GENERALIZATIONS OF STILLMAN'S CONJECTURE VIA TWISTED COMMUTATIVE ALGEBRA

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Abstract. Combining recent results on noetherianity of twisted commutative algebras by Draisma and the resolution of Stillman's conjecture by Ananyan–Hochster, we prove a vast generalization of Stillman's conjecture. Our theorem yields an array of boundedness results in commutative algebra that only depend on the degrees of the generators of an ideal, and not the number of variables in the ambient polynomial ring.

1. Introduction

The introduction of noetherianity conditions in commutative algebra streamlined and generalized boundedness results in invariant theory. We revisit this theme: using Draisma's recent noetherianity result for twisted commutative algebras, and the resolution of Stillman's conjecture by Ananyan–Hochster, we prove a boundedness result for a large class of ideal invariants. This can be seen as as a far reaching generalization of Stillman's conjecture.

1.1. **Statement of results.** Fix an algebraically closed field $\mathbf{k}$. An **ideal invariant** is a rule ν that associates to every homogeneous ideal I in every standard-graded polynomial ring $A = \mathbf{k}[x_1, \dots, x_n]$ a quantity $\nu(I) \in \mathbf{Z} \cup \{\infty\}$ such that $\nu(I)$ only depends on the pair (A, I) up to isomorphism. There are countless examples of ideal invariants: degree, projective dimension, regularity, the (i, j) Betti number, etc. See §4 for many examples.

Let $\mathbf{d} = (d_1, \dots, d_r)$ be a tuple of positive integers. We say that an ideal I is **type d** if it is generated by $f_1, \dots, f_r$ where f_i is homogeneous of degree d_i. We say that an ideal invariant ν is **bounded in degree d** if there exists $B \in \mathbf{Z}$ such that for every type $\mathbf{d}$ ideal $I \subset A$ we have $\nu(I) \leq B$ or $\nu(I) = \infty$. We say ν is **degreewise bounded** if it is bounded in degree $\mathbf{d}$ for all $\mathbf{d}$. The main point of this definition is that the bound is independent of the number of variables.

There are two "niceness" conditions we require on our ideal invariants. We say that ν is **cone-stable** if $\nu(I[x]) = \nu(I)$ for all (A, I), that is, adjoining a new variable does not affect the invariant. We say that ν is **weakly upper semi-continuous** if the following holds: given a polynomial ring A, a variety S over $\mathbf{k}$, and a homogeneous ideal sheaf $\mathcal{I}$ of $\mathcal{A} = \mathcal{O}_S \otimes_{\mathbf{k}} A$ such that $\mathcal{A}/\mathcal{I}$ is $\mathcal{O}_S$-flat, the map $s \mapsto \nu(\mathcal{I}_s)$ is upper semi-continuous for $s \in S(\mathbf{k})$; that is, for each n, the locus $\{s \mid \nu(\mathcal{I}_s) \geq n\}$ is Zariski-closed. (Here $\mathcal{I}_s$ denotes the fiber of $\mathcal{I}$ at s, which is an ideal of A by the flatness assumption.)

Stillman's conjecture is exactly the statement that the ideal invariant "projective dimension" (which is cone-stable and weakly upper semi-continuous) is degreewise bounded. Our main theorem vastly generalizes this:

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Theorem 1.1. *Any ideal invariant that is cone-stable and weakly upper semi-continuous is degreewise bounded.*

We discuss some consequences of this result in §4. We highlight one example here.

Corollary 1.2. *Fix positive integers d and c. There exists an integer B with the following property. Suppse that $Y \subset \mathbf{P}^n$ is a hypersurface of degree d (with n arbitrary) such that all codimension c linear subspaces it contains are rigid (see §4.1). Then the number of these subspaces is at most B.*

1.2. **Structure of proof.** We introduce a topological space $Y_{\mathbf{d}}$ that parametrizes isomorphism classes of type $\mathbf{d}$ ideals in the infinite polynomial ring $\mathbf{k}[x_1, x_2, \ldots]$. We construct $Y_{\mathbf{d}}$ as a quotient of an infinite dimensional variety $X_{\mathbf{d}}$ that parametrizes generating sets $(f_1, \ldots, f_r)$ of type $\mathbf{d}$ ideals. Theorem 1.1 is deduced from two results about $Y_{\mathbf{d}}$.

Theorem 1.3. *The space $Y_{\mathbf{d}}$ is noetherian.*

Theorem 1.4. *The space $Y_{\mathbf{d}}$ admits a finite stratification $\{Y_{\mathbf{d}}^{\lambda}\}_{\lambda \in \Lambda}$ such that the universal quotient ring is flat over each stratum (see §3.6 for the precise meaning of this).*

The primary input into the proof of Theorem 1.3 is Draisma's theorem [Dra17] on noetherianity of polynomial representations, while the primary input into the proof of Theorem 1.4 is the resolution of Stillman's conjecture by Ananyan–Hochster [AH16].

We now sketch the proof of Theorem 1.1. Suppose ν is a cone-stable weakly upper semi-continuous ideal invariant. By cone-stability, ν defines a function on $Y_{\mathbf{d}}$. Let $Z_n \subset Y_{\mathbf{d}}$ be the locus where $\nu \geq n$. These loci form a descending chain. By weak upper semi-continuity and Theorem 1.4, $Z_n \cap Y_{\mathbf{d}}^{\lambda}$ is closed in Y^{λ}. By Theorem 1.3, $Y_{\mathbf{d}}^{\lambda}$ is noetherian, and so $Z_{\bullet} \cap Y_{\mathbf{d}}^{\lambda}$ stabilizes for each λ. Thus $Z_{\bullet}$ stabilizes, which shows that ν is bounded.

Remark 1.5. As should be clear, our proof of Theorem 1.1 uses the proof of Stillman's conjecture, and thus does not provide an independent proof of it. A direct proof of Theorem 1.4 would lead to an independent proof of Stillman's conjecture, however. □

1.3. **Outline.** In §2, we collect the results we need about certain infinite dimensional varieties. In §3, we prove the main theorems of the paper. In §4, we give examples of degreewise bounded invariants. Finally, §5 has some further comments and generalizations.

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2. Varieties defined by polynomial functors

2.1. **Setup.** Recall that a **polynomial functor** is an endofunctor of the category of $\mathbf{k}$-vector spaces that is a subquotient of a direct sum of tensor power functors. In characteristic 0, every polynomial functor is a direct sum of Schur functors. In positive characteristic, the structure of polynomial functors is more complicated. The category of polynomial functors is abelian, and comes equipped with a tensor product defined by $(F \otimes G)(U) = F(U) \otimes G(U)$.

Fix a finitely generated algebra object $\underline{R}$ in the category of polynomial functors. Examples of such algebra objects are easy to come by: if $\underline{V}$ is a finite length polynomial functor then

$\mathrm{Sym}(\underline{V})$ is such an algebra object. In fact, these are the only examples relevant to this paper. Let $R_n = \underline{R}(\mathbf{k}^n)$, a finitely generated $\mathbf{k}$-algebra, and let $X_n = \mathrm{Spec}(R_n)$.

The standard inclusion $\mathbf{k}^n \to \mathbf{k}^{n+1}$ induces an inclusion $R_n \to R_{n+1}$ and thus a projection $X_{n+1} \to X_n$. We let R be the direct limit of the R_n, and let $\widehat{X}$ be the inverse limit of the X_n's in the category of schemes, which is simply the affine scheme $\mathrm{Spec}(R)$. We let $|\widehat{X}|$ be the topological space underlying the scheme $\widehat{X}$, which is the inverse limit of the spaces $|X_n|$.

