# FINITENESS OF K3 SURFACES AND THE TATE CONJECTURE

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ABSTRACT. Given a finite field $k$ of characteristic $p \geq 5$, we show that the Tate conjecture holds for K3 surfaces over $\overline{k}$ if and only if there are finitely many K3 surfaces defined over each finite extension of $k$.

CONTENTS



## 1. INTRODUCTION

Given a class of algebraic varieties, it is reasonable to ask if there are only finitely many members defined over a given finite field. While this is clearly the case when the appropriate moduli functor is bounded, matters are often not so simple. For example, consider the case of abelian varieties of a given dimension $g$. There is no single moduli space parameterizing them; rather, for each integer $d \geq 1$ there is a moduli space parameterizing abelian varieties of dimension $g$ with a polarization of degree $d$. It is nevertheless possible to show (see, for example, [Mi, Cor. 13.13]) that there are only finitely many abelian varieties over a given finite field, up to isomorphism. Another natural class of varieties where this difficulty arises is the case of K3 surfaces. As with abelian varieties, there is not a single moduli space but rather a moduli space for each even integer $d \geq 2$, parametrizing K3 surfaces with a polarization of degree $d$.

In this paper, we consider the finiteness question for K3 surfaces over finite fields. Given a K3 surface $X$ defined over a finite field $k$ of characteristic $p$, the Tate conjecture predicts that the natural map

$$\mathrm{Pic}(X) \otimes \mathbf{Q}_\ell \to \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_\ell(1))^{\mathrm{Gal}(\overline{k}/k)}$$

is surjective for $\ell \neq p$. It admits many alternate formulations; for example, it is equivalent to the statement that the Brauer group of $X$ is finite. We say that $X/k$ *satisfies the Tate conjecture over some extension* $k'/k$ if the Tate conjecture holds for the base change $X_{k'}$.

Our main result is that this conjecture is essentially equivalent to the finiteness of the set of K3 surfaces over $k$. To state our result precisely, let us fix some notation. Let $k$ be a finite field of characteristic $p$ and let $K$ be the extension of $k$ of degree

$$N = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 465585120.$$

The relevance of this number is easy to explain: if $X$ is a K3 surface over $k$ then all Tate classes and line bundles on $X_{\overline{k}}$ descend to $X_K$.

Our main theorem is the following:

**Main Theorem.** *Assume the characteristic* $p$ *is at least 5.*

(1) *There are only finitely many isomorphism classes of K3 surfaces over* $k$ *which satisfy the Tate conjecture over* $K$.
(2) *If there are only finitely many isomorphism classes of K3 surfaces over* $K$ *then every K3 surface over* $k$ *satisfies the Tate conjecture over* $K$.

*In particular, the Tate conjecture holds for all K3 surfaces over* $\overline{k}$ *if and only if there are only finitely many K3 surfaces defined over each finite extension of* $k$.

*Remark.* It is not difficult to see that a K3 surface defined over $k$ satisfies the Tate conjecture over $K$ if and only if it satisfies it over every finite extension of $k$.

As the Tate conjecture is known for K3 surfaces of finite height in characteristic at least 5 [NO], we obtain the following unconditional corollary:

**Corollary.** *Assume* $p \geq 5$. *There are only finitely many isomorphism classes of K3 surfaces of finite height defined over* $k$.

Our argument proceeds as follows. To obtain finiteness from Tate, it suffices to prove the existence of low-degree polarizations on K3 surfaces over $k$. In order to do this, we use the Tate conjecture in both $\ell$-adic and crystalline cohomology to control the possibilities of the Néron-Severi lattice. For the other direction, we use the finiteness statement and the existence of infinitely many Brauer classes to create a K3 surface with infinitely many twisted Fourier-Mukai partners. Since this cannot happen in characteristic zero, we obtain a contradiction by proving a lifting result.

**Notation.** Throughout, $k$ denotes a finite field of characteristic $p$ and cardinality $q = p^f$. We fix an algebraic closure $\overline{k}$ of $k$ and let $K$ denote the unique extension of $k$ within $\overline{k}$ of degree $N = 465585120$.

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## 2. Tate implies finiteness

2.1. **Discriminant bounds for the étale and crystalline lattices.** In this section, we produce bounds on the discriminants of certain lattices constructed from the étale and crystalline cohomologies of K3 surfaces over $\mathbf{F}_q$. We begin by recalling some terminology. Let $A$ be a principal ideal domain. By a *lattice* over $A$, we mean a finite free $A$ module $M$ together with a symmetric $A$-linear form $(,) : M \otimes_A M \to A$. We say that $M$ is *non-degenerate* (resp. *unimodular*) if the map $M \to \mathrm{Hom}_A(M, A)$ provided by the pairing is injective (resp. bijective). The *discriminant* of a lattice $M$, denote $\mathrm{disc}(M)$, is the determinant of the matrix $(e_i, e_j)$, where $\{e_i\}$ is a basis for $M$ as an $A$-module; it is a well-defined element of $A/(A^\times)^2$. The lattice $M$ is non-degenerate (resp. unimodular) if and only if its discriminant is non-zero (resp. a unit). Note that the valuation of $\mathrm{disc}(M)$ at a maximal ideal of $A$ is well-defined.

We will need a simple lemma concerning discriminants:

**Lemma 2.1.1.** *Let $A$ be a discrete valuation ring with uniformizer $t$. Let $M$ be a lattice over $A$ and let $M' \subset M$ be an $A$-submodule such that $M/M'$ has length $r$ as an $A$-module. Regard $M'$ as a lattice by restricting the form from $M$. Then $\mathrm{disc}(M') = t^{2r}\,\mathrm{disc}(M)$ up to units of $A$.*

*Proof.* Let $e_1, \dots, e_n$ be a basis for $M$ and let $f_1, \dots, f_n$ be a basis for $M'$. Let $B$ be the matrix $(e_i, e_j)$ and let $B'$ be the matrix $(f_i, f_j)$. Thus $\mathrm{disc}(M) = \det B$ and $\mathrm{disc}(M') = \det B'$. Let $C \in M_n(A)$ be the change of basis matrix, so that $f_i = Ce_i$. Then $\det(C) = t^r$ up to units of $A$. As $B' = C^t BC$, the result follows. □

Let $X$ be a K3 surface over $k$. For a prime number $\ell \neq p$ put

$$M_\ell(X) = \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell), \qquad N_\ell(X) = M_\ell(X)^{\phi=q}.$$

Then $M_\ell(X)$ is a free $\mathbf{Z}_\ell$-module of rank 22, and the cup product gives it the structure of a unimodular lattice. The space $M_\ell(X)$ admits a natural $\mathbf{Z}_\ell$-linear automorphism $\phi$, the geometric Frobenius element of $\mathrm{Gal}(\overline{k}/k)$. The map $\phi$ does not quite preserve the form, but satisfies $(\phi x, \phi y) = q^2(x, y)$. It is known [D] that the action of $\phi$ on $M_\ell(X)$ is semi-simple. We give $N_\ell(X)$ the structure of a lattice by restricting the form from $M_\ell(X)$.

**Proposition 2.1.2.** *There exist constants $C_1 = C_1(k)$ and $C_2 = C_2(k)$ with the following properties. Let $X$ be a K3 surface over $k$ and let $\ell \neq p$ be a prime number. Then*

1. *For $\ell > C_1$, the discriminant of $N_\ell(X)$ has $\ell$-adic valuation zero.*
2. *The discriminant of $N_\ell(X)$ has $\ell$-adic valuation at most $C_2$.*

*Proof.* We first define the constants $C_1$ and $C_2$. Let $\mathcal{W}$ be the set of all Weil integers of weight $q$ and degree at most 22. It is easy to bound the coefficients of the minimal polynomial of an element of $\mathcal{W}$, and so one sees that $\mathcal{W}$ is a finite set. Let $S$ denote the set of elements of $\mathbf{Z}[T]$ which are monic of degree 22 and whose roots belong to $\mathcal{W}$. Clearly, $S$ is a finite set; enumerate its elements as $f_1(T), \dots, f_m(T)$. We can factor each $f_i(T)$ as $g_i(T)h_i(T)$, where $g_i(T)$ is a power of $T - q$ and $h_i(T)$ is an element of $\mathbf{Z}[T]$ which does not have $q$ as a root. For each $i$, pick rational polynomials $a_i(T)$ and $b_i(T)$ such that

$$a_i(T)g_i(T) + b_i(T)h_i(T) = 1.$$

Let $N$ be the least common multiple of the denominators of the coefficients of $a_i(T)$ and $b_i(T)$. Let $r$ be the maximal integer such that $\ell^r$ divides $N$, for some prime $\ell$, and let $\ell_0$ be the largest prime dividing $N$. We claim that we can take $C_1 = \ell_0$ and $C_2 = 44r$.

We now prove these claims. Thus let $X$ and $\ell$ be as in the statement of the proposition and put $M = M_\ell(X)$ and $N = N_\ell(X)$. The characteristic polynomial of $\phi$ belongs to $S$, and is thus equal to $f_i(T)$ for some $i$. Put $M_1 = h_i(\phi)M$ and $M_2 = g_i(\phi)M$. One easily sees that $M_1 \oplus M_2$ is a finite index $\mathbf{Z}_\ell$-submodule of $M$ and that $M_1$ and $M_2$ are orthogonal. Furthermore, $M_1$ is contained in $N$, since $\phi$ is semi-simple, and has finite index.

