REPRESENTATIONS OF CATEGORIES OF G-MAPS

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Abstract. We study representations of wreath product analogues of categories of finite sets. This includes the category of finite sets and injections (studied by Church, Ellenberg, and Farb) and the opposite of the category of finite sets and surjections (studied by the authors in previous work). We prove noetherian properties for the injective version when the group in question is polycyclic-by-finite and use it to deduce general twisted homological stability results for such wreath products and indicate some applications to representation stability. We introduce a new class of formal languages (quasi-ordered languages) and use them to deduce strong rationality properties of Hilbert series of representations for the surjective version when the group is finite.

Contents

1. Introduction

In [CEF], Church, Ellenberg, and Farb studied representations of the category **FI**, consisting of finite sets with injective maps, and found numerous applications to topology and algebra. In [SS2], we studied representations of the closely related category $\mathbf{FS}^{\mathrm{op}}$ (in addition to many other examples), where **FS** is the category of finite sets with surjective maps, and also found several applications (e.g., to the Lannes–Schwartz artinian conjecture and to Δ-modules). In this paper, we study generalizations of **FI** and $\mathbf{FS}^{\mathrm{op}}$ in which a group G has been added to the mix. Our main tool is the theory developed in [SS2]; in fact, the example $\mathbf{FS}_G^{\mathrm{op}}$ was a primary source of motivation for that paper. In the rest of the introduction, we summarize our motivations and results.

1.1. ***G*-maps.** Let G be a group. A **G-map** between finite sets R and S is a pair (f, ρ) consisting of functions $f\colon R \to S$ and $\rho\colon R \to G$. If $(f, \rho)\colon R \to S$ and $(g, \sigma)\colon S \to T$ are two G-maps, their composition $(h, \tau)\colon R \to T$ is defined by $h = g \circ f$ and $\tau(x) = \sigma(f(x))\rho(x)$ (the product taken in G). We let $\mathbf{FA}_G$ be the category whose objects are finite sets and whose morphisms are G-maps. We let $\mathbf{FI}_G$ (resp. $\mathbf{FS}_G$) be the subcategory where the function f is injective (resp. surjective). We note that the automorphism group of the set $[n] = \{1, \dots, n\}$ in any of these categories is the wreath product $S_n \wr G$. Thus a representation of any of these

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categories can be thought of as a sequence $(M_n)_{n\geq 0}$, where M_n is a representation of $S_n \wr G$, equipped with certain transition maps between M_n and M_{n+1}. (The kind of transition maps depends on the category.) These representations are the subject of this paper.

1.2. **The category $\mathbf{FI}_G$.** Usually, noetherianity is the first property one wants to establish about the representation theory of a category. (See §2.1 for the definition of "noetherian" in this context.) For **FI**, this was proved in [Sn, CEF, CEFN, SS2], in varying levels of generality. Our main result about $\mathbf{FI}_G$ is a characterization of when representations are noetherian:

Theorem 1.2.1. *Let $\mathbf{k}$ be a ring. Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ is noetherian if and only if the group algebra $\mathbf{k}[G^n]$ is left-noetherian for all $n \geq 0$.*

Recall that a group G is **polycyclic** if it has a finite composition series $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ such that G_i/G_{i-1} is cyclic for $i = 1, \ldots, r$, and it is **polycyclic-by-finite** if it contains a polycyclic subgroup of finite index. It is known [Hal, §2.2, Lemma 3] that the group ring of a polycyclic-by-finite group over a left-noetherian ring is left-noetherian (there it is stated for the integral group ring, but the proof works for any left-noetherian coefficient ring). In fact, there are no other known examples of noetherian group algebras, but see [Iv] for related results. As the product of two polycyclic-by-finite groups is again polycyclic-by-finite the above theorem gives:

Corollary 1.2.2. *Let G be a polycyclic-by-finite group and let $\mathbf{k}$ be a left-noetherian ring. Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ is noetherian.*

When G is a finite group, we prove a stronger result:

Theorem 1.2.3. *If G is finite then the category $\mathbf{FI}_G$ is quasi-Gröbner.*

"Quasi-Gröbner" is a purely combinatorial condition on a category, introduced in [SS2] (and recalled in §2.1 below), that implies noetherianity of the representation category. Thus Theorem 1.2.3 implies Theorem 1.2.1 when G is finite, as stated. However, quasi-Gröbner gives more than just noetherianity: it implies that representations admit a theory of Gröbner bases, in an appropriate sense, and thus computations with representations can be carried out algorithmically (at least in principle).