The standard projection $\mathbf{k}^{n+1} \to \mathbf{k}^n$ induces a surjection $R_{n+1} \to R_n$ and thus a closed immersion $X_n \to X_{n+1}$. We let $\widehat{R}$ be the inverse limit of the R_n's, regarded as a topological ring, and we let X be the ind-scheme defined by the directed system $\{X_n\}$. Let $|X|$ be the direct limit of the sets $|X_n|$. We define the **ind-topology** on $|X|$ to be the direct limit topology, and let $|X|^{\mathrm{ind}}$ denote the resulting topological space. The set $|X|$ is canonically identified with the set of open prime ideals in the topological ring $\widehat{R}$. In this way, $|X|$ is a subset of $\mathrm{Spec}(\widehat{R})$, and one can give it the subspace topology. We call this the **Zariski topology**, and denote the resulting space by $|X|^{\mathrm{Zar}}$. The Zariski topology is, in many situations, the "correct" topology, since its closed sets are zero loci of equations, but it is often easier to check that a set is closed in the ind-topology, since this can be checked in each X_n separately. Every Zariski closed set is ind-closed, but the converse is not true in general; see [And11] for an example.

By definition, R_n maps to R. However, there is also a canonical map in the opposite direction. Indeed, for each $m \geq n$ the standard projection $\mathbf{k}^m \to \mathbf{k}^n$ induces a ring homomorphism $R_m \to R_n$. These are compatible, and thus define a map $R \to R_n$. The composition $R_n \to R \to R_n$ is the identity, and so the map $R_n \to R$ is injective while the map $R \to R_n$ is surjective. These surjections are compatible as n varies, and thus define a map $R \to \widehat{R}$. By the same reasoning, we get a map $X \to \widehat{X}$.

Recall that a topological space Y equipped with an action of a group G is **topologically G-noetherian** if every descending chain of G-stable closed subsets stabilizes. Draisma [Dra17] proved that $|\widehat{X}|$ is topologically $\mathbf{GL}$-noetherian, where here, and in what follows, $\mathbf{GL}$ denotes the group $\bigcup_{n \geq 1} \mathbf{GL}_n(\mathbf{k})$. The goal of this section is to transfer Draisma's result to the spaces $|X|^{\mathrm{Zar}}$ and $|X|^{\mathrm{ind}}$ and some related spaces.

2.2. Comparison of X and $\widehat{X}$.

We begin by comparing $\widehat{X}$ to the Zariski topology on X.

Proposition 2.1. *We have a bijection*

$$\{\mathbf{GL}\text{-stable closed subsets of } |\widehat{X}|\} \to \{\mathbf{GL}\text{-stable closed subsets of } |X|^{\mathrm{Zar}}\}$$

given by $Z \mapsto Z \cap |X|$. In particular, $|X|^{\mathrm{Zar}}$ is $\mathbf{GL}$-noetherian.

We refer to elements of the image of the map $R \to \widehat{R}$ as **finite polynomials**.

Lemma 2.2. *Suppose $I \subset \widehat{R}$ is a closed, $\mathbf{GL}$-stable ideal. Then every element of I is the limit of finite polynomials that also belong to I.*

Proof. We note that any continuous endomorphism of $\widehat{\mathbf{k}}^\infty = \varprojlim \mathbf{k}^n$ acts on $\widehat{R}$, and that any $\mathbf{GL}$-stable subspace of $\widehat{R}$ (such as I) is automatically stable by these additional operators. Let $f \in I$, and let f_n be its image under $\widehat{R} \to R_n \to \widehat{R}$. The composite map $\widehat{R} \to \widehat{R}$ is the action of a continuous endomorphism $\widehat{\mathbf{k}}^\infty \to \mathbf{k}^n \to \widehat{\mathbf{k}}^\infty$, and so it maps I into itself. We thus see that $f_n \in I$. Since f is the limit of the f_n's, the result follows. $\square$

Lemma 2.3. *Let $Z \subset \widehat{X}$ be a* $\mathbf{GL}$*-stable closed subset. Then every point of Z is a limit of points of $Z \cap X$.*

Proof. Let $I \subset R$ be the ideal defining Z. As in the previous proof, any endomorphism of $\mathbf{k}^\infty$ acts on R and carries I to itself. Let $f_n \colon \widehat{X} \to \widehat{X}$ be the map induced by the projection $\mathbf{k}^\infty \to \mathbf{k}^n \to \mathbf{k}^\infty$. We thus see $f_n(Z) \subset Z$. Let $x \in Z$ and put $x_n = f_n(x)$. The map f_n factors as $\widehat{X} \to X_n \to X \to \widehat{X}$, and so $x_n \in Z \cap X$ for all n. Since x is the limit of the x_n, the result follows. □

Proof of Proposition 2.1. Define a map in the opposite direction by sending a $\mathbf{GL}$-stable Zariski closed set Z of $|X|$ to its closure in $|\widehat{X}|$. We claim that the two maps are inverse.

Start with $Z \subseteq |X|$. Let Z' be the closure of Z in $|\widehat{X}|$ and let $Z'' = Z' \cap |X|$. Then clearly $Z \subset Z''$. Let x be a point of Z''. We must show that every infinite polynomial that vanishes on Z also vanishes on x. If f is some infinite polynomial vanishing on Z, then by Lemma 2.2, it is a limit of finite polynomials f_n vanishing on Z. Each f_n vanishes on x, and hence so does the limiting polynomial f. So $x \in Z$.

Now consider $W \subseteq |\widehat{X}|$. Let W' be the closure of $W \cap |X|$ in $|\widehat{X}|$. Then clearly $W' \subset W$. Let x be a point of W. Lemma 2.3 shows that every point in W is a limit of points from $W \cap Y$ and hence we obtain the opposite containment as well.

We have thus shown that $Z \mapsto Z \cap |X|$ defines a bijection as in the statement of the proposition. Draisma's theorem shows that the source of this bijection satisfies the descending chain condition. Thus the target does as well, which shows that $|X|^{\mathrm{Zar}}$ is $\mathbf{GL}$-noetherian. □

2.3. **Comparison of the ind- and Zariski topologies.** The Zariski and ind-topologies on an ind-scheme are typically very different. Fortunately, this distinction disappears in our situation, where we focus on $\mathbf{GL}$-stable subsets.

Proposition 2.4. *A* $\mathbf{GL}$*-stable subset of $|X|$ is Zariski closed if and only if it is ind-closed. In particular, $|X|^{\mathrm{ind}}$ is* $\mathbf{GL}$*-noetherian.*

Lemma 2.5. *Let I be a* $\mathbf{GL}$*-stable radical ideal of R. Then its image in R_n is radical, for any n.*

Proof. Let I_n be the image of I in R_n. We claim that $I_n = I \cap R_n$. Since the composite $R_n \to R \to R_n$ is the identity, it is clear that any element of $I \cap R_n$ is contained in I_n. Conversely, the projection map $R_m \to R_n \subset R_m$ is obtained by applying the projection map $\mathbf{k}^m \to \mathbf{k}^n \subset \mathbf{k}^m$, which can be realized as the limit of elements of $\mathbf{GL}_m$, and so maps $I \cap R_m$ into itself. We thus see that any element of I_n belongs to I, which shows $I_n \subset I \cap R_n$.