Suppose that $\ell > C_1$. Then $a_i(T)$ and $b_i(T)$ belong to $\mathbf{Z}_\ell[T]$ and so $M = M_1 \oplus M_2$. Thus $\mathrm{disc}(M) = \mathrm{disc}(M_1)\,\mathrm{disc}(M_2)$. Since $M$ is unimodular, $\mathrm{disc}(M)$ has $\ell$-adic valuation 0, and so it follows that $\mathrm{disc}(M_1)$ has $\ell$-adic valuation 0 as well. As $N$ and $M_1$ are saturated in $M$ and $M_1 \subset N$, we have $N = M_1$, and so (a) follows.

Now suppose that $\ell$ is arbitrary. Then $\ell^r a_i(T)$ and $\ell^r b_i(T)$ belong to $\mathbf{Z}_\ell[T]$. It follows that $M_1 \oplus M_2$ contains $\ell^r M$, and thus has index at most $\ell^{22r}$ in $M$. By Lemma 2.1.1, we find that $\operatorname{disc}(M_1)\operatorname{disc}(M_2)$ divides $\ell^{44r}\operatorname{disc}(M) = \ell^{44r}$. The lemma also shows that $\operatorname{disc}(N)$ divides $\operatorname{disc}(M_1)$. It follows that $\operatorname{disc}(N)$ has $\ell$-adic valuation at most $44r = C_2$, which proves (b). □

We also need a version of the above result at $p$. Let $W = W(k)$ be the Witt ring of $k$. Put

$$M_p(X) = \mathrm{H}^2_{\mathrm{cris}}(X/W), \qquad N_p(X) = M_p(X)^{\phi_0 = p}.$$

Then $M_p(X)$ is a free $W$-module of rank 22, and the cup product gives it the structure of a unimodular lattice. The lattice $M_p(X)$ admits a natural semilinear automorphism $\phi_0$, the crystalline Frobenius. The map $\phi = \phi_0^f$ is $W$-linear (where $q = p^f$). We have $(\phi_0 x, \phi_0 y) = p^2\phi_0((x, y))$. (Note: the $\phi_0$ on the right is the Frobenius on $W$.) Since $\phi_0$ is only semi-linear, $N_p(X)$ is not a $W$-module, but a $\mathbf{Z}_p$-module. We give $N_p(X)$ the structure of a lattice via the form on $M_p(X)$.

We say that an eigenvalue $\alpha$ of a linear map is semi-simple if the $\alpha$-eigenspace coincides with the $\alpha$-generalized eigenspace. We now come to the main result at $p$:

**Proposition 2.1.3.** *There exists a constant $C_3 = C_3(k)$ with the following property. Let $X$ be a K3 surface over $k$. Assume that $q$ is a semi-simple eigenvalue of $\phi$ on $M_p(X)[1/p]$. Then the discriminant of $N_p(X)$ has $p$-adic valuation at most $C_3$.*

*Proof.* Let $X$ be given, and put $N' = M_p(X)^{\phi=q}$, so that $N_p(X) = (N')^{\phi_0=p}$. By exactly the same reasoning used in the proof of Proposition 2.1.2, we can bound the valuation of the discriminant of $N'$ (regarded as a lattice over $W$); in fact, the valuation is still bounded by $C_2$. The following lemma (which defines a constant $C_4$) now shows that the $p$-adic valuation of $\operatorname{disc}(N_p(X))$ is bounded by $C_3 = C_2 f + 44C_4 f$. □

**Lemma 2.1.4.** *There exists a constant $C_4 = C_4(k)$ with the following property. Let $N'$ be a lattice over $W$ of rank $n$. Let $\phi_0$ be a semi-linear endomorphism of $N'$ satisfying $\phi_0^f = q$ and $(\phi_0 x, \phi_0 y) = p^2\phi_0((x, y))$, and put $N = (N')^{\phi_0=p}$. Then $v_p(\operatorname{disc}(N))$ is at most $f v_p(\operatorname{disc}(N')) + 2C_4 n f$.*

*Proof.* Let $g(T) = T - p$ and let $h(T) = (T^f - q)/(T - p)$, two monic polynomials in $\mathbf{Z}_p[T]$. Pick polynomials $a(T)$ and $b(T)$ in $\mathbf{Q}_p[T]$ such that

$$a(T)g(T) + b(T)h(T) = 1.$$

Let $r$ be such that $p^r a(T)$ and $p^r b(T)$ belong to $\mathbf{Z}_p[T]$. We claim that we can take $C_4 = r$.

Thus let $N'$ with $\phi_0$ be given. Let $L'$ be $N'$, regarded as a $\mathbf{Z}_p$-lattice with pairing $\langle , \rangle$ given by $\langle x, y\rangle = \operatorname{tr}_{W/\mathbf{Z}_p}(x, y)$. We have $\operatorname{rk}(L') = fn$ and $\operatorname{disc}(L') = \mathbf{N}(\operatorname{disc}(N'))$, where $\mathbf{N}$ is the norm from $W$ to $\mathbf{Z}_p$. We regard $\phi_0$ as a linear map of $L'$. As such, it is semi-simple (when $p$ is inverted) with minimal polynomial $T^f - q$. The identity $\langle \phi_0 x, \phi_0 y\rangle = p^2\langle x, y\rangle$ holds. Let $L$ be the $p$ eigenspace of $\phi_0$, regarded as a sublattice of $L'$. Then $L$ and $N$ are the same subset of $N'$, but the form on $L$ is that on $N$ scaled by $f$. It follows that $\operatorname{disc}(N)$ divides $\operatorname{disc}(L)$.

Put $L_1 = h(\phi_0)L'$ and $L_2 = g(\phi_0)L'$. Then $L_1$ and $L_2$ are orthogonal under $\langle , \rangle$ and $L_1 \oplus L_2$ has index at most $p^{fnr}$ in $L'$. It follows from Lemma 2.1.1 that $\operatorname{disc}(L_1)$ divides $p^{2fnr}\operatorname{disc}(L')$. Since $L_1$ is contained in $L$, we find that $\operatorname{disc}(L)$ divides $p^{2fnr}\operatorname{disc}(L')$ as well. Finally, we see that $\operatorname{disc}(N)$ divides $p^{2fnr}\mathbf{N}(\operatorname{disc}(N'))$, and so $v_p(\operatorname{disc}(N))$ is bounded by $2fnr + f v_p(\operatorname{disc} N')$. □

*Remark* 2.1.5. As far as we are aware, it is not known if the action of $\phi$ on $M_p(X)[1/p]$ is semi-simple. See [O2] for a partial result. However, for K3 surfaces which satisfy the Tate conjecture, semi-simplicity is known, and can be easily be deduced from the aforementioned result of Ogus.

2.2. **Controlling the Néron–Severi lattice.** We assume for the rest of §2 that $p \geq 5$. Let $X$ be a K3 surface over $k$. Write $\mathrm{NS}(X)$ for the Néron–Severi group of $X$, which is a lattice (over $\mathbf{Z}$) under the intersection pairing. The main result of this section is the following:

**Proposition 2.2.1.** *There exists a finite set $\mathscr{L} = \mathscr{L}(k)$ of lattices over $\mathbf{Z}$ with the following property: if $X$ is a K3 surface over $k$ which satisfies the Tate conjecture (over $k$) then $\mathrm{NS}(X)$ is isomorphic to a member of $\mathscr{L}$.*

We handle the finite and infinite height cases separately. In the finite height case, the Tate conjecture is known. We will need integral refinements of it, both at $\ell$ and at $p$, which we now give.

**Lemma 2.2.2.** *Let $X$ be a K3 surface over $k$ which satisfies the Tate conjecture and let $\ell \neq p$ be a prime. Then the map*
$$c_1 : \mathrm{NS}(X) \otimes \mathbf{Z}_\ell \to N_\ell(X)(1)$$
*is an isomorphism.*

*Proof.* Since $X$ satisfies the Tate conjecture, $c_1$ is an isomorphism after inverting $\ell$; as is well-known, the Kummer sequence then implies that $c_1$ itself is an isomorphism. □

**Lemma 2.2.3.** *Let $X$ be a K3 surface of finite height over $k$. Then the map*
$$c_1 : \mathrm{NS}(X) \otimes \mathbf{Z}_p \to N_p(X)\{1\}$$
*is an isomorphism.*

*Proof.* Let $W = W(k)$ be the Witt ring and let $E$ be the field of fractions of $W$. The main theorem of [NO] shows that $c_1$ is an isomorphism after inverting $p$. To show that $c_1$ itself is an isomorphism, it is enough to show that the image of $\mathrm{NS}(X) \otimes W\{-1\}$ in $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ is saturated. (We write $\{1\}$ for the crystalline version of Tate twist, so that $W\{-1\}$ is $W$ with $\phi_0$ acting by $1 \mapsto p$ and filtration concentrated in degree 1.) Let $\mathfrak{X}$ be a lift of $X$ to $W$ such that the restriction map $\mathrm{NS}(\mathfrak{X}) \to \mathrm{NS}(X)$ is an isomorphism; such a lift exists by [LM, Cor 4.2]. Of course, it suffices to show that the image of $\mathrm{NS}(\mathfrak{X}) \otimes W\{-1\}$ in $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ is saturated.

