** *Fix $c$ and $e$. There exist positive integers $d_1, \dots, d_r$, depending only on $c$ and $e$, such that any homogeneous, equidimensional, and radical ideal $Q \subseteq \mathbf{k}[x_1, \dots, x_n]$ (with $\mathbf{k}$ perfect) of codimension $c$ and degree $e$ can be generated (not necessarily minimally) by homogeneous polynomials $f_1, \dots, f_r$ and where $\deg(f_i) \le d_i$. Neither $r$ nor the $d_i$ depend on $n$ or $\mathbf{k}$.*

*Proof.* For a homogeneous ideal $I$ we write $\nu(I)$ for the sum of the degrees of the minimal generators of $I$. If the statement were false, then for each $j \in \mathbf{N}$ we could find a homogeneous, equidimensional, and radical ideal $Q'_j \subset \mathbf{k}_j[x_1, x_2, \dots]$ of codimension $c$ and degree $e$, such that $\nu(Q'_j) \to \infty$ as $j \to \infty$. Choose some function $m \colon \mathcal{I} \to \mathbf{N}$ where $m(i)$ is unbounded in any neighborhood of $*$. For each $i \in \mathcal{I}$, choose some $j$ such that $\nu(Q'_j) \ge m(i)$ and set $Q_i = Q'_j$. By construction, the function $i \mapsto \nu(Q_i)$ is unbounded in every neighborhood of $*$.

By [EHV92, Proposition 3.5], each $Q_i$ can be generated up to radical by a regular sequence $f_{1,i}, f_{2,i}, \dots, f_{c,i} \in \mathbf{k}_i[x_1, x_2, \dots]$ with $\deg(f_{j,i}) = e$ for all $i, j$. Let $f_j = (f_{j,i}) \in \mathbf{S}$ and let $J = (f_1, f_2, \dots, f_c) \subseteq \mathbf{S}$. Since $P = \sqrt{J}$ is finitely generated by Theorem 2.3, we can write $P = (g_1, \dots, g_r) \subseteq \mathbf{S}$. We let $P_i = (g_{1,i}, \dots, g_{r,i})$ be the corresponding ideal in $\mathbf{k}_i[x_1, x_2, \dots]$.

For any $x = (x_i) \in \mathbf{S}$, we have:

$$\begin{array}{c} x_i \in P_i \\ \text{for } i \text{ near } * \end{array} \iff x \in P \iff \begin{array}{c} x^n \in J \\ \text{for some } n \end{array} \iff \begin{array}{c} x_i^n \in J_i \\ \text{for } i \text{ near } * \end{array} \iff \begin{array}{c} x_i \in Q_i \\ \text{for } i \text{ near } * \end{array}.$$

Thus $P_i = Q_i$ for $i$ near $*$. It follows that in a neighborhood of $*$, $\nu(Q_i)$ is bounded by $\sum_{k=1}^r \deg(g_k)$, providing our contradiction. $\square$

## 3. Proof of the main result

**Theorem 3.1.** *There is a function $N(c, e)$ such that if $Q \subseteq \mathbf{k}[x_1, \dots, x_n]$ is a homogeneous, equidimensional ideal of codimension $c$ and degree $e$ and if $V(Q)$ is nonsingular in codimension $\ge N(c, e)$, then $Q$ is defined by a regular sequence of length $c$. In particular, $N(c, e)$ does not depend on $n$ or on the field $\mathbf{k}$.*

**Remark 3.2.** Since an equidimensional ideal of codimension $c$ that is nonsingular in codimension $2c + 1$ must be prime, it would be equivalent to rephrase Theorem 3.1 in terms of prime ideals. We stick with equidimensional and radical ideals because some of the auxiliary results in this paper might be of interest with this added generality. $\square$

*Proof of Theorem 3.1.* We reduce to the case where $\mathbf{k}$ is perfect. Extending the field will change neither the minimal number of generators of $Q$, nor the codimension of the singular locus. By taking $N(c, e) \ge 1$, we can also assume that $Q$ is radical, even after extending to the perfect case. Finally, since a field extension will not change the codimension of any minimal prime of $Q$ [Stacks, 00P4], we can assume that $\mathbf{k}$ is perfect and that $Q$ is radical and equidimensional of codimension $c$.

Suppose that the theorem were false. Then for some fixed $c, e$ and for each $j \in \mathbf{N}$ we can find an equidimensional, radical ideal $Q'_j \subseteq \mathbf{k}_j[x_1, x_2, \dots]$ (with $\mathbf{k}_j$ perfect) of codimension $c$ and degree $e$ that is not a complete intersection, but where the codimension of the singular locus of $V(Q'_j)$ tends to $\infty$ as $j \to \infty$. Since the singular locus of $V(Q'_j)$ is defined by

$Q'_j + J_c(Q'_j)$, this implies that codim $J_c(Q'_j) \to \infty$ as $j \to \infty$. We choose a function $m\colon \mathcal{I} \to \mathbf{N}$ where $m(i)$ is unbounded in each neighborhood of $*$. For each $i \in \mathcal{I}$, define $Q_i$ to be any of the $Q'_j$ satisfying codim $J_c(Q'_j) \geq m(i)$. By construction, codim $J_c(Q_i)$ is unbounded in every neighborhood of $*$.

By Lemma 2.4, there are positive integers $d_1, \dots, d_r$ satisfying: for each $i \in \mathcal{I}$, there are homogeneous $f_{1,i}, \dots, f_{r,i}$ of degrees $d_1, \dots, d_r$ which generate $Q_i$. Let $f_1 = (f_{1,i}), \dots, f_r = (f_{r,i})$ be the corresponding elements in $\mathbf{S}$ and let $Q = (f_1, \dots, f_r)$. By Lemma 2.1, we can assume that the strength of $f_k$ is the collective strength of $f_1, \dots, f_k$ for each $1 \leq k \leq r$.

If $f_c$ had strength at most $s$, then the same holds for $f_{c,i}$ in a neighborhood of $*$; for if $f_c = \sum_{j=0}^{s} a_j h_j$ then $f_{c,i} = \sum_{j=0}^{s} (a_j)_i (h_j)_i$ for $i$ near $*$. But by Lemma 2.2, this would imply that codim $J_c(Q_i)$ is bounded in a neighborhood of $*$. Since this cannot happen, $f_c$ must have infinite strength. Thus the collection $f_1, \dots, f_c$ has infinite collective strength and so by Theorem 2.3, $f_1, \dots, f_c$ are independent variables in $\mathbf{S}$. In particular, $(f_1, \dots, f_c)$ defines a prime ideal of codimension $c$ and we must have $f_{c+1} = \cdots = f_r = 0$. However, this implies that in a neighborhood of $*$, each $Q_i$ is a complete intersection, contradicting our original assumption. □

*Proof of Theorem 1.1.* As in the beginning of the proof of Theorem 3.1, we can quickly reduce to the case where $\mathbf{k}$ is perfect. For a fixed $c$ and $e$, we let $N$ equal the bound from Theorem 3.1. Fix some $X \subseteq \mathbf{P}^n$ satisfying the hypotheses of Theorem 1.1, and let $Q \subseteq \mathbf{k}[x_1, \dots, x_{n+1}]$ be the defining ideal of $X$. By Theorem 3.1, $Q$ is defined by a regular sequence, and thus $X$ is a complete intersection. □

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