Now suppose $x \in R_n$ and $x^k \in I_n$. Since I is a radical ideal and $x^k \in I$, we have $x \in I$. Thus $x \in I \cap R_n = I_n$, and so I_n is a radical ideal of R_n. □

Proof of Proposition 2.4. Every Zariski closed set is ind-closed (even without $\mathbf{GL}$-stability), so it suffices to show that if $Z \subset |X|$ is $\mathbf{GL}$-stable and ind-closed then it is Zariski closed. Let $Z_n = Z \cap X_n$, a Zariski closed subset of X_n. Let $I_n \subset R_n$ be the unique radical ideal such that $V(I_n) = Z_n$. Then I_n is $\mathbf{GL}_n$-stable. Let $I^{(n)}$ be the smallest $\mathbf{GL}$-stable radical ideal of R that contains I_n. Its zero locus is the intersection of the $\mathbf{GL}$-translates of the inverse image of Z_n in X. Hence, if $m \geq n$, the image of $I^{(m)}$ in R_n defines Z_n, since $Z_m \cap X_n = Z_n$. But this image is radical, by Lemma 2.5, so $I^{(m)} \cap R_n = I_n$. If $m < n$, the image of $I^{(m)}$ in R_n is a closed subset of Z_n. Again, $I^{(m)} \cap R_n$ is radical by Lemma 2.5, so $I^{(m)} \cap R_n \supseteq I_n$. Now let $J = \bigcap_{n \geq 1} I^{(n)}$. Then $J \cap R_n = I_n$, so $V(J) = Z$ and we see that Z is Zariski-closed. □

2.4. **Some variants.** Let $R_n^* = \underline{R}((\mathbf{k}^n)^*)$, where $(\mathbf{k}^n)^*$ denotes the dual space to $\mathbf{k}^n$. Of course, R_n^* is isomorphic to R_n as a $\mathbf{k}$-algebra, but the action of $\mathbf{GL}_n$ is different. Let X^* and $\widehat{X}^*$ be defined analogously to before. Given an action of $\mathbf{GL}$ or $\mathbf{GL}_n$ on some object, we define the **conjugate action** as the precomposition with the automorphism $g \mapsto {}^t g^{-1}$.

Proposition 2.6. *We have isomorphisms of (ind-)schemes $\widehat{X} \to \widehat{X}^*$ and $X \to X^*$ that are $\mathbf{GL}$-equivariant for the conjugate action on the source and the standard action on the target.*

Proof. Let $e_1, \dots, e_n$ be the standard basis for $\mathbf{k}^n$ and $e_1^*, \dots, e_n^*$ the dual basis for $(\mathbf{k}^n)^*$. We have a linear isomorphism $i_n \colon \mathbf{k}^n \to (\mathbf{k}^n)^*$ taking e_i to e_i^*. This is $\mathbf{GL}_n$-equivariant using the conjugate action on the source and the standard action on the target. The i_n thus induce the requisite isomorphisms $\widehat{X} \to \widehat{X}^*$ and $X \to X^*$. □

A subset of $|\widehat{X}|$ or $|X|$ is stable under the standard $\mathbf{GL}$-action if and only if it is stable under the conjugate $\mathbf{GL}$-action. Thus, by Proposition 2.4 and its corollary, we find:

Corollary 2.7. *A $\mathbf{GL}$-stable subset of $|X^*|$ is ind-closed if and only if it is Zariski closed.*

Corollary 2.8. *The spaces $|\widehat{X}^*|$, $|X^*|^{\mathrm{Zar}}$, and $|X^*|^{\mathrm{ind}}$ are topologically $\mathbf{GL}$-noetherian.*

Example 2.9. Let $\underline{V}$ be a finite length polynomial functor and let $\underline{R} = \mathrm{Sym}(\underline{V})$. Let $V_n = \underline{V}(\mathbf{k}^n)$ and let $V = \varinjlim V_n$. Also let $V_* = \varinjlim V_n^*$ be the restricted dual of V, where the limit is taken with respect to the standard inclusions of $(\mathbf{k}^n)^*$ into $(\mathbf{k}^{n+1})^*$. Then we have canonical identifications

$$\widehat{X} = V^*, \qquad X = V_*, \qquad \widehat{X}^* = (V_*)^*, \qquad X^* = V,$$

where here V^* and $(V_*)^*$ are the usual linear duals of the spaces V and V_*. □

Remark 2.10. The moral of this section is that all limit topological spaces one can sensibly form from $\underline{R}$ are essentially equivalent when working equivariantly. This heuristic does *not* hold in some similar situations; see [ES, §4] for examples. □

3. The main theorems

3.1. **Notation.** Let A_n be the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ and A be the infinite polynomial ring $\mathbf{k}[x_1, x_2, \dots]$. For a tuple $\mathbf{f} = (f_1, \dots, f_r)$ of elements of A_n, we let $I_{\mathbf{f},n}$ be the ideal of A_n they generate, and $I_{\mathbf{f}}$ the ideal of A they generate. Let $B_{\mathbf{f},n} = A_n/I_{\mathbf{f},n}$ and $B_{\mathbf{f}} = A/I_{\mathbf{f}}$.

For an integer $d > 0$, let $X_{d,n} = \mathrm{Sym}^d(\mathbf{k}^n)$, regarded as an affine scheme. For a degree tuple $\mathbf{d} = (d_1, \dots, d_r)$, we let

$$X_{\mathbf{d},n} = X_{d_1,n} \times \cdots \times X_{d_r,n}$$

and we let $X_{\mathbf{d}}$ be the ind-scheme defined by the system $\{X_{\mathbf{d},n}\}$. This fits into the variant setup of the previous section: in fact, $X_{\mathbf{d}}$ is the scheme X^* from Example 2.9 with $\underline{V} = \mathrm{Sym}^{d_1} \oplus \cdots \oplus \mathrm{Sym}^{d_r}$. Let $\mathcal{A}_{\mathbf{d},n}$ be the sheaf of algebras $A_n \otimes \mathcal{O}_{X_{\mathbf{d},n}}$ on $X_{\mathbf{d},n}$. A closed point of $X_{\mathbf{d},n}$ corresponds to a tuple $\mathbf{f} = (f_1, \dots, f_d)$ of elements of A_n. The family of ideals $I_{\mathbf{f},n}$ assembles to an ideal sheaf $\mathcal{I}_{\mathbf{d},n}$ of $\mathcal{A}_{\mathbf{d},n}$ (meaning that the image of the fiber $\mathcal{I}_{\mathbf{d},n}(\mathbf{f})$ in the fiber $\mathcal{A}_{\mathbf{d},n}(\mathbf{f}) = A_n$ is $I_{\mathbf{f},n}$). We let $\mathcal{B}_{\mathbf{d},n}$ be the quotient sheaf; its fiber at $\mathbf{f}$ is $B_{\mathbf{f},n}$.

3.2. **The space $Y_{\mathbf{d}}$.** Let $Y_{\mathbf{d}}$ be the set of isomorphism classes of type $\mathbf{d}$ ideals in A, where we say that ideals I and J are isomorphic if there exists an isomorphism $\sigma\colon A \to A$ of graded rings with $\sigma(I) = J$. Let $X^\circ_{\mathbf{d}}$ be the set of closed points in $X_{\mathbf{d}}$ and let $\pi\colon X^\circ_{\mathbf{d}} \to Y_{\mathbf{d}}$ be the map taking a tuple $\mathbf{f}$ to the class of the ideal $I_{\mathbf{f}}$ it generates. The map π is surjective and $\mathbf{GL}$-invariant. We give $Y_{\mathbf{d}}$ the induced topology, using the Zariski topology on $X^\circ_{\mathbf{d}}$. Thus a subset Z of $Y_{\mathbf{d}}$ is closed if and only if $\pi^{-1}(Z)$ is Zariski closed in $X^\circ_{\mathbf{d}}$. Since $\pi^{-1}(Z)$ is $\mathbf{GL}$-stable, it is Zariski closed if and only if it is ind-closed (Proposition 2.4), so the ind-topology on $X^\circ_{\mathbf{d}}$ induces the same topology on $Y_{\mathbf{d}}$.