We now need to make use of some integral $p$-adic Hodge theory, specifically, the theories of Fontaine–Laffaille and Fontaine–Messing. See [BM, §3] for a review of these theories. For a strongly divisible module $M$ over $W$, we let $T^\vee(M)$ be the $\mathbf{Z}_p$-lattice $\mathrm{Hom}_{W,\mathrm{Fil},\phi}(M, A_{\mathrm{cris}})$ and we let $T(M)$ be the $\mathbf{Z}_p$-dual of $T^\vee(M)$. The map $c_1 : \mathrm{NS}(X) \otimes W\{-1\} \to \mathrm{H}^2_{\mathrm{cris}}(\mathfrak{X}/W)$ is a map of strongly divisible modules. By basic properties of the functor $T$, we can check if the image of $c_1$ is saturated after applying $T$. Now, we have a commutative diagram
$$\begin{tikzcd}
\mathrm{NS}(\mathfrak{X}) \otimes \mathbf{Z}_p(-1) \arrow[r] & \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\overline{E}}, \mathbf{Q}_p) \\
\mathrm{NS}(\mathfrak{X}) \otimes T(W\{-1\}) \arrow[r] \arrow[u] & T(\mathrm{H}^2_{\mathrm{cris}}(X/W)) \arrow[u]
\end{tikzcd}$$
The left arrow is just the isomorphism $T(W\{-1\}) = \mathbf{Z}_p(-1)$, while the right arrow is the Fontaine–Messing isomorphism (which exists since $p-1$ is greater than the degree of the cohomology group in consideration, namely 2). The top arrow is the usual étale Chern class while the bottom arrow is $T$ applied to the crystalline Chern class map. The commutativity of the diagram is a basic property of the $p$-adic comparison map. It thus suffices to show that the image of the top map is saturated. Now, $\mathrm{NS}(\mathfrak{X}) = \mathrm{NS}(\mathfrak{X}_E)$, and $\mathrm{NS}(\mathfrak{X}_E)$ is saturated in $\mathrm{NS}(\mathfrak{X}_{\overline{E}})$. Thus it suffices to show that $\mathrm{NS}(\mathfrak{X}_{\overline{E}})(-1)$ is saturated in $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\overline{E}}, \mathbf{Q}_p)$. However, this is well-known (replace $\overline{E}$ by $\mathbf{C}$ and use the exponential sequence). □

We now return to the proof of the proposition.

*Proof of Proposition 2.2.1.* Let $X$ be a finite height K3 surface over $k$. Let $\ell \neq p$ be a prime number. By Lemma 2.2.2, $\mathrm{NS}(X) \otimes \mathbf{Z}_\ell$ is isomorphic, as a lattice, to $N_\ell(X)(1)$. Since $N_\ell(X)(1)$ is isomorphic, as a lattice, to $N_\ell(X)$, we find that $\mathrm{disc}(\mathrm{NS}(X))$ and $\mathrm{disc}(N_\ell(X))$ have the same $\ell$-adic valuations. Similarly, appealing to Lemma 2.2.3, we find that $\mathrm{disc}(\mathrm{NS}(X))$ and $\mathrm{disc}(N_p(X))$ have the same $p$-adic valuations. Applying Propositions 2.1.2 and 2.1.3, we find that $|\,\mathrm{disc}(\mathrm{NS}(X))|$ is at most $p^{C_3} \prod_{\ell \leq C_1} \ell^{C_2}$. As there are only finitely many isomorphism classes of lattices of a given rank and discriminant [Cas, Ch. 9, Thm. 1.1], it follows that there are only finitely many possibilities for $\mathrm{NS}(X)$ (up to isomorphism).

Now let $X$ be an infinite height K3 surface over $k$ which satisfies the Tate conjecture. By [O, §1.7] the discriminant of $\mathrm{NS}(X_{\overline{k}})$ is $-p^a$ where $1 \leq a \leq 20$ is even. The Frobenius element $\phi$ of $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathrm{NS}(X_{\overline{k}})$ with finite order; it is therefore semi-simple and its eigenvalues are roots of unity (of a bounded degree). Since $\mathrm{NS}(X) = \mathrm{NS}(X_{\overline{k}})^{\phi=1}$, we can bound $\mathrm{disc}(\mathrm{NS}(X))$ using an argument similar to that of Proposition 2.1.2. Appealing again to the finiteness of lattices with given rank and discriminant, we find that there are only finitely many possibilities for $\mathrm{NS}(X)$ (up to isomorphism). □

2.3. **Constructing low-degree ample line bundles.** The purpose of this section is to prove the following proposition:

**Proposition 2.3.1.** *There exists a constant $C_5 = C_5(k)$ with the following property: if $X$ is a K3 surface over $k$ which satisfies the Tate conjecture over $K$ then $X$ admits an ample line bundle of degree at most $C_5$ defined over $K$.*

We begin with a lemma.

**Lemma 2.3.2.** *Let $X$ and $X'$ be K3 surfaces over algebraically closed fields such that $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic as lattices. Then the set of degrees of ample line bundles on $X$ and $X'$ coincide.*

*Proof.* Put $N = \mathrm{NS}(X)$ and $N_{\mathbf{R}} = N \otimes \mathbf{R}$. Let $\Delta$ be the set of elements $\delta$ in $N$ such that $(\delta, \delta) = -2$. For $\delta \in \Delta$, let $r_\delta : N_{\mathbf{R}} \to N_{\mathbf{R}}$ be the reflection given by $r_\delta(x) = x + (x, \delta)\delta$. Let $\Gamma$ be the group generated by the $r_\delta$, with $\delta \in \Delta$. Finally, let $V$ be the set of elements $x$ in $N_{\mathbf{R}}$ such that $(x, x) > 0$ and $(x, \delta) \neq 0$ for all $\delta \in \Delta$. Then $V$ is an open subset of $N_{\mathbf{R}}$ and the group $\pm\Gamma$ acts transitively on the set of connected components of $V$ [O, Prop. 1.10]. Furthermore, there exists a unique connected component $V_0$ of $V$ such that an element of $N$ is ample if and only if it lies in $V_0$ [O, p. 371].

Let $N'$, etc., be as above but for $X'$. Choose an isomorphism of lattices $i : N \to N'$. Clearly, $i$ induces a bijection $\Delta \to \Delta'$ and a homeomorphism $V \to V'$. Thus $i(V_0)$ is some connected component of $V'$. We can find an element $\gamma$ of $\pm\Gamma'$ such that $\gamma(i(V_0)) = V_0'$. Thus, replacing $i$ by $\gamma i$, we can assume that $i(V_0) = V_0'$. It then follows that $i$ induces a bijection between the set of ample elements in $N$ and the set of ample elements in $N'$. Since $i$ preserves degree, this proves the lemma. □

**Corollary 2.3.3.** *For every lattice $L$ there is an integer $d(L)$ with the following property: if $X$ is a K3 surface over an algebraically closed field such that $\mathrm{NS}(X)$ is isomorphic to $L$ then $X$ admits an ample line bundle of degree $d(L)$.*

We now return to the proof of Proposition 2.3.1.

*Proof of Proposition 2.3.1.* Let $X$ be a K3 surface over $k$. The Frobenius element $\phi$ of the absolute Galois group of $k$ acts on $\mathrm{NS}(X_{\overline{k}})$ as a finite order endomorphism. It is therefore semi-simple. Furthermore, since its characteristic polynomial has degree 22 and has at least one eigenvalue equal to 1, its remaining eigenvalues which are roots of unity of degree at most 21. At this point, we use the defining property of $K$. By construction, any integer $m$ with

$\varphi(m) \leq 21$ must divide $N = \deg[K : k]$. It follows that $\phi^N = 1$ holds. In particular, we see that $\mathrm{NS}(X_{\overline{k}}) = \mathrm{NS}(X_K)$.

Let $\mathscr{L} = \mathscr{L}(K)$ be the set of lattices provided by Proposition 2.2.1. Let $C_5$ be the maximum value of $d(L)$ for $L \in \mathscr{L}$, where $d(L)$ is as defined in Corollary 2.3.3. Let $X$ be a K3 surface over $k$ satisfying the Tate conjecture over $K$. Then $\mathrm{NS}(X_K)$, and thus $\mathrm{NS}(X_{\overline{k}})$, belongs to $\mathscr{L}$. It follows that $X_{\overline{k}}$, and thus $X_K$, admits an ample line bundle of degree at most $C_5$. This proves the proposition. □

2.4. **Finiteness of twisted forms.** The purpose of this section is to establish the following result.