The proof of Theorem 1.2.3 is an easy consequence of the theory developed in [SS2]. The proof of Theorem 1.2.1 is more involved. The key input is the fact that $\mathrm{Rep}_{\mathbf{k}'}(\mathbf{FI})$ is noetherian whenever $\mathbf{k}'$ is left-noetherian. This is applied with $\mathbf{k}' = \mathbf{k}[G^n]$, so even if one only cares about Theorem 1.2.1 when $\mathbf{k}$ is a field, the proof uses **FI**-modules over non-commutative rings. In fact, this is the first real application of **FI**-modules over non-commutative rings that we know of.

There is one additional result on $\mathbf{FI}_G$-modules worth mentioning:

Theorem 1.2.4. *Suppose G is finite and $\mathbf{k}$ is a field in which the order of G is invertible. Then representations of $\mathbf{FI}_G$ are equivalent to representations of $\mathbf{FI} \times \mathbf{FB}^r$, where r is the number of non-trivial irreducible representations of G over $\mathbf{k}$.*

Here **FB** is the groupoid of finite sets, i.e., the category of finite sets with bijections as morphisms. Thus, in the context of the theorem, the theory of $\mathbf{FI}_G$-modules reduces to the theory of ordinary **FI**-modules.

1.3. **The category $\mathbf{FS}_G$.** We only study $\mathbf{FS}_G$ when G is a finite group. The first result is about noetherianity, and is an easy consequence of the theory of [SS2]:

Theorem 1.3.1. *If G is finite then $\mathbf{FS}_G^{\mathrm{op}}$ is quasi-Gröbner. In particular, if $\mathbf{k}$ is left-noetherian, then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$ is noetherian.*

We next turn to Hilbert series. Suppose that M is an $\mathbf{FS}_G^{\mathrm{op}}$-module over a field $\mathbf{k}$, and suppose that G has r irreducible representations over $\mathbf{k}$. We define a power series $\mathrm{H}_M(\mathbf{t}) \in \mathbf{Q}[\![t_1, \ldots, t_r]\!]$, called the **Hilbert series** of M, that records the class of $M([n])$ in the representation ring of G^n for all $n \geq 0$. Our main result is a rationality result for this series. The strongest and most general result takes some preparation to state, so we confine ourselves to the following simplified form here:

Theorem 1.3.2. *Let M be a finitely generated representation of $\mathbf{FS}_G^{\mathrm{op}}$ over an algebraically closed field $\mathbf{k}$. Let N be the exponent of the group G. Then $\mathrm{H}_M(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$, where f and g are polynomials in the t_i with coefficients in $\mathbf{Q}(\zeta_N)$, and g factors over $\overline{\mathbf{Q}}$ as $\prod_{k=1}^n (1-\lambda_k)$, where λ_k is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i. (And $\zeta_N \in \overline{\mathbf{Q}}$ is a primitive Nth root of unity.)*

To paraphrase: if M is a finitely generated $\mathbf{FS}_G^{\mathrm{op}}$-module then the representations $M([n])$ of G^n satisfy recursive relations of a very particular form.

This is by far the deepest result of the paper, and much of our work goes into its proof. The idea is to first use Brauer's theorem to reduce to the case where G is a cyclic group whose order is invertible in $\mathbf{k}$. Representations of G are identified with vector spaces graded by the character group Λ of G, and, in a similar fashion, representations of $\mathbf{FS}_G^{\mathrm{op}}$ with those of $\mathbf{FWS}_\Lambda^{\mathrm{op}}$, a certain category whose objects are finite sets in which each element is weighted by an element of Λ. An ordered version of this category is amenable to the theory of lingual structures developed in [SS2], which produces rationality results as above for Hilbert series.