Remark 3.1. If $\mathbf{d} = (d, \dots, d)$, then two tuples $\mathbf{f}, \mathbf{g} \in X_{\mathbf{d}}$ generate the same ideal in A if and only if they differ by an element of $\mathbf{GL}_r$, and generate isomorphic ideals if and only if they differ by an element of $\mathbf{GL}_\infty \times \mathbf{GL}_r$. We thus see that $Y_{\mathbf{d}}$ is the quotient of $X^\circ_{\mathbf{d}}$ by the group $\mathbf{GL}_\infty \times \mathbf{GL}_r$. For general $\mathbf{d}$, it is more complicated to describe $Y_{\mathbf{d}}$ directly. □

We define $Y_{\mathbf{d},n}$ as the set of isomorphism classes of ideals in A_n given the topology induced by the surjection $\pi_n\colon X^\circ_{\mathbf{d},n} \to Y_{\mathbf{d},n}$. There are natural maps $Y_{\mathbf{d},n} \to Y_{\mathbf{d},n+1}$ and $Y_{\mathbf{d},n} \to Y_{\mathbf{d}}$.

Proposition 3.2. *The space $Y_{\mathbf{d}}$ is the direct limit of the spaces $Y_{\mathbf{d},n}$.*

Proof. It is clear that the set $Y_{\mathbf{d}}$ is the direct limit of the sets $Y_{\mathbf{d},n}$. If Z is a subset of $Y_{\mathbf{d}}$, then Z is closed if and only if $\pi^{-1}(Z)$ is closed in $X^\circ_{\mathbf{d}}$, which is equivalent to $\pi^{-1}(Z) \cap X^\circ_{\mathbf{d},n}$ being closed for all n, which is equivalent to $\pi_n^{-1}(Z \cap Y_{\mathbf{d},n})$ being closed for all n, which is equivalent to $Z \cap Y_{\mathbf{d},n}$ being closed for all n, which is equivalent to Z being closed in the direct limit topology. Thus the topology on $Y_{\mathbf{d}}$ is the direct limit topology. □

Theorem (Theorem 1.3)**.** The space $Y_{\mathbf{d}}$ is noetherian.

Proof. Suppose $Z_\bullet$ is a descending chain of closed subsets in $Y_{\mathbf{d}}$. Then $\pi^{-1}(Z_\bullet)$ is a descending chain of $\mathbf{GL}$-stable Zariski closed subsets of $X^\circ_{\mathbf{d}}$, and thus stabilizes by Corollary 2.8. It follows that $Z_\bullet$ stabilizes, and so $Y_{\mathbf{d}}$ is noetherian. □

Remark 3.3. An ideal invariant induces a function $Y_{\mathbf{d},n} \to \mathbf{Z} \cup \{\infty\}$ for each n. An ideal invariant is cone-stable if and only if it is compatible with the transition maps $Y_{\mathbf{d},n} \to Y_{\mathbf{d},n+1}$. It follows that a cone-stable ideal invariant induces a function $Y_{\mathbf{d}} \to \mathbf{Z} \cup \{\infty\}$. □

3.3. **Finiteness of initial ideals.** For an ideal $I \subset A_n$ we let $\operatorname{gin}(I)$ denote the generic initial ideal of I under the revlex order. We note that $\operatorname{gin}(\sigma(I)) = \operatorname{gin}(I)$ for $\sigma \in \mathbf{GL}_n$, essentially by definition. The proof of the following theorem crucially depends on the theorem of Ananyan–Hochster [AH16] (Stillman's conjecture).

Theorem 3.4. *Given $\mathbf{d}$ there exist B and C such that for any n and any type $\mathbf{d}$ ideal I of A_n the ideal $\operatorname{gin}(I)$ is generated by monomials of degree at most C in the variables $x_1, \dots, x_B$.*

Proof. By Stillman's conjecture for $\mathbf{d}$, there exists B such that $\operatorname{pdim}(A_n/I) \le B$ for any type $\mathbf{d}$ ideal I of A_n. By a theorem of Caviglia [Pee11, Theorem 29.5], this implies that there exists C such that $\operatorname{reg}(I) \le C$ for any such I. Both B and C are independent of n.

Let I be a type $\mathbf{d}$ ideal of A_n and put $s = \operatorname{depth}(A_n/I)$. After a general change of coordinates, we can assume that $x_n, x_{n-1}, \dots, x_{n-s+1}$ form a regular sequence on A_n/I. Since the projective dimension of A_n/I is at most B, the Auslander–Buchsbaum Theorem [Eis95, Theorem 19.9] implies that $n-s \le B$. By the Bayer–Stillman criterion [BS87, Theorem 2.4], the same x_i form a regular sequence on the quotient of A_n by the revlex initial ideal of I. Thus, the revlex generic initial ideal of I is definable in at most B variables. Moreover [BS87, Corollary 2.5] implies that the revlex generic initial ideal is definable in degree at most C. □

It is a consequence of [BS87, Lemma 2.2] that formation of revlex gin is cone-stable, and thus $\mathrm{gin}(I)$ is well-defined for any finitely generated homogeneous ideal $I \subset A$, and $\mathrm{gin}(I)$ is a finitely generated monomial ideal of A. We define a **type d revlex generic initial ideal** as an ideal of A of the form $\mathrm{gin}(I)$ where I is a type $\mathbf{d}$ ideal. Theorem 3.4, yields:

Corollary 3.5. *There are only finitely many type* $\mathbf{d}$ *revlex generic initial ideals.*

Remark 3.6. The above corollary requires the revlex term order. See §5.5. □

3.4. **Hilbert numerators.** For a homogeneous ideal I of A_n, the Hilbert series $\mathrm{H}_{A_n/I}(t)$ can be written as a rational function $Q(t)/(1-t)^n$ with $Q(t) \in \mathbf{Z}[t]$. We call $Q(t)$ the **Hilbert numerator** of A_n/I, and we denote it by $\mathrm{HN}_{A_n/I}(t)$. Terminology for $\mathrm{HN}_{A_n/I}(t)$ varies in the literature: our usage follows [KR00, p. 282], but it is also called the K-polynomial [MS05, p. 156], and has other names elsewhere. Note that the Hilbert numerator is not necessarily the numerator of $\mathrm{H}_{A_n/I}(t)$ when written in lowest terms.

The advantage of the Hilbert numerator is that it is cone-stable: $\mathrm{HN}_{A_{n+1}/IA_{n+1}}(t) = \mathrm{HN}_{A_n/I}(t)$. We can thus define the Hilbert numerator of A/I for any finitely generated homogeneous ideal I of A. We define a **type d Hilbert numerator** to be a polynomial of the form $\mathrm{HN}_{A/I}(t)$ for I a type $\mathbf{d}$ ideal of A.

Theorem 3.7. *There are only finitely many type* $\mathbf{d}$ *Hilbert numerators.*

Proof. The Hilbert series associated to an ideal $I \subset A_n$ coincides with that of $\mathrm{gin}(I)$. It follows that the Hilbert numerator associated to an ideal $I \subset A$ coincides with that of $\mathrm{gin}(I)$, and so the result follows from Corollary 3.5. □

3.5. **Two partial orders.** Given polynomials $f, g \in \mathbf{R}[t]$, define $f < g$ if $f(x) < g(x)$ for all $0 < x < 1$. We use $f(x) \le g(x)$ to mean that either $f = g$ or that $f < g$, which is *not* the same as $f(x) \le g(x)$ for all $0 < x < 1$. Given series $f = \sum_{i \ge 0} a_i t^i$ and $g = \sum_{i \ge 0} b_i t^i$ in $\mathbf{R}[\![t]\!]$, define $f \preceq g$ if $a_i \le b_i$ for all i. Note that $f \le g \iff 0 \le g - f$, and similarly for $\preceq$.