**Proposition 2.4.1.** *Let $X$ be a K3 surface over the finite field $k$. Then $X$ has only finitely many twisted forms, up to isomorphism.*

Recall that a *twisted form* of $X$ is a K3 surface over $k$ which is isomorphic to $X$ over $\overline{k}$. The set of isomorphism classes of twisted forms of $X$ is in bijection with $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}_{\overline{k}}(X_{\overline{k}}))$, and so to prove the proposition it suffices to show finiteness of this cohomology set. We begin with two lemmas. In what follows, $\widehat{\mathbf{Z}}$ denotes the profinite completion of $\mathbf{Z}$ and $\phi$ a topological generator. Suppose $\widehat{\mathbf{Z}}$ acts continuously on a discrete group $E$. A 1-cocycle for this action is given by an element $x \in E$ such that $xx^{\phi} \cdots x^{\phi^{n-1}} = 1$ for all sufficiently divisible integers $n$. Moreover, cocycles represented by $x$ and $y$ are cohomologous if there exists an element $h \in E$ such that $x = h^{-1}yh^{\phi}$

**Lemma 2.4.2.** *Let $G$ and $G'$ be discrete groups on which $\widehat{\mathbf{Z}}$ acts continuously and let $f : G \to G'$ be a $\widehat{\mathbf{Z}}$-equivariant homomorphism whose kernel is finite and whose image has finite index. If $\mathrm{H}^1(\widehat{\mathbf{Z}}, G')$ is finite then $\mathrm{H}^1(\widehat{\mathbf{Z}}, G)$ is finite.*

*Proof.* Let $x_1, \ldots, x_n \in G'$ be cocycles representing the elements of $\mathrm{H}^1(\widehat{\mathbf{Z}}, G')$ and $y_1, \ldots, y_j \in G'$ representatives for the right cosets of $f(G)$ in $G'$. Given a cocycle $x \in G$, there is $i$ and $h \in G'$ such that

$$h^{-1}f(x)h^{\phi} = x_i.$$

Choosing $j$ and $z \in G$ such that $h = f(z)y_j$ yields

$$y_j^{-1}f(z)^{-1}f(x)f(z)^{\phi}y_j^{\phi} = x_i,$$

whence

$$f(z^{-1}xz^{\phi}) = y_jx_i(y_j^{\phi})^{-1}.$$

Since $f$ has finite kernel, each element $y_jx_i(y_j^{\phi})^{-1}$ has finitely many preimages in $G$, and we see that the union of this finite set of finite sets contains cocycles representing all of $\mathrm{H}^1(\widehat{\mathbf{Z}}, G)$, as desired. □

**Lemma 2.4.3.** *Let $G$ be a discrete group and let $\widehat{\mathbf{Z}} \to G$ be a continuous homomorphism; regard $\widehat{\mathbf{Z}}$ as acting on $G$ by inner automorphisms via the homomorphism. Assume that $G$ has only finitely many conjugacy classes of finite order elements. Then $H^1(\widehat{\mathbf{Z}}, G)$ is finite.*

*Proof.* Let $g$ be the image of $\phi$ under the map $\widehat{\mathbf{Z}} \to G$. Continuity forces $g$ to have finite order. Because $\widehat{\mathbf{Z}}$ acts by conjugation, the cocycle condition for an element $x \in G$ simply amounts to $xg$ having finite order, and $x$ and $y$ are cohomologous if $xg$ and $yg$ are conjugate. We thus find that multiplication by $g$ gives a bijection between $\mathrm{H}^1(\widehat{\mathbf{Z}}, G)$ and the set of conjugacy classes of finite order elements of $G$. This completes the proof. □

We now prove the proposition.

*Proof of Proposition 2.4.1.* Let $N = \mathrm{NS}(X_{\overline{k}})$ and let $N' \subset N_{\mathbf{R}}$ be the nef cone. Let $G'$ be the group of automorphisms of the lattice $N$ which map $N'$ to itself, let $G = \mathrm{Aut}_{\overline{k}}(X_{\overline{k}})$, let $G^\circ = \mathrm{Aut}_k(X_{\overline{k}})$, and let $\Gamma = \mathrm{Aut}(\overline{k}/k) \cong \widehat{\mathbf{Z}}$. Since the natural action of $G^\circ$ on $N$ preserves $N'$, there are homomorphisms

$$\Gamma \to G^\circ \to G'.$$

In addition, conjugation by the image of $\phi$ in $G^\circ$ preserves $G$ and gives the natural Frobenius action on the automorphism group. Thus, the natural map

$$f : G \to G'$$

is $\widehat{\mathbf{Z}}$-equivariant with respect to the natural action on $G$ and the conjugation action on $G'$. The map $f$ has finite kernel and its image has finite index by [LM, Thm 6.1]. Furthermore, $G'$ has finitely many conjugacy classes of finite order elements (see [T, §6]). The above two lemmas thus imply that $\mathrm{H}^1(\Gamma, G)$ is finite, which completes the proof. □

2.5. **Proof of Main Theorem (1).** We now complete the proof of the first part of the main theorem. Let $M_d$ be the stack over $k$ of pairs $(X, L)$ where $X$ is a K3 surface and $L$ is a polarization of degree $d$. It follows from Artin's representability theorem that $M_d$ is Deligne–Mumford and locally of finite type over $k$; since the third power of any polarization is very ample [SD], the stack is of finite type. Let $C_5$ be the constant produced by Proposition 2.3.1. Consider the diagram

$$\begin{tikzcd} \coprod_{d=1}^{C_5} M_d(K) \arrow[r, "\alpha"] & \{\text{isomorphism classes of K3's over } K\} \\ & \{\text{isomorphism classes of K3's over } k \text{ satisfying Tate over } K\} \arrow[u, "\beta"] \end{tikzcd}$$

By the definition of $C_5$, any element in the image of $\beta$ is also in the image of $\alpha$. Since the domain of $\alpha$ is finite, it follows that the image of $\beta$ is finite. Any two elements of a fiber of $\beta$ are twisted forms of each other, and so the fibers of $\beta$ are finite by Proposition 2.4.1. We thus find that the domain of $\beta$ is finite, which completes the proof.

## 3. Finiteness implies Tate

3.1. **Twisted sheaves.** We use the notions and notation from [L1], [L2], and the references therein. Recall the basic definition. Fix a $\boldsymbol{\mu}_r$-gerbe over an algebraic space $\mathscr{X} \to Z$.

**Definition 3.1.1.** A sheaf $\mathscr{F}$ of (left) $\mathscr{O}_{\mathscr{X}}$-modules is *$\lambda$-fold $\mathscr{X}$-twisted* if the natural left inertial $\boldsymbol{\mu}_r$-action $\boldsymbol{\mu}_r \times \mathscr{F} \to \mathscr{F}$ of a section $\rho$ of $\boldsymbol{\mu}_r$ is scalar multiplication by the $\rho^\lambda$.

*Notation* 3.1.2. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to Z$, write $\mathrm{D}^{\mathrm{tw}}(\mathscr{X})$ for the derived category of perfect complexes of $\mathscr{X}$-twisted sheaves and $\mathrm{D}^{-\mathrm{tw}}(\mathscr{X})$ for the derived category of perfect complexes of $(-1)$-fold $\mathscr{X}$-twisted sheaves.

3.2. **$\ell$-adic $B$-fields.** Let $Z$ be a separated scheme of finite type over $k$. The following is an "$\ell$-adification" of a notion familiar from mathematical physics. For the most part, this rephrases well-known results in a form that aligns them with the literature on twisted Mukai lattices, to be developed $\ell$-adically in the next section.

**Definition 3.2.1.** An *$\ell$-adic $B$-field on $Z$* is an element

$$B \in \mathrm{H}^2_{\text{ét}}(Z, \mathbf{Q}_\ell(1)).$$

We can write any $B$-field as $\alpha/\ell^n$ with

$$\alpha \in \mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1))$$

a primitive element. *When we write $B$ in this form we will always assume (unless noted otherwise) that $\alpha$ is primitive.*

**Definition 3.2.2.** Given a $B$-field $\alpha/\ell^n$ on $Z$, the *Brauer class associated to $B$* is the image of $\alpha$ under the map

$$\mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(Z, \boldsymbol{\mu}_{\ell^n}) \to \mathrm{Br}(Z)[\ell^n].$$

*Notation* 3.2.3. Given $\alpha \in \mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1))$ we will write $\alpha_n$ for the image in $\mathrm{H}^2_{\text{ét}}(Z, \boldsymbol{\mu}_{\ell^n})$. We will use brackets to indicate the map $\mathrm{H}^2_{\text{ét}}(Z, \boldsymbol{\mu}_{\ell^n}) \to \mathrm{Br}(X)$.

Thus, the Brauer class associated to the $B$-field $\alpha/\ell^n$ is written $[\alpha_n]$.