Remark 1.3.3. We did not state any theorems about Hilbert series of $\mathbf{FI}_G$-modules. When G is finite, one can use the fact that the Hilbert series of a finitely generated $\mathbf{FI}_G$-module is also the Hilbert series of a finitely generated $\mathbf{FI}$-module (use Proposition 5.1.1), together with known results on Hilbert series of ordinary $\mathbf{FI}$-modules [CEFN, Theorem B], [SS2, Corollary 7.1.5]. □

1.4. **Applications and motivations.**

- The category $\mathbf{FI}_{\mathbf{Z}/2\mathbf{Z}}$ is equivalent to the category $\mathbf{FI}_{\mathrm{BC}}$ defined in [W2, Defn. 1.2]. It is possible to define and prove properties about modified versions of our categories to include her category $\mathbf{FI}_{\mathrm{D}}$; we leave the modifications to the reader. We point the reader to [W1, W2] for applications of the category $\mathbf{FI}_{\mathrm{BC}}$.
- In §5.3, we use the machinery developed in [PS] applied to the category $\mathbf{FI}_G$ to prove general twisted homological stability results for wreath products $S_n \ltimes G^n$ when G is a polycyclic-by-finite group.
- Let M be a simply-connected manifold of dimension at least 3. In [KM], it is shown that the rational homotopy groups of the configuration spaces of M satisfy representation stability, i.e., are finitely generated $\mathbf{FI}$-modules (see [CEF]). In §5.4, we outline how this result might be extended when we drop the assumption that M is simply-connected by using the category $\mathbf{FI}_{\pi_1(M)}$.
- As we explain in §6.2, representations of the category $\mathbf{FS}_G^{\mathrm{op}}$ when G is a symmetric group are essentially equivalent to Δ-modules, in the sense of [Sn]. This observation

was our original source of motivation for studying the category $\mathbf{FS}_G^{\mathrm{op}}$: since $\mathbf{FS}_G^{\mathrm{op}}$-modules are much easier to think about than Δ-modules, this point of view could be profitable. Indeed, our results on $\mathbf{FS}_G^{\mathrm{op}}$-modules imply the main theorems about Δ-modules from [Sn], and more, and are less abstruse than the proofs given there. These theorems apply for any G, though, and so represent a significant generalization of the theory of Δ-modules.

- The main theorem on Hilbert series of Δ-modules in [Sn], Theorem B, was suspected to be suboptimal. Our original motivation in proving Theorem 1.3.2 was to improve [Sn, Theorem B], which it does. We subsequently found an even stronger improvement, which appears in [SS2, Theorem 9.1.3]. However, [SS2, Theorem 9.1.3] is very specific to Δ-modules, while Theorem 1.3.2 applies to any group G.

1.5. **Open problems.**

1.5.1. *Optimal results for Hilbert series of* $\mathbf{FS}_G^{\mathrm{op}}$*-modules.* Let G be a finite group whose order is invertible in the field $\mathbf{k}$, and let M be a finitely generated $\mathbf{FS}_G^{\mathrm{op}}$-module. Consider the minimal subfield F of $\mathbf{C}$ with the following property: $\mathrm{H}_M(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$ where $f \in F[\mathbf{t}]$ and $g(\mathbf{t})$ factors as $\prod(1-\lambda_i)$ where each λ_i is a linear combination of the $\mathbf{t}$ with coefficients in the ring of integers $\mathcal{O}_F$.

When $\mathbf{k}$ is algebraically closed, we prove $F \subseteq \mathbf{Q}(\zeta_N)$, where N is the exponent of G. When G is the symmetric group, we show (Corollary 6.5.4) that $F = \mathbf{Q}$. It would be interesting to determine F in general. This is related to finding optimal good collections of subgroups of G in the sense of §6.4, although it is probably necessary to find an alternative approach.

1.5.2. *A reconstruction problem.* Theorem 1.2.4 shows that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ knows very little of G (only the number of irreducible representations). In contrast, $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$ knows a lot about G: for instance, it knows about tensor products of G representations. It seems reasonable to think one could recover G from $\mathrm{Rep}_{\mathbf{k}}(G)$.

Here is a precise question. Let G and H be finite groups and let $\mathbf{k}$ be an algebraically closed field. Suppose $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$ and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_H^{\mathrm{op}})$ are equivalent as $\mathbf{k}$-linear abelian categories. Are G and H isomorphic?