Proposition 3.8. *Let* $f \in \mathbf{R}[t]$*. The following are equivalent:*

(a) $0 \le f$.
(b) *f can be expressed in the form $\sum_{0 \le i,j \le N} c_{i,j} x^i (1-x)^j$ for some N and non-negative coefficients $c_{i,j} \in \mathbf{R}$.*
(c) *There exists $N \ge 0$ such that $0 \preceq (1-t)^{-n} f(t)$ for all $n \ge N$.*
(d) *There exists $n \ge 0$ such that $0 \preceq (1-t)^{-n} f(t)$.*

Proof. (a) ⇒ (b). After a change of coordinates, this is [PS98, Part 6, §6, Problem 49].

(b) ⇒ (c). Write $f = \sum_{0 \le i,j \le N} c_{i,j} x^i (1-x)^j$ as in (b). Let $n \ge N$. Then $(1-t)^{-n} f(t) = \sum_{0 \le i,j \le N} c_{i,j} x^i (1-t)^{-(n-j)}$. Since $n - j \ge 0$ for all j in the sum, the series $(1-t)^{-(n-j)}$ has non-negative coefficients. Since the $c_{i,j}$ are non-negative, it follows that $(1-t)^{-n} f(t)$ has non-negative coefficients.

(c) ⇒ (d). Obvious.

(d) ⇒ (a). If $f = 0$ then obviously (a) holds. Thus suppose $f \ne 0$. Let n be such that $0 \preceq (1-t)^{-n} f(t)$, and let $x \in (0,1)$. The series $(1-t)^{-n} f(t)$ is non-zero, has non-negative coefficients, and converges at $t = x$. Thus its value at $t = x$ is positive. Since $1 - x$ is also positive, we conclude that $f(x)$ is positive, and so (a) holds. □

3.6. **The flattening stratification.** Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be the set of all type $\mathbf{d}$ Hilbert numerators, where Λ is a finite index set. Define a partial order on Λ by $\mu < \lambda$ if $H_\mu < H_\lambda$, that is, if $H_\mu(x) < H_\lambda(x)$ for all $0 < x < 1$. For $\lambda \in \Lambda$, let $Y_{\mathbf{d}}^\lambda$ be the locus in $Y_{\mathbf{d}}$ where the corresponding ideal class has Hilbert numerator H_λ. Let $Y_{\mathbf{d},n}^\lambda = Y_{\mathbf{d}}^\lambda \cap Y_{\mathbf{d},n}$.

Proposition 3.9. *For any $\lambda \in \Lambda$, the set $\bigcup_{\mu\geq\lambda} Y_{\mathbf{d}}^\mu$ is closed in $Y_{\mathbf{d}}$.*

Proof. It suffices to show that $Z_n = \bigcup_{\mu\geq\lambda} \pi_n^{-1}(Y_{\mathbf{d},n}^\mu)$ is closed in $X_{\mathbf{d},n}^\circ$ for all $n \gg 0$. Let N be such that for any $\lambda, \mu \in \Lambda$ and $n \geq N$ we have $\lambda \leq \mu$ if and only if $(1-t)^{-n}H_\lambda(t) \preceq (1-t)^{-n}H_\mu(t)$; this exists by Proposition 3.8 and the fact that Λ is finite. Let $n \geq N$ and let $\mathbf{f}$ be a point of $X_{\mathbf{d},n}^\circ$. Then $\mathbf{f} \in Z_n$ if and only if $H_\lambda(t) \leq \mathrm{HN}_{B_{\mathbf{f}}}(t)$, which in turn is equivalent to $(1-t)^{-n}H_\lambda(t) \preceq (1-t)^{-n}\,\mathrm{HN}_{B_{\mathbf{f}}}(t)$. Since $(1-t)^{-n}\,\mathrm{HN}_{B_{\mathbf{f}}}(t)$ is the Hilbert series $\mathrm{H}_{B_{\mathbf{f},n}}(t) = \mathrm{H}_{\mathcal{B}_{\mathbf{d},n}(\mathbf{f})}(t)$, we see

$$Z_n = \{\mathbf{f} \in X_{\mathbf{d},n}^\circ \mid (1-t)^{-n}H_\lambda(t) \preceq \mathrm{H}_{\mathcal{B}_{\mathbf{d},n}(\mathbf{f})}(t)\},$$

which is closed by the usual semi-continuity property of Hilbert series. □

Corollary 3.10. *Each $Y_{\mathbf{d}}^\lambda$ is locally closed in $Y_{\mathbf{d}}$.*

Let $\overline{X}_{\mathbf{d},n}^\lambda$ be the closure of $\bigcup_{\mu\geq\lambda} \pi_n^{-1}(Y_{\mathbf{d},n}^\mu)$ in $X_{\mathbf{d},n}$, endowed with the reduced subscheme structure. Let $X_{\mathbf{d},n}^\lambda$ be the complement of $\bigcup_{\mu>\lambda} \overline{X}_{\mathbf{d},n}^\mu$ in $\overline{X}_{\mathbf{d},n}^\lambda$, considered as an open subscheme of $\overline{X}_{\mathbf{d},n}^\lambda$. The set $X_{\mathbf{d},n}^{\lambda,\circ}$ of closed points in $X_{\mathbf{d},n}^\lambda$ is exactly $\pi_n^{-1}(Y_{\mathbf{d},n}^\lambda)$.

For simplicity, we have only defined $Y_{\mathbf{d}}$ as a topological space (as opposed to an ind-stack). To make sense of the flatness statement from Theorem 1.4, we thus pass to the cover $X_{\mathbf{d}}$, and interpret Theorem 1.4 as the following concrete statement:

Proposition 3.11. *The restriction of $\mathcal{B}_{\mathbf{d},n}$ to $X = X_{\mathbf{d},n}^\lambda$ is $\mathcal{O}_X$-flat.*

Proof. For $\mathbf{f} \in X$ we have $\mathrm{H}_{\mathcal{B}_{\mathbf{d},n}(\mathbf{f})}(t) = \mathrm{H}_{B_{\mathbf{f},n}}(t) = (1-t)^{-n}H_\lambda(t)$. Thus the Hilbert series of the fiber of $\mathcal{B}_{\mathbf{d},n}$ is constant on X. Let $\mathcal{F}$ be one of the graded pieces of $\mathcal{B}_{\mathbf{d},n}$. Then $\mathcal{F}$ is a coherent sheaf on X whose fiber at all closed points has the same dimension, say dimension d. By semi-continuity of fibral dimension, the locus of (not necessarily closed) points where the fibral dimension is $\neq d$ is the union of a closed set (where the dimension is $> d$) and an open set (where the dimension is $< d$). Yet this locus contains no closed points, and since X is finite type over a field, this implies that the fiber of $\mathcal{F}$ has the same dimension on the non-closed points as well, which in turn implies that $\mathcal{F}$ is locally free [Eis95, Ex. 20.14(b)]. □

Corollary 3.12. *Let ν be a cone-stable weakly upper semi-continuous ideal invariant, and let $n \in \mathbf{Z}$ be given. Then $Z = \{I \in Y_{\mathbf{d}}^\lambda \mid \nu(I) \geq n\}$ is a closed subset of $Y_{\mathbf{d}}^\lambda$.*