**Lemma 3.2.4.** *Given $\alpha \in \mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1))$ and positive integers $n, n'$, we have that*

$$\ell^n[\alpha_{n+n'}] = [\alpha_{n'}] \in \mathrm{Br}(Z).$$

*Proof.* Consider the commutative diagram of Kummer sequences

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_{\ell^{n+n'}} \arrow[r] \arrow[d, "\ell^n"'] & \mathbf{G}_m \arrow[r, "\ell^{n+n'}"] \arrow[d, "\ell^n"'] & \mathbf{G}_m \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_{\ell^{n'}} \arrow[r] & \mathbf{G}_m \arrow[r, "\ell^{n'}"] & \mathbf{G}_m \arrow[r] & 1.
\end{tikzcd}$$

The induced map

$$\ell^n : \mathrm{H}^2_{\text{ét}}(Z, \boldsymbol{\mu}_{\ell^{n+n'}}) \to \mathrm{H}^2_{\text{ét}}(Z, \boldsymbol{\mu}_{\ell^{n'}})$$

is identified with the reduction map

$$\mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1) \otimes \mathbf{Z}/\ell^{n+n'}\mathbf{Z}) \to \mathrm{H}^2_{\text{ét}}(Z, \mathbf{Z}_\ell(1) \otimes \mathbf{Z}/\ell^{n'}\mathbf{Z}),$$

so it sends $\alpha_{n+n'}$ to $\alpha_{n'}$. On the other hand, $\ell^n$ acts by multiplication by $\ell^n$ on $\mathrm{H}^2_{\text{ét}}(Z, \mathbf{G}_m)$. The result follows from the resulting diagram of cohomology groups. ☐

**Lemma 3.2.5.** *The Brauer classes associated to $\ell$-adic $B$-fields on $X$ form a subgroup*

$$\mathrm{Br}^B_\ell(Z) \subset \mathrm{Br}(Z)(\ell)$$

*of the $\ell$-primary part of the Brauer group of $Z$.*

*Proof.* Let $\alpha/\ell^n$ and $\beta/\ell^m$ be $B$-fields with Brauer classes $[\alpha_n]$ and $[\beta_m]$. By Lemma 3.2.4 we have that $[\alpha_n] = \ell^m[\alpha_{n+m}]$ and $[\beta_m] = \ell^n[\beta_{n+m}]$. Let $\gamma = \ell^m\alpha + \ell^n\beta$. We have that

$$[\gamma_{n+m}] = \ell^m[\alpha_{n+m}] + \ell^n[\beta_{n+m}] = [\alpha_n] + [\beta_m],$$

as desired. ☐

Given a smooth projective geometrically connected algebraic surface $X$ over $k$, the intersection pairing defines a map

$$\mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)) \times \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)) \to \mathrm{H}^4_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(2)) = \mathbf{Z}_\ell.$$

There is a cycle class map

$$\mathrm{Pic}(X) \otimes \mathbf{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)).$$

Write $P(X, \mathbf{Z}_\ell)$ for its image, which is a $\mathbf{Z}_\ell$-sublattice.

**Definition 3.2.6.** The *$\ell$-adic transcendental lattice of $X$* is

$$T(X, \mathbf{Z}_\ell) := P(X, \mathbf{Z}_\ell)^\perp \subset \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1))$$

**Lemma 3.2.7.** *The map $\alpha \otimes (1/\ell^n) \mapsto [\alpha_n]$ defines an isomorphism*

$$T(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \xrightarrow{\sim} \mathrm{Br}_\ell^B(X).$$

*Proof.* One easily checks that the map is well-defined. It is surjective by definition. Suppose $\alpha/\ell^n$ maps to $0$ in $\mathrm{Br}(X)$. We have that $[\alpha_n] = 0$, so that

$$\alpha_n \in \mathrm{Pic}(X)/\ell^n \mathrm{Pic}(X) \subset \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n}).$$

Taking the cohomology of the exact sequence

$$0 \to \mathbf{Z}_\ell(1) \to \mathbf{Z}_\ell(1) \to \boldsymbol{\mu}_{\ell^n} \to 0,$$

we see that

$$\alpha \in \ell^n \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)),$$

so that

$$\alpha/\ell^n = 0 \in T(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell,$$

as desired. $\square$

**Proposition 3.2.8.** *If $X$ is a smooth projective surface over the finite field $k$ of characteristic $p$ then the following are equivalent.*

(1) $\mathrm{Br}(X)$ *is infinite*
(2) $\mathrm{Br}_\ell^B(X) \neq 0$ *for all* $\ell$
(3) $\mathrm{Br}_\ell^B(X) \neq 0$ *for some* $\ell$
(4) $T(X, \mathbf{Z}_\ell) \neq 0$

*Proof.* By [LLR], $\mathrm{Br}(X)$ is infinite if and only if $\mathrm{Br}(X)(\ell)$ is infinite for one $\ell$ if and only if $\mathrm{Br}(X)(\ell)$ is infinite for all $\ell$. To prove the Proposition it suffices to prove that $\mathrm{Br}_\ell^B(X) \neq 0$ if $\mathrm{Br}(X)(\ell)$ is infinite. So suppose $\alpha_n \in \mathrm{Br}(X)(\ell^n)$ is a sequence of classes. Choose lifts

$$\widetilde{\alpha}_n \in \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^n})$$

for each $n$. Let $\beta_0 = 0 \in \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^0})$. Assume we have constructed $\beta_1, \ldots, \beta_m$, $\beta_i \in \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^i})$ such that $\ell\beta_{i+1} = \beta_i$ and $\ell^j \widetilde{\alpha}_{m+j} = \beta_m$ for all $j > 0$. The group $\mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^{m+1}})$ is finite, so there is one element $\beta_{m+1}$ that is a multiple of infinitely many $\widetilde{\alpha}_n$ such that $\ell\beta_{m+1} = \beta_m$. Replacing the sequence of $\widetilde{\alpha}_n$ with the subsequence mapping to $\beta_{m+1}$, we see that we can proceed by induction, yielding an element

$$\beta \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1))$$

giving infinitely many distinct elements of $\mathrm{Br}_\ell^B(X)$. $\square$

3.3. **Twisted $\ell$-adic Mukai lattices.** Fix a K3 surface over a field $k$ and an element $\alpha \in T(X, \mathbf{Z}_\ell)$. Fix a $B$-field $\alpha/r$ with $r = \ell^n$ for some $n$. Note that because $X$ is simply connected it follows from the Leray spectral sequence that

$$\mathrm{H}^2(X, \mathbf{Z}_\ell(2)) = \mathrm{H}^2(X_{\overline{k}}, \mathbf{Z}_\ell(2)) = \mathbf{Z}_\ell.$$

**Definition 3.3.1.** The $\ell$-adic Mukai lattice of $X$ is the free $\mathbf{Z}_\ell$-module

$$\mathrm{H}_{\text{ét}}(X, \mathbf{Z}_\ell) := \mathrm{H}^0_{\text{ét}}(X, \mathbf{Z}_\ell) \oplus \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)) \oplus \mathrm{H}^4_{\text{ét}}(X, \mathbf{Z}_\ell(2))$$

with the intersection pairing

$$(a, b, c) \cdot (a', b', c') = bb' - ac' - a'c \in \mathrm{H}^4_{\text{ét}}(X, \mathbf{Z}_\ell(2)) = \mathbf{Z}_\ell.$$

The algebraic part of the cohomology gives a sublattice

$$\mathrm{CH}(X, \mathbf{Z}_\ell) = \mathbf{Z}_\ell \oplus P(X, \mathbf{Z}_\ell) \oplus \mathbf{Z}_\ell.$$

Is is easy to see that

$$\mathrm{CH}(X, \mathbf{Z}_\ell)^\perp = T(X, \mathbf{Z}_\ell),$$

as sublattices of $\mathrm{H}_{\text{ét}}(X, \mathbf{Z}_\ell)$. We will write $\mathrm{H}_{\text{ét}}(X, \mathbf{Q}_\ell)$ (resp. $\mathrm{CH}(X, \mathbf{Q}_\ell)$, resp. $T(X, \mathbf{Q}_\ell)$) for $\mathrm{H}_{\text{ét}}(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ (resp. $\mathrm{CH}(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell$), resp. $T(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell$). The lattice $\mathrm{CH}(X, \mathbf{Z}_\ell)$ has an integral structure:

$$\mathrm{CH}(X, \mathbf{Z}) = \mathbf{Z} \oplus \mathrm{Pic}(X) \oplus \mathbf{Z}$$

such that $\mathrm{CH}(X, \mathbf{Z}) \otimes \mathbf{Z}_\ell = \mathrm{CH}(X, \mathbf{Z}_\ell)$.

Following [Y], we consider the map

$$T_{-\alpha/r} : \mathrm{H}_{\text{ét}}(X, \mathbf{Z}_\ell) \to \mathrm{H}_{\text{ét}}(X, \mathbf{Q}_\ell)$$

that sends $x$ to the cup product $x \cup e^{-\alpha/r}$.

**Definition 3.3.2.** The *$\alpha/r$-twisted Chow lattice of $X$* is

$$\mathrm{CH}^{\alpha/r}(X, \mathbf{Z}_\ell) := (T_{-\alpha/r})^{-1}(\mathrm{CH}(X, \mathbf{Q}_\ell)).$$

There is also an integral structure on the twisted Chow lattice.