1.6. **Outline.** In §2, we review material that we will use often, especially the main results from [SS2]. In §3, we introduce a class of formal languages, the **quasi-ordered languages**, and prove results about their Hilbert series. In §4, we discuss representations of the category $\mathbf{FWS}_\Lambda$ of finite weighted sets. Quasi-ordered languages are used to establish the main result about Hilbert series of representations of this category, and this result about Hilbert series is the key input to the proof of Theorem 1.3.2. In §5 we investigate the category $\mathbf{FI}_G$, and prove Theorems 1.2.1, 1.2.3, and 1.2.4. In §6, we study $\mathbf{FS}_G$. We prove Theorems 1.3.1 and 1.3.2, discuss the connection to Δ-modules, and give some examples.

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2. Background

2.1. **Representations of categories.** In this section, we recall the main combinatorial preliminaries that we need from [SS2]. We also prove some additional results.

Let $\mathcal{C}$ be an essentially small category. We denote by $|\mathcal{C}|$ the set of isomorphism classes in $\mathcal{C}$. We say that $\mathcal{C}$ is **directed** if every self-map in $\mathcal{C}$ is the identity. If $\mathcal{C}$ is directed, then $|\mathcal{C}|$ is naturally a poset by defining $x \le y$ if there exists a morphism $x \to y$.

Fix a nonzero ring $\mathbf{k}$ (not necessarily commutative) and let $\mathrm{Mod}_{\mathbf{k}}$ denote the category of left $\mathbf{k}$-modules. A **representation** of $\mathcal{C}$ (or a **$\mathcal{C}$-module**) over $\mathbf{k}$ is a functor $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$. A map of $\mathcal{C}$-modules is a natural transformation. We write $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ for the category of representations, which is abelian.

Let x be an object of $\mathcal{C}$. Define a representation P_x of $\mathcal{C}$ by $P_x(y) = \mathbf{k}[\mathrm{Hom}(x,y)]$, i.e., $P_x(y)$ is the free left $\mathbf{k}$-module with basis $\mathrm{Hom}(x,y)$. If M is another representation then $\mathrm{Hom}(P_x, M) = M(x)$. This shows that $\mathrm{Hom}(P_x, -)$ is an exact functor, and so P_x is a projective object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$. We call it the **principal projective** at x. A $\mathcal{C}$-module is **finitely generated** if it is a quotient of a finite direct sum of principal projective objects.

An object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every ascending chain of subobjects stabilizes; this is equivalent to every subrepresentation being finitely generated. The category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every finitely generated object in it is.

Let $\Phi\colon \mathcal{C} \to \mathcal{C}'$ be a functor. There is then a pullback functor on representations

$$\Phi^*\colon \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}') \to \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}).$$

Definition 2.1.1. We say that Φ satisfies **property (F)** if the following condition holds: given any object x of $\mathcal{C}'$ there exist finitely many objects $y_1, \dots, y_n$ of $\mathcal{C}$ and morphisms $f_i\colon x \to \Phi(y_i)$ in $\mathcal{C}'$ such that for any object y of $\mathcal{C}'$ and any morphism $f\colon x \to \Phi(y)$ in $\mathcal{C}'$, there exists a morphism $g\colon y_i \to y$ in $\mathcal{C}$ such that $f = \Phi(g) \circ f_i$. $\square$

Proposition 2.1.2. *Suppose that $\Phi\colon \mathcal{C} \to \mathcal{C}'$ is an essentially surjective functor. Let M be an object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ such that $\Phi^*(M)$ is finitely generated (resp. noetherian). Then M is finitely generated (resp. noetherian).*

Proposition 2.1.3. *Suppose $\Phi\colon \mathcal{C} \to \mathcal{C}'$ satisfies property (F). Then Φ^* takes finitely generated objects of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ to finitely generated objects of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Proposition 2.1.4. *Consider functors $\Phi\colon \mathcal{C}_1 \to \mathcal{C}_2$ and $\Psi\colon \mathcal{C}_2 \to \mathcal{C}_3$.*