Proof. It suffices to show that $Z' = \pi_n^{-1}(Z \cap Y_{\mathbf{d},n})$ is a closed subset of $X_{\mathbf{d},n}^{\lambda,\circ}$ for all n. We have $Z' = \{\mathbf{f} \in X_{\mathbf{d},n}^{\lambda,\circ} \mid \nu(I_{\mathbf{f},n}) \geq n\}$. This is closed by the definition of weakly upper semi-continuous, since $\mathcal{B}_{\mathbf{d},n}$ is flat over $X_{\mathbf{d},n}^\lambda$. □

3.7. **Proof of Theorem 1.1.** Fix a cone-stable weakly upper semi-continuous ideal invariant ν, and let $\mathbf{d}$ be given. Let $Z_k \subset Y_{\mathbf{d}}$ be the locus defined by $\nu \geq k$. Observe that $Z_k \cap Y_{\mathbf{d}}^\lambda$ is closed in $Y_{\mathbf{d}}^\lambda$, by Corollary 3.12, and that the space $Y_{\mathbf{d}}^\lambda$ is noetherian, being a subspace of the noetherian space $Y_{\mathbf{d}}$ (see Theorem 1.3). We thus see that the descending chain $Z_\bullet \cap Y_{\mathbf{d}}^\lambda$ stabilizes. Since there are only finitely many λ, it follows that the chain $Z_\bullet$ stabilizes. Let N

be such that $Z_k = Z_N$ for all $k \geq N$. We thus see that $\nu \geq N$ implies $\nu \geq k$ for all $k \geq N$; thus $\nu \geq N$ implies $\nu = \infty$. We therefore find that $\nu < N$ or $\nu = \infty$ holds at all points in $Y_{\mathbf{d}}$, and so ν is bounded in degree $\mathbf{d}$. □

4. Examples of ideal invariants

4.1. **The number of linear subspaces in a variety.** A smooth cubic surface in $\mathbf{P}^3$ contains exactly 27 lines. An arbitrary cubic surface $Y \subset \mathbf{P}^3$ can contain fewer than 27 lines or it can contain an infinite number of lines (e.g., if Y is reducible); but if Y contains a *finite* number of lines, then it contains at most 27 lines [Mil, Theorem 9.48]. In this section, we prove a sort of generalization of this.

Fix a non-negative integer c. Let $\mathbf{Gr}_c(\mathbf{k}^n)$ be the Grassmannian of c codimensional subspaces of $\mathbf{k}^n$. For a homogeneous ideal $I \subset A_n$, let $\mathfrak{U}_I$ be the closed subscheme of $\mathbf{Gr}_c(\mathbf{k}^n)$ whose T-points are those families of subspaces of $\mathbf{k}^n$ scheme-theoretically contained in $T \times \operatorname{Spec}(A_n/I)$. Thus $\mathfrak{U}_I(\mathbf{k})$ is exactly the set of subspaces of $\mathbf{k}^n$ of codimension c contained in $V(I)$. We say that a point x of $\mathfrak{U}_I(\mathbf{k})$ is **rigid** if the Zariski tangent space to $\mathfrak{U}_I$ at x vanishes. Define an ideal invariant ν as follows: $\nu(I) = \infty$ if $\mathfrak{U}_I$ contains a non-rigid point; otherwise, $\nu(I) = \#\mathfrak{U}_I(\mathbf{k})$. We note that $\nu(I) < \infty$ if and only if $\mathfrak{U}_I$ is étale over $\mathbf{k}$.

Proposition 4.1. *The ideal invariant ν is degreewise bounded.*

Proof. We first show that ν is "eventually" cone-stable. Let $I \subseteq A_n$ with $n > c$ and let $I' = IA_{n+1}$. We show $\nu(I) = \nu(I')$. We have $V(I') = V(I) \times \mathbf{A}^1$, and so we have a map

$$\mathfrak{U}_I \to \mathfrak{U}_{I'}, \qquad U \mapsto U \times \mathbf{A}^1. \tag{4.1a}$$

This map is clearly a closed immersion. We claim that if $\mathfrak{U}_I$ is finite over $\mathbf{k}$ then it is an isomorphism on $\mathbf{k}$-points. Let $U' \in \mathfrak{U}_{I'}(\mathbf{k})$, and let U be its projection to $\mathbf{A}^n$. Then U is a linear subspace of $V(I) \subset \mathbf{A}^n$ containing the origin. If U has codimension c then $U' = U \times \mathbf{A}^1$, and so U' is in the image of (4.1a). If not, U has codimension strictly less than c, and thus dimension at least 2 (since $c < n$), and thus contains an infinite number of linear subspaces of codimension c in $\mathbf{A}^n$, which contradicts the finiteness of $\mathfrak{U}_I$.

If $\nu(I) = \infty$ then $\mathfrak{U}_I$ is not étale, and so $\mathfrak{U}_{I'}$ is not étale, and so $\nu(I') = \infty$. Suppose now that $\nu(I)$ is finite. Then (4.1a) is an isomorphism on $\mathbf{k}$-points. A $\mathbf{k}$-point x of $\mathfrak{U}_I$ corresponds to a surjection $B = A_n/I \to \mathbf{k}[t_1, \ldots, t_{n-c}] = C$ of graded rings. The Zariski cotangent space of $x \in \mathfrak{U}_I$ is identified with $\operatorname{Hom}_B(J/J^2, C)$, where J is the kernel of $B \to C$, and the maps are taken as graded B-modules. Since $\nu(I)$ is finite, this Hom space therefore vanishes. It follows that $\operatorname{Hom}_{B[t_{n+1}]}(J/J^2[t_{n+1}], C[t_{n+1}])$ also vanishes, which is the cotangent space of the image of x under (4.1a). We thus see that $\mathfrak{U}_{I'}$ is étale, and so (4.1a) is an isomorphism of schemes. Thus $\nu(I) = \nu(I')$.

We now prove semi-continuity. Suppose that $\mathcal{I} \subset \mathcal{O}_S \otimes A_n$ is a family of ideals, and write $\mathcal{I}_s$ for the ideal of A_n at $s \in S(\mathbf{k})$. The construction of $\mathfrak{U}$ works in families: we have a scheme $\mathfrak{U}_{\mathcal{I}}$ over S whose fiber at $s \in S(\mathbf{k})$ is $\mathfrak{U}_{\mathcal{I}_s}$. Since $\mathfrak{U}_{\mathcal{I}}$ is a closed subscheme of the Grassmannian, it is proper over S.

By Lemma 4.2 below, the set of points $s \in S(\mathbf{k})$ at which $(\mathfrak{U}_{\mathcal{I}})_s$ is étale is open. We thus see that the locus $\nu \geq \infty$ is closed.

We now show $\nu \geq k$ is closed, for $k < \infty$. Since the locus $\nu \geq \infty$ is closed, we can discard it from S. Thus we can assume $(\mathfrak{U}_{\mathcal{I}})_s$ is étale over $\mathbf{k}$ for all $s \in S(\mathbf{k})$. We thus see that $\mathfrak{U}_{\mathcal{I}}$ is quasi-finite over S, and thus finite. It therefore follows that the fibral dimension of $\mathcal{O}_{\mathfrak{U}_{\mathcal{I}}}$, which is just ν, is upper semi-continuous.