**Lemma 3.3.3.** *The subgroup*

$$\mathrm{CH}^{\alpha/r}(X, \mathbf{Z}) := T^{-1}_{-\alpha/r}(\mathrm{CH}(X, \mathbf{Q})) \subset \mathrm{CH}^{\alpha/r}(X, \mathbf{Z}_\ell)$$

*induces an isomorphism*

$$\mathrm{CH}^{\alpha/r}(X, \mathbf{Z}) \otimes \mathbf{Z}_\ell \xrightarrow{\sim} \mathrm{CH}^{\alpha/r}(X, \mathbf{Z}_\ell).$$

*Proof.* We immediately reduce to the case in which $\alpha$ is primitive, in which case we see that

$$\mathrm{CH}^{\alpha/r}(X, \mathbf{Z}) = \{(ar, D + a\alpha, c\} \mid a, c \in \mathbf{Z}, D \in \mathrm{Pic}(X).\}$$

The result is then a simple calculation. □

The (integral) twisted Chow lattice is a natural recipient of Chern classes for twisted sheaves. Let $\pi : \mathscr{X} \to X$ be a $\boldsymbol{\mu}_r$-gerbe representing the class $[\alpha_n]$ associated to the $B$-field $\alpha/r$, $r = \ell^n$. Suppose $\mathscr{P}$ is a perfect complex of $\mathscr{X}$-twisted sheaves.

**Definition 3.3.4.** The *Chern character of $\mathscr{P}$* is the unique element

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{P}) \in \mathrm{CH}(X, \mathbf{Q})$$

such that

$$\mathrm{rk}\,\mathrm{ch}_{\mathscr{X}}(\mathscr{P}) = \mathrm{rk}\,\mathscr{P}$$

and

$$\mathrm{ch}_{\mathscr{X}}(\mathscr{P})^r = \mathrm{ch}\left(\mathbf{R}\pi_*(\mathscr{P}^{\overset{\mathbf{L}}{\otimes} r})\right) \in \mathrm{CH}(X, \mathbf{Q}).$$

A less *ad hoc* approach is to define rational Chern classes using a splitting principle, etc. This can be done and yields the same result.

**Definition 3.3.5.** Given a perfect complex $\mathscr{P}$ of $\mathscr{X}$-twisted sheaves as above, we define the *twisted Mukai vector of $\mathscr{P}$* to be

$$v^{\alpha/r}(\mathscr{P}) := e^{\alpha/r}\,\mathrm{ch}_{\mathscr{X}}(\mathscr{P})\sqrt{\mathrm{Td}_X}.$$

**Lemma 3.3.6.** *Given perfect complexes $\mathscr{P}$ and $\mathscr{Q}$ of $\mathscr{X}$-twisted sheaves as above, the following hold.*

(1) *the element $v^{\alpha/r}(\mathscr{P})$ lies in the integral subring $\mathrm{CH}^{\alpha/r}(X, \mathbf{Z})$*

(2) *we have that*

$$\chi(\mathscr{P}, \mathscr{Q}) = -v^{\alpha/r}(\mathscr{P}) \cdot v^{\alpha/r}(\mathscr{Q}).$$

*Proof.* To see the integrality, we may assume that the base field is algebraically closed. Suppose that there is an invertible sheaf $\mathscr{L}$ such that

$$[\alpha_n] = c_1(\mathscr{L}) \in \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_n).$$

In this case we have a class $\beta \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1))$ such that

$$\alpha = c_1(\mathscr{L}^\vee) + r\beta.$$

(Warning to the casual reader: note the confusing sign!) Choosing an invertible $\mathscr{X}$-twisted sheaf $\mathscr{M}$ with an isomorphism $\mathscr{M}^{\otimes r} \xrightarrow{\sim} \mathscr{L}$, we have that

$$\mathscr{L}^\vee \otimes \mathbf{R}\pi_* \left( \mathscr{P}^{\overset{\mathbf{L}}{\otimes} r} \right) = \left( \mathbf{R}\pi_*(\mathscr{P} \overset{\mathbf{L}}{\otimes} \mathscr{M}^\vee) \right)^{\otimes r},$$

so that

$$e^{\alpha/r - \beta} \operatorname{ch}_{\mathscr{X}}(\mathscr{P}) = \operatorname{ch}\left(\mathbf{R}\pi_*(\mathscr{P} \otimes \mathscr{M}^\vee)\right),$$

so

$$e^{\alpha/r} \operatorname{ch}_{\mathscr{X}}(\mathscr{P}) = e^{\beta} \operatorname{ch}\left(\mathbf{R}\pi_*(\mathscr{P} \otimes \mathscr{M}^\vee)\right)$$

is an integral class.

In the general case, we can deform and specialize $X$ so that

$$\alpha \in \operatorname{Pic}(X) + r\,\mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1)).$$

This reduces us to the case just treated. For example, we can work with the universal deformation of $X$ over which a chosen ample sheaf and $\det \mathscr{P}$ remain algebraic, so that the class of $\mathscr{P}$ in $K$-theory deforms. The generic point of the universal deformation (which is now in characteristic $0$!) specializes to a surface on which any initial segment of the $\ell$-adic expansion of $\alpha$ is algebraic. Finally, the intersection product, Chern classes, etc., are constant in families. A similar argument can be found in Section 3.2 of [Y].

To see the second statement, note that

$$\begin{aligned}
-v^{\alpha/r}(\mathscr{P}) \cdot v^{\alpha/r}(\mathscr{Q}) &= \deg\left( \operatorname{ch}_{\mathscr{X}}(\mathscr{P}^\vee \overset{\mathbf{L}}{\otimes} \mathscr{Q}) \operatorname{Td}_X \right) \\
&= \deg\left( \operatorname{ch}(\mathbf{R}\pi_*(\mathbf{R}\mathscr{H}om(\mathscr{P}, \mathscr{Q})) \operatorname{Td}_X \right) \\
&= \chi(\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathscr{P}, \mathscr{Q})) \\
&= \chi(\mathscr{P}, \mathscr{Q})
\end{aligned}$$

□

3.4. **Moduli spaces of twisted sheaves.** We recall the essential details of the moduli theory of twisted sheaves on K3 surfaces, building on the foundational work in [Mu]. The reader can consult [Y] for a development of the theory using a stack-free formulation of the notion of twisted sheaf. The material on obstructions is related to various ideas developed [Mu] and [Căl].

Fix a K3 surface $X$ over $k$.

**Proposition 3.4.1.** *Given an $\ell$-adic $B$-field $\alpha/\ell^n$ and a Mukai vector*

$$v \in \mathrm{CH}^{\alpha/\ell^n}(X, \mathbf{Z})$$

*such that* $\operatorname{rk} v > 0$ *and* $v^2 = 0$*, we have that*

(1) *the stack $\mathscr{M}_{\mathscr{X}}(v)$ of stable $\mathscr{X}$-twisted sheaves with Mukai vector $v$ is a $\boldsymbol{\mu}_{\ell^n}$-gerbe over a K3 surface $M_{\mathscr{X}}(v)$;*

(2) *the universal sheaf* $\mathscr{E}$ *on* $\mathscr{X} \times \mathscr{M}_{\mathscr{X}}(v)$ *defines an equivalence of derived categories*

$$\mathrm{D}^{\mathrm{tw}}(\mathscr{X}) \xrightarrow{\sim} \mathrm{D}^{-\mathrm{tw}}(\mathscr{M}_{\mathscr{X}}(v));$$

(3) *if there is a vector* $u \in \mathrm{CH}^{\alpha/\ell^n}(X, \mathbf{Z})$ *such that* $(u \cdot v, \ell) = 1$ *then the Brauer class of the gerbe*

$$\mathscr{M}_{\mathscr{X}}(v) \to M_{\mathscr{X}}(v)$$

*is trivial. In particular, there is an equivalence of derived categories*

$$\mathrm{D}^{-\mathrm{tw}}(\mathscr{M}_{\mathscr{X}}(v)) \xrightarrow{\sim} \mathrm{D}(M_{\mathscr{X}}(v)).$$

Here stability is taken with respect to a general polarization that will be suppressed from the notation.

*Proof.* The non-emptiness, smoothness, symplectic structure, etc., of the moduli problem can be deduced from Yoshioka's results in characteristic $0$ [Y] by nearly identical arguments: lift the Mukai vector and polarization over the Witt vectors using Deligne, see that it is enough (by Langton) to prove the result over the geometric generic point, then deform (as does Yoshioka does) to the case in which the $B$-field is algebraic (i.e., there is an invertible sheaf $L$ such that $\alpha - c_1(L) \in \ell^n \mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell(1))$).

To prove the last statement, it suffices to prove the following lemma.

**Lemma 3.4.2.** *Given a Mukai vector*

$$u \in \mathrm{CH}^{\alpha/\ell^n}(X, \mathbf{Z}),$$

*there is a perfect complex* $\mathscr{P}$ *of* $\mathscr{X}$*-twisted sheaves such that*

$$v^{\alpha/\ell^n}(\mathscr{P}) = u.$$

Let us accept the lemma for a moment and see why this implies the result. Write $\mathscr{E}$ for the universal sheaf on

$$\mathscr{X} \times \mathscr{M}_{\mathscr{X}}(v).$$

This sheaf is simultaneously $\mathscr{X}$- and $\mathscr{M}_{\mathscr{X}}(v)$-twisted. Given a complex $\mathscr{P}$ as in Lemma 3.4.2, consider the perfect complex of $\mathscr{M}_{\mathscr{X}}(v)$-twisted sheaves

$$\mathscr{Q} := \mathbf{R}q_*(\mathbf{L}p^*\mathscr{P}^\vee \overset{\mathbf{L}}{\otimes} \mathscr{E}).$$

The rank of this complex over a geometric point $m$ of $\mathscr{M}$ is calculated by

$$\chi(\mathscr{P}, \mathscr{E}_m) = -v^{\alpha/\ell^n}(\mathscr{P}) \cdot v = -u \cdot v,$$

which is relatively prime to $\ell$. By standard results, we have that the Brauer class of $\mathscr{M}_{\mathscr{X}}(v)$ satisfies

$$[\mathscr{M}_{\mathscr{X}}(v)] \in \mathrm{Br}(M_{\mathscr{X}}(v))[u \cdot v].$$

On the other hand, $\mathscr{M}_{\mathscr{X}}(v) \to M_{\mathscr{X}}(v)$ is a $\boldsymbol{\mu}_{\ell^n}$-gerbe, which implies that

$$[\mathscr{M}_{\mathscr{X}}(v)] \in \mathrm{Br}(M_{\mathscr{X}}(v))[\ell^n].$$

Combining the two statements yields the result.