(a) *If Φ, Ψ satisfy property (F), then the composition $\Psi \circ \Phi$ satisfies property (F).*
(b) *If Φ is essentially surjective and $\Psi \circ \Phi$ satisfies property (F), then Ψ satisfies property (F).*

A **norm** on $\mathcal{C}$ is a function $\nu\colon |\mathcal{C}| \to \mathbf{N}^I$, where I is a finite set. A **normed category** is a category equipped with a norm. Fix a category $\mathcal{C}$ with a norm ν with values in $\mathbf{N}^I$; let $\{t_i\}_{i \in I}$ be indeterminates. Let M be a representation of $\mathcal{C}$ over a field $\mathbf{k}$. We define the **Hilbert series** of M as

$$\mathrm{H}_{M,\nu}(\mathbf{t}) = \sum_{x \in |\mathcal{C}|} \dim_{\mathbf{k}} M(x) \cdot \mathbf{t}^{\nu(x)},$$

when this makes sense. We omit the norm ν from the notation when possible.

2.2. **Gröbner bases.** A poset X is **noetherian** if for every sequence $x_1, x_2, \dots$ of elements in X, there exists $i < j$ such that $x_i \le x_j$. See [SS2, §2] for basic facts.

For an object x, let $S_x\colon \mathcal{C} \to \mathbf{Set}$ be the functor given by $S_x(y) = \mathrm{Hom}(x,y)$. Note that $P_x = \mathbf{k}[S_x]$. An **ordering** on S_x is a choice of well-order on $S_x(y)$, for each $y \in \mathcal{C}$, such that for every morphism $y \to z$ in $\mathcal{C}$ the induced map $S_x(y) \to S_x(z)$ is strictly order-preserving. We write $\preceq$ for an ordering; S_x is **orderable** if it admits an ordering.

Set $\widetilde{S}_x = \bigcup_{y \in \mathcal{C}} S_x(y)$. Given $f \in S(y)$ and $g \in S(z)$, define $f \le g$ if there exists $h \colon y \to z$ with $h_*(f) = g$. Define an equivalence relation on $\widetilde{S}_x$ by $f \sim g$ if $f \le g$ and $g \le f$. The poset $|S_x|$ is the quotient of $\widetilde{S}_x$ by $\sim$, with the induced partial order.

Definition 2.2.1. We say that $\mathcal{C}$ is **Gröbner** if, for all objects x, the functor S_x is orderable and the poset $|S_x|$ is noetherian. We say that $\mathcal{C}$ is **quasi-Gröbner** if there exists a Gröbner category $\mathcal{C}'$ and an essentially surjective functor $\mathcal{C}' \to \mathcal{C}$ satisfying property (F). □

Theorem 2.2.2. *Let $\mathcal{C}$ be a quasi-Gröbner category. Then for any left-noetherian ring $\mathbf{k}$, the category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian.*

Proof. See [SS2, Theorem 5.2.2]. □

Proposition 2.2.3. *Suppose that $\Phi \colon \mathcal{C}' \to \mathcal{C}$ is an essentially surjective functor satisfying property (F) and $\mathcal{C}'$ is quasi-Gröbner. Then $\mathcal{C}$ is quasi-Gröbner.*

Proposition 2.2.4. *The cartesian product of finitely many (quasi-)Gröbner categories is (quasi-)Gröbner.*

3. Quasi-ordered languages

Let Λ be a finite abelian group and let $\varphi \colon \Sigma \to \Lambda$ be a function. Extend φ to a monoid homomorphism on $\Sigma^\star$. Given a subset S of Λ, let $\Sigma^\star_{\varphi,S}$ be the set of all words $w \in \Sigma^\star$ for which $\varphi(w) \in S$. We say that a language $\mathcal{L} \subset \Sigma^\star$ is a **congruence language** if it is of the form $\Sigma^\star_{\varphi,S}$ for some Λ, φ and S. The **modulus** of a congruence language is the exponent of the group Λ. (Recall that the **exponent** of a group is the least common multiple of the orders of all elements in the group.)

Let $F(\mathbf{t})$ be a power series in variables $\mathbf{t} = (t_1, \ldots, t_r)$. An **$N$-cyclotomic translate** of F is a series of the form $F(\zeta_