The proof of Theorem 1.1 shows that $I \mapsto \nu(I)$ is degreewise bounded for ideals I in A_n with $n > c$. For $n \leq c$ we have $\nu(I) \leq 1$. Thus ν is degreewise bounded. □

Lemma 4.2. *Let $X \to S$ be a proper map of schemes, with $S/\mathbf{k}$ of finite type. Then the locus of points $s \in S(\mathbf{k})$ where X_s is étale over $\mathbf{k}$ is open.*

Proof. We leave the proof as an exercise to the reader. □

Remark 4.3. We believe the ideal invariant $I \mapsto \#\mathfrak{U}_I(\mathbf{k})$ is degreewise bounded. It is cone-stable, but not upper semi-continuous. We also believe that the more refined invariant $I \mapsto \operatorname{len} \mathfrak{U}_I$ is degreewise bounded. (For a scheme $X/\mathbf{k}$ we put $\operatorname{len}(X) = \dim_{\mathbf{k}}(\mathcal{O}_X)$ if X is finite and $\operatorname{len}(X) = \infty$ otherwise.) It is upper semi-continuous, but not cone-stable. □

4.2. **Invariants of singularities.** Our ideal invariants are integer-valued. However, there are interesting ideal invariants that are not integer-valued. The methods used in the proof of Theorem 1.1 can sometimes be applied to these invariants. Here is an example:

Proposition 4.4. *Suppose $\mathbf{k}$ has characteristic 0, and fix $\mathbf{d}$. Let Λ be the set of rational numbers occurring as the log-canonical threshold (at the cone point) of some type $\mathbf{d}$ ideal. Then Λ satisfies the descending chain condition.*

Proof. Cone-stability follows from the definition and lower semi-continuity follows from [Laz04, Section 9.3]. For each $\lambda \in \Lambda$ and each $\mathbf{f} \in X_{\mathbf{d},n}$ we say that $\mathbf{f}$ lies Z_λ if and only if the log canonical threshold at the cone point $V(f_1, \ldots, f_r) \subset \mathbf{A}^n$ is at most λ. Since Z is $\mathbf{GL}$-invariant, and since $Z_\lambda \cap X_{\mathbf{d},n}$ is closed by semicontinuity, it follows that Z_λ is closed. An infinite decreasing chain of values in Λ would thus yield an infinite decreasing chain of $\mathbf{GL}$-stable closed subsets in $X_{\mathbf{d}}$, contradicting Theorem 1.3. □

Remark 4.5. A similar statement holds for F-pure thresholds in positive characteristic. Here the semi-continuity follows from [MY09, Theorem 5.1]. □

4.3. **Previously known degreewise bounded invariants.** Many ideal invariants have been previously shown to be degreewise bounded. We catalogue some here to hint at the ubiquity of the phenomenon.

Proposition 4.6. *The following invariants of an ideal I are known to be degreewise bounded by previous results in the literature:*

(1) *The degree of I.*
(2) *The maximal codimension of a minimal or associated prime of I.*
(3) *The projective dimension of I.*
(4) *The Castelnuovo–Mumford regularity of I.*
(5) *The Betti number $\beta_{i,j}(I)$ for any i, j.*
(6) *The sum of the degrees of: the minimal primes of I, the minimal primary components of I, or the embedded primes of I; the rth arithmetic degree of I.*[1]
(7) *The number of: minimal primes, embedded primes, or associated primes of I.*
(8) *The degree of $\sqrt{I}$.*
(9) *The minimal B such that the symbolic power $I^{(Br)}$ belongs to I^r for all r.*
(10) *The minimal B such that $\sqrt{I}^B \subseteq I$.*

[1]The embedded primary components of an ideal are not uniquely defined, but one can make sense of the sum of the degrees of all of the embedded primary components of an ideal of a given dimension by using the notion of arithmetic degree [BM93, Definition 3.4].

(11) *The largest degree of a generator, or the number of generators, for: any associated prime of I, the radical $\sqrt{I}$, or the symbolic power $I^{(r)}$ for any integer r.*

Proof. (1) Refined Bezout's Theorem [Ful98, Theorem 12.3].

(2) The principal ideal theorem [Eis95, Theorem 10.2].
(3) Stillman's conjecture [AH16].
(4) Stillman's conjecture combined with Caviglia's theorem [Pee11, Theorem 29.5].
(5) We can combine (3) and (4) with Boij–Söderberg theory [ES09, Theorems 0.1, 0.2] to see that only finitely many Betti tables for $\beta(S/I)$ are possible.
(6) For minimal primes, this is Refined Bezout's Theorem [Ful98, Theorem 12.3]; for associated primes, use [BM93, Proposition 3.6], noting that the exponent $n-r$ in that formula is bounded by (2) above.
(7) Follows from (6).
(8) Follows from (6).
(9) Follows from (2) plus [HH02, Theorem 1.1(c)].
(10) Follows from the Effective Nullstellensatz [Bro87, Kol88, Som99].
(11) For an associated prime P of I, or for a minimal primary component Q of I, this follows from [AH16, Theorem D(b)]. Since $\sqrt{I}$ is the intersection of the minimal primes of I, it suffices to show that we can bound the number and degree of defining equations of $J \cap J'$ in terms of the number and degree of defining equations of J and J'. Using parts (3) and (4) above, we can bound the regularity and projective dimension of J, J' and $J + J'$. Using the exact sequence relating these ideals to $J \cap J'$, we can bound the regularity and projective dimension of $J \cap J'$ as well. There are thus only a finite number of possible revlex gins of $J \cap J'$, yielding the desired bounds. This proves the statement for $\sqrt{I}$; a similar argument works for symbolic powers. □

Remark 4.7. Combining parts (11) and (4) answers [PS09, Problem 3.9]. See also [CD03, Rav90] for related work on the degreewise boundedness of taking radicals. □

5. Additional comments

5.1. **A converse theorem.** Before Draisma's paper [Dra17] appeared, we had Theorem 1.1 in the form "if sums of symmetric powers are topologically noetherian then ideal invariants are bounded." In fact, we also had a proof of the converse statement, and the main theorem of an earlier version of this paper was the equivalence of these two statements. The proof of the converse direction relies on a general equivariant version of the Hilbert basis theorem that we believe to be of independent interest. This will appear in [ESS].

5.2. **An improvement to Theorem 1.3.** Let $Y_{\le d}$ be the set of all isomorphism classes of finitely generated homogeneous ideals of $A = \mathbf{k}[x_1, x_2, \ldots]$ that are generated in degrees at most d (but with no condition on the number of generators). We topologize $Y_{\le d}$ similarly to $Y_{\mathbf{d}}$. The following conjectural statement greatly strengthens Theorem 1.3:

Conjecture 5.1. *The space $Y_{\le d}$ is noetherian.*

We give two consequences of the conjecture.

Proposition 5.2. *Let $\mathcal{H}$ be the set of all polynomials of the form $\mathrm{HN}_{A/I}(t)$ with I a homogeneous ideal finitely generated in degrees at most d. Endow $\mathcal{H}$ with the partial order $\le$ from §3.5. Then $\mathcal{H}$ satisfies the ascending chain condition.*

We say that an ideal invariant is **strongly upper semi-continuous** if for any family of ideals $\mathcal{I}$ over S (with no flatness condition imposed), the function $s \mapsto \nu(\mathcal{I}_s)$ is upper semi-continuous. (Here $\mathcal{I}_s$ denotes the ideal at s, which is a homomorphic image of the fiber.) We say that ν is **strongly degreewise bounded** if for every d there exists a B such that $\nu(I) \le B$ for any homogeneous ideal I finitely generated in degrees at most d.