It remains to prove Lemma 3.4.2.

*Proof of Lemma 3.4.2.* We know that

$$u = (ra, D + a\alpha, c),$$

so we seek a perfect complex of rank $ra$, determinant $D$, and appropriate second Chern class. By Theorem 4.3.1.1 of [L2], there is a locally free $\mathscr{X}$-twisted sheaf $V$ of rank $r$. Moreover, Lang's theorem implies that for any curve $C \subset X$, the restriction $\mathscr{X} \times_X C$ has trivial Brauer class, hence supports twisted sheaves $L_C$ of rank $1$. Since $\det L_C \cong \mathscr{O}(C)$, we see that we can get any determinant by adding (in the derived category) a sum of shifts of invertible sheaves supported on curves. Finally, Tsen's theorem yields invertible twisted sheaves supported at points, and they have twisted Mukai vector $(0, 0, 1)$, so there is a sum of (shifted) points giving the desired second Chern class. Since any bounded complex on $\mathscr{X}$ is perfect ($\mathscr{X}$ being regular), the lemma is proven. □

□

3.5. **Twisted partners of K3 surfaces over a finite field.** Fix a K3 surface $X$ over a finite field $k$ of characteristic $p \geq 5$. Let $K$ be the extension of $k$ of degree 465585120, as in the introduction.

**Lemma 3.5.1.** *If* $\mathrm{Br}(X_K)$ *is infinite then for any odd prime* $\ell$ *relatively prime to* $\operatorname{disc}\mathrm{Pic}(X_K)$ *there is* $\alpha \in T(X_K, \mathbf{Z}_\ell)$ *such that* $\alpha^2 = 1$.

*Proof.* First, since we know that K3 surfaces of finite height have finite Brauer groups [NO], we have that $X$ is supersingular, so that every eigenvalue of Frobenius $\phi$ acting on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))$ is a root of unity. By [D], the Frobenius action on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))$ is semisimple.

As in the proof of Proposition 2.3.1, by the definition of $N$ and $K$, we have that the characteristic polynomial of the geometric Frobenius $\phi^N$ acting on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))$ is

$$\det(\phi^N - T|\,\mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))) = (T-1)^{22}.$$

On the other hand, we have that

$$\mathrm{H}^2_{\text{ét}}(X_K, \mathbf{Z}_\ell(1)) = \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))^{F^N}.$$

We conclude that $\mathrm{H}^2_{\text{ét}}(X_K, \mathbf{Z}_\ell(1))$ is a unimodular $\mathbf{Z}_\ell$-lattice of rank 22. Since $\mathrm{Br}(X_K)$ is infinite, we have that $T(X_K, \mathbf{Z}_\ell) \neq 0$ by Proposition 3.2.8. Finally, we know by [A] that if

$$\rho(X_{\overline{k}}) = \rho(X_K) \geq 5$$

then $\rho(X_K) = 22$ and

$$P(X_K, \mathbf{Z}_\ell) = \mathrm{H}^2_{\text{ét}}(X_K, \mathbf{Z}_\ell(1)),$$

contradicting our assumption that $T(X_K, \mathbf{Z}_\ell(1)) \neq 0$. Thus, we may assume that

$$\operatorname{rk} T(X_K, \mathbf{Z}_\ell) \geq 17.$$

The hypothesis that $\ell$ does not divide $\operatorname{disc}\mathrm{Pic}(X_K)$ implies that there is an orthogonal decomposition of the unimodular lattice

$$\mathrm{H}^2_{\text{ét}}(X_K, \mathbf{Z}_\ell(1)) = P(X_K, \mathbf{Z}_\ell) \perp T(X_K, \mathbf{Z}_\ell).$$

It follows that $T(X_K, \mathbf{Z}_\ell)$ is a unimodular lattice of rank at least 17 over $\mathbf{Z}_\ell$. By Chevalley's theorem, the non-degenerate quadratic space $T(X_K, \mathbf{Z}_\ell) \otimes \mathbf{F}_\ell$ has a non-trivial zero, whence it contains a hyperbolic plane and thus takes all values. In other words, there is some $\beta \in T(X_K, \mathbf{Z}_\ell)$ such that $\beta^2 \in 1 + \ell\mathbf{Z}_\ell$. By Hensel's Lemma (applied to the non-singular quadratic form on $T(X_K, \mathbf{Z}_\ell)$), a suitable $\mathbf{Z}_\ell$-multiple of $\beta$ will have square 1, as desired. □

Now assume that $X$ is a K3 surface over the finite field $k$ such that $\mathrm{Br}(X)$ is infinite. Fix a primitive ample divisor $D \in \mathrm{Pic}(X)$ and choose an odd prime $\ell$ relatively prime to $\operatorname{disc}\mathrm{Pic}(X)$ such that $-D^2$ and 2 are non-zero squares in $\mathbf{Z}/\ell\mathbf{Z}$.

**Definition 3.5.2.** Given a relative K3 surface $\mathscr{Y} \to S$, an invertible sheaf $\mathscr{L} \in \mathrm{Pic}(\mathscr{Y})$, and positive integers $r$ and $s$, let

$$\mathbf{Sh}_{\mathscr{Y}/S}(w) \to S$$

be the stack of simple locally free sheaves with Mukai vector $w = (r, \mathscr{L}, s)$ on each fiber.

**Lemma 3.5.3.** *If $w$ is primitive and $w^2 = 0$ then $\mathbf{Sh}_{\mathcal{Y}/S}(w)$ is a $\boldsymbol{\mu}_r$-gerbe over a smooth algebraic space $\mathrm{Sh}_{\mathcal{Y}/S}(w)$ of relative dimension 2 over $S$ with non-empty geometric fibers.*

*Proof.* Non-emptiness of the geometric fibers is proven in Section 3 of [Y]. The smoothness follows from the fact that that obstruction theory of a locally free sheaf $V$ with determinant $\mathscr{L}$ on a fiber $\mathcal{Y}_s$ is given by the kernel of the trace map

$$\mathrm{Ext}^2(V, V) \to \mathrm{H}^2_{\text{ét}}(\mathcal{Y}_s, \mathscr{O}),$$

which vanishes when $V$ is simple. The relative dimension is

$$\dim \mathrm{Ext}^1(V, V) = w^2 + 2 = 2,$$

as desired. $\square$

When $S = \operatorname{Spec} L$ and the base field is understood, we will write simply $\mathbf{Sh}_{\mathcal{Y}}(w)$ and $\mathrm{Sh}_{\mathcal{Y}}(w)$.

**Proposition 3.5.4.** *In the above situation, there is an infinite sequence of pairs $(\gamma_n, M_n)$ such that*

(1) $\gamma_n \in \mathrm{H}^2_{\text{ét}}(X_K, \boldsymbol{\mu}_{\ell^n})$ *with* $[\gamma_n]$ *of exact order* $\ell^n$ *in* $\mathrm{Br}(X_K)$*;*
(2) $M_n$ *is a K3 surface over $K$ such that $\rho(M_n) = \rho(X)$ and*
$$\operatorname{rk} T(M_n, \mathbf{Z}_\ell) = \operatorname{rk} T(X_K, \mathbf{Z}_\ell);$$
(3) *given a gerbe $\mathscr{X}_n \to X_K$ representing $\gamma_n$, we have that $M_n$ is a fine moduli space of stable $\mathscr{X}_n$-twisted sheaves with determinant $D$;*
(4) *for each $n$, there is a (classical) Mukai vector $w_n \in \mathrm{CH}(M_n)$ such that*
   (a) *a tautological sheaf $\mathscr{E}$ on $\mathscr{X}_n \times M_n$ induces an open immersion*
   $$\mathscr{X}_n \hookrightarrow \mathbf{Sh}_{M_n}(w_n);$$
   (b) *given a complete dvr $R$ with residue field $K$ and a relative K3 surface $\mathcal{M}$ over $R$ with $\mathcal{M} \otimes_R K = M_n$ such that $\mathrm{Pic}(\mathcal{M}) = \mathrm{Pic}(M_n)$, there is a relative K3 surface $\mathcal{X} \to \operatorname{Spec} R$ such that $\mathcal{X} \otimes K = X$, a $\boldsymbol{\mu}_{\ell^n}$-gerbe $\mathfrak{X} \to \mathcal{X}$ such that $\mathfrak{X} \otimes_R K = \mathscr{X}_n$, and a locally free $\mathfrak{X} \times \mathcal{M}$-twisted sheaf $\mathfrak{E}$ such that $\mathfrak{E} \otimes_R K = \mathscr{E}$. For any inclusion $R \hookrightarrow \mathbf{C}$, the base changed family $(\mathfrak{X}_{\mathbf{C}}, \mathcal{M}_{\mathbf{C}}, \mathfrak{E}_{\mathbf{C}})$ yields a twisted Fourier-Mukai partnership*
   $$(\mathcal{X}_{\mathbf{C}}, [\mathfrak{X}_{\mathbf{C}}]) \sim \mathcal{M}_{\mathbf{C}},$$
   *where $[\mathfrak{X}_{\mathbf{C}}] \in \mathrm{Br}(\mathcal{X}_{\mathbf{C}})$ is a Brauer class of exact order $\ell^n$.*

*Proof.* By Lemma 3.5.1 and the fact that $2$ is a square in $\mathbf{Z}_\ell$, there is some $\alpha \in T(X_K, \mathbf{Z}_\ell)$ with $\alpha^2 = 2$. By assumption $-\frac{1}{2}D^2$ is a square modulo $\ell$, so by Hensel's lemma there is a $q \in 1 + \ell\mathbf{Z}_\ell$ such that $2q^2 = -D^2$. Let $\gamma = q\alpha$, and consider the Mukai vector

$$v_n := (\ell^n, \gamma + D, 0) \in \mathrm{CH}^{\gamma/\ell^n}(X_K, \mathbf{Z}_\ell).$$

We have that

$$v_n^2 = \gamma^2 + D^2 = 2q^2 + D^2 = 0$$

and

$$v_n \cdot (\ell^n, \gamma, 0) = \gamma^2 = 2q^2 = -D^2 \not\equiv 0 \mod \ell.$$

It is easy to see that $\gamma$ is primitive. Let $\mathscr{X}_n$ be a gerbe representing $\gamma_n$. Since the natural map

$$\iota : \mathrm{H}^2_{\text{ét}}(X_K, \mathbf{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell(1))$$

is an isomorphism (and thus injective), we have that the image of $[\gamma_n]$ in $\mathrm{Br}(X_{\overline{k}})$ has order exactly $\ell^n$ (since otherwise some non-zero multiple would lie in the kernel of $\iota$). Applying Proposition 3.4.1, we have that $\mathscr{M}_{\mathscr{X}_n}(v_n)$ is a $\boldsymbol{\mu}_{\ell^n}$-gerbe over a K3 surface $M_{\mathscr{X}_n}(v_n)$ whose Brauer class is killed by

$$v_n \cdot (\ell^n, \gamma, 0) = -D^2,$$

hence is trivial. This yields an equivalence

$$\mathrm{D}^{\mathrm{tw}}(\mathscr{X}_n) \xrightarrow{\sim} \mathrm{D}(M_{\mathscr{X}_n}(v_n)).$$

Finally, the universal sheaf $\mathscr{E}$ on $\mathscr{X}_n \times \mathscr{M}_{\mathscr{X}_n}(v_n)$ induces an isometry

$$\mathrm{H}_{\text{ét}}(X_K, \mathbf{Q}_\ell) \xrightarrow{\sim} \mathrm{H}_{\text{ét}}(M_{\mathscr{X}_n}(v_n), \mathbf{Q}_\ell)$$

that restricts to an isometry

$$\mathrm{CH}(X_K, \mathbf{Q}_\ell) \xrightarrow{\sim} \mathrm{CH}(M_{\mathscr{X}_n}(v_n), \mathbf{Q}_\ell)$$

(see Section 4.1.1 of [Y] for the proof, written using Yoshioka's notation), establishing the first three statements of the Proposition. (In fact, one can see that $M_{\mathscr{X}_n}(v_n)$ is also supersingular without recourse to [NO].)

Let $\mathscr{E}$ be a tautological sheaf on $\mathscr{X}_n \times M_n$. The determinant of $\mathscr{E}$ is naturally identified with the pullback of an invertible sheaf

$$L_{X_n} \boxtimes L_{M_n} \in \mathrm{Pic}(X_n \times M_n),$$

and each geometric fiber $\mathscr{E}_x$ has a second Chern class $c \in \mathbf{Z}$. Letting

$$w_n = (\ell^n, L_{M_n}, c),$$

the sheaf $\mathscr{E}$ gives a morphism of $\boldsymbol{\mu}_{\ell^n}$-gerbes

$$\mathscr{X}_n \to \mathbf{Sh}_{M_n}(w_n).$$

Since $\mathscr{E}$ is a Fourier-Mukai kernel, this morphism is an étale monomorphism (see e.g. Section 4 of [LO]), so it is an open immersion.

Now fix a lift $\mathcal{M}$ over $R$; since $\mathrm{Pic}(M_n)$ lifts, so does $w_n$ and $\mathbf{Sh}_{\mathcal{M}/R}(w_n)$ is a $\boldsymbol{\mu}_{\ell^n}$-gerbe over a smooth algebraic space $\mathrm{Sh}_{\mathcal{M}/R}(w_n)$ over $R$. Write $\mathscr{V}$ for the universal sheaf on $\mathbf{Sh}_{\mathcal{M}/R}(w_n) \times_R \mathcal{M}$. We can write

$$\det \mathscr{V} = \mathscr{U} \boxtimes L_{M_n}$$

with

$$\mathscr{U} \in \mathrm{Pic}(\mathrm{Sh}_{\mathcal{M}/R}(w_n)),$$

and by assumption the pullback of $\mathscr{U}$ along the map

$$X_K \to \mathrm{Sh}_{M_n}(w_n) \hookrightarrow \mathrm{Sh}_{\mathcal{M}/R}(w_n)$$

is isomorphic to $D$, hence is ample (by the assumption on $D$). Since $\mathscr{X}_n$ is open in $\mathbf{Sh}_{\mathcal{M}/R}(w_n)$, the induced open formal substack

$$\mathscr{Z} \subset \widehat{\mathbf{Sh}}_{\mathcal{M}/R}(w_n) \to \mathrm{Spf}\, R$$

is a $\boldsymbol{\mu}_{\ell^n}$-gerbe over a formal deformation $\mathfrak{Z}$ of $X_K$ over $\mathrm{Spf}\, R$. The determinant of the universal formal sheaf gives a formal lift of $D$ over $\mathfrak{Z}$, hence a polarization. It follows that $\mathscr{Z} \to \mathfrak{Z}$ is algebraizable, giving a $\boldsymbol{\mu}_{\ell^n}$-gerbe over a relative K3 surface

$$\mathscr{G} \to \mathcal{X} \to \mathrm{Spec}\, R$$

and an open immersion

$$\mathscr{G} \hookrightarrow \mathbf{Sh}_{\mathcal{M}/R}(w_n)$$

extending

$$\mathscr{X}_n \hookrightarrow \mathbf{Sh}_{M_n}(w_n).$$

Restricting the universal sheaf gives the desired deformation $\mathfrak{E}$ of $\mathscr{E}$. The usual functorialities show that $\mathfrak{E}$ is the kernel of a relative Fourier-Mukai equivalence (i.e., the relevant adjunction maps in the derived category are quasi-isomorphisms). Foundational details are contained in Section 3 of [LO], and a similar deformation argument is contained in Section 6 of [LO].

□

3.6. **Proof of Main Theorem (2).** Fix a finite field $k$ of characteristic $p$. We now prove the second part of the main theorem from the introduction.

**Theorem 3.6.1.** *Suppose $p \geq 5$. Let $K$ be the extension of $k$ of degree 465585120, as in the introduction. If there are only finitely many K3 surfaces over $K$ then for any K3 surface $X$ over $k$, the Brauer group $X_K$ is finite, i.e., the Tate conjecture holds for $X_K$.*

*Proof.* If there is a K3 surface $X$ over $k$ such that $\mathrm{Br}(X_K)$ is infinite, then we know by Proposition 3.5.4 that there is an infinite sequence of Brauer classes $\gamma_n \in \mathrm{Br}(X_K)$ such that

(1) $\gamma_n$ has exact order $\ell^n$ over $\overline{K}$, and
(2) the twisted K3 surface $(X_K, \gamma_n)$ is Fourier-Mukai equivalent to a K3 surface $M_n$ over $K$.

If there are only finitely many K3 surfaces over $K$, there is a subsequence $n_i$ such that all of the $M_{n_i}$ are the same surface, say $M$. Since $M$ has infinite Brauer group (by Proposition 3.5.4(2)), we know that it must be supersingular but not Shioda-supersingular. There is thus a lift $\mathcal{M}$ of $M$ over a complete dvr $R$ with residue field finite over $K$ such that such that $\mathrm{Pic}(\mathcal{M}) = \mathrm{Pic}(M)$. Applying Proposition 3.5.4(4)(b) we end up with a complex K3 surface $\mathcal{M}_{\mathbf{C}}$ with a sequence of twisted derived partners $(X_i, \eta_i)$ with $\eta_i \in \mathrm{Br}(X_i)$ of exact order $\ell^{n_i}$ for a strictly increasing sequence $n_i$. In particular, $\mathcal{M}_{\mathbf{C}}$ has infinitely many twisted partners. But this is a contradiction by Corollary 4.6 of [HS]. □

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