Proposition 5.3. *Suppose Conjecture 5.1 holds. Then any cone-stable strongly upper semi-continuous ideal invaraint is strongly degreewise bounded.*

Remark 5.4. Theorem 1.4 fails for $Y_{\le d}$. Even $Y_{\le 1}$ would need a separate strata for each integer c consisting of the isomorphism class of the ideal $\langle x_1, x_2, \dots, x_c \rangle$. □

Remark 5.5. Let Y be the set of isomorphism classes of all finitely generated ideals in A. We claim that Y is not noetherian. Indeed, if it were then the set of all polynomials of the form $\mathrm{HN}_{A/I}$, with I any finitely generated homogeneous ideal, would satisfy the ascending chain condition. But it does not: indeed, $1 - t^d$ is a Hilbert numerator for any $d \ge 0$, and these form an ascending chain. □

5.3. **Boundedness of tca ideal invariants.** Draisma's theorem states that if V is any finite length polynomial representation of $\mathbf{GL}_\infty$ then $\mathrm{Spec}(\mathrm{Sym}(V))$ is topologically noetherian. In our proof of Theorem 1.3, we only applied this result with V being a finite sum of symmetric powers. It is natural to wonder, therefore, if the remaining cases of Draisma's theorem have implications for ideal invariants. We now give one possible answer to this question.

Recall that a **twisted commutative algebra** (tca) over $\mathbf{k}$ is a commutative associative unital graded $\mathbf{k}$-algebra $A = \bigoplus_{k \ge 0} A_k$ equipped with an action of the symmetric group S_k on A_k satisfying certain conditions, the most important being that multiplication is commutative up to a "twist" by S_k; see [SS12] for the full definition. Let $A_n = \mathbf{k}\langle x_1, \dots, x_n \rangle$ be the tca freely generated by n indeterminates of degree one. Explicitly, A_n is the tensor algebra on $\mathbf{k}^n$, equipped with the natural action of S_k on the kth tensor power.

We define a **tca ideal invariant** to be a rule associating to every ideal I in any A_n a quantity $\nu(I) \in \mathbf{Z} \cup \{\infty\}$, depending only on (A_n, I) up to isomorphism. One can then prove:

Theorem 5.6. *Any cone-stable strongly upper semi-continuous tca ideal invariant is degreewise bounded.*

The proof is similar to before. We define a space $Y_{\mathbf{d}}$ parametrizing ideals of type $\mathbf{d}$ in $A = \mathbf{k}\langle x_1, x_2, \dots \rangle$. Using Draisma's theorem (now applied to arbitrary polynomial functors), we deduce that $Y_{\mathbf{d}}$ is noetherian. We no longer have the analog of Theorem 1.4 (this is an interesting open problem), which is why we restrict to strongly upper semi-continuous invariants in the theorem: this ensures that the locus $Z_n \subset Y_{\mathbf{d}}$ where $\nu \ge n$ is closed.

The projective dimension of any non-trivial ideal in the tca A_n is infinite. However, it is known [SS17b, GL17] that the regularity of any ideal is finite. One can therefore formulate the following tca analog of Stillman's conjecture:

Question 5.7 (TCA Stillman)**.** *Is the tca ideal invariant "regularity" degreewise bounded?*

Regularity is only weakly semi-continuous, so Theorem 5.6 does not apply to this question. We do not know if a positive answer to this question would imply a version of Theorem 1.4 in this setting, as the tools used in the proof of Theorem 3.4 do not yet exist for tca's.

5.4. **Degreewise bounded invariants of modules.** Fix a doubly indexed sequence $\mathbf{d} = (d_{1,1}, d_{1,2}, \dots, d_{1,b}, d_{2,1}, \dots, d_{a,b})$ in $\mathbf{Z}^{a\times b}$. We say that a graded module is **type d** if it admits a presentation matrix

$$\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,b} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,b} \\ \vdots & & \ddots & \vdots \\ f_{a,1} & f_{a,2} & \cdots & f_{a,b} \end{pmatrix}$$

where $\deg(f_{i,j}) = d_{i,j}$ for all i, j. A **module invariant** is a rule that associates to every graded module M over any standard-graded polynomial ring A a quantity in $\mathbf{Z} \cup \{\infty\}$ that only depends on the pair (A, M) up to graded isomorphism. We extend the notions of cone-stability, weak upper semi-continuity, and degreewise boundedness in the natural way.

Theorem 5.8. *Any module invariant that is cone-stable and weakly upper semi-continuous is degreewise bounded.*

The proof is similar to that of Theorem 1.1, so we omit the details.

5.5. **Boundedness questions for Gröbner bases.** One natural question, first posed to us by Hochster, is whether there exist degreewise bounds for invariants of initial ideals, where the bounds are in terms of the generating degrees of the original ideal.

Question 5.9. *For a term order $>$, can the regularity (or projective dimension or other invariants) of the initial ideal $\mathrm{in}_>(I)$ be bounded in terms of the generating degrees of I?*

If one uses the reverse lexicographic order and works in generic coordinates, then the regularity and projective dimension of an initial ideal are the same as for the original ideal by [BS87, Corollary 2.5]. Thus, by [AH16] these invariants are degreewise bounded. But there appears to be no work on this question without passing to generic coordinates.

We show that Question 5.9 has a negative answer if one works with lexicographic order.

Proposition 5.10. *For an ideal I, none of the following invariants of $\mathrm{in}_{\mathrm{lex}}(I)$ can be bounded in terms of the defining degrees of I: regularity, projective dimension, number of generators, or maximal degree of a generator.*

Sketch of Proof. We will show that for ideals generated by a single quadric and a single cubic, each of these invariants can go to ∞. Let $q_n = x_1^2+x_2^2+\cdots+x_n^2$ and $c_n = x_1^3+x_2^3+\cdots+x_n^3$ in A_n. Let L_n be the lexicographic initial ideal of $\langle q_n, c_n\rangle$. By semi-continuity of Betti numbers under reduction mod p, it suffices to assume that the ground field has characteristic 0.

Define a map $\varphi\colon A_{n-1} \to A_n$ by shifting indices up by one, so that $\varphi(x_j) := x_{j+1}$. We claim that $(L_n : x_1) \subseteq \langle x_1\rangle + \varphi(L_{n-1})$. For the first inclusion, we observe that $x_1^2 \in L_n$ and then we let $m \in L_{n-1}$ be the lex-leading term of $aq_{n-1} + bc_{n-1}$. One can check

$$(x_1\varphi(a) - x_1^2\varphi(b) - \varphi(bq_{n-1}))q_n + (x_1\varphi(b) - \varphi(a))c_n = x_1\varphi(aq_{n-1}+bc_{n-1}) - \varphi(bq_{n-1}^2) - \varphi(ac_{n-1}).$$

Since any polynomial of the form $\varphi(f)$ does not involve the variable x_1, the lex-leading term here is x_1 times the lex-leading term of $\varphi(aq_{n-1} + bc_{n-1})$. We thus have $\varphi(L_{n-1}) + \langle x_1\rangle \subseteq (L_n : x_1)$. We may confirm that $(L_n : x_1) = \langle x_1\rangle + \varphi(L_{n_1})$ by an involved but elementary Hilbert series computation, which we omit.

We now prove by induction that $m_n := x_1x_2\cdots x_n$ is a socle element of A_n/L_n for $n \geq 2$. For $n = 2$ this is a direct computation (here we use characteristic $\neq 2$), and for $n > 2$ we combine the fact that $(L_n : x_1) = \varphi(L_{n-1}) + \langle x_1 \rangle$ with the short exact sequence

$$0 \to \frac{A_n}{(L_n : x_1)}(-1) \xrightarrow{\cdot x_1} \frac{A_n}{L_n} \longrightarrow \frac{A_n}{\langle x_1, x_2^6 \rangle} \to 0.$$

The statements of the proposition can then be derived. □

In light of Proposition 5.10 and [BS87], Question 5.9 may be most interesting for the reverse lexicographic order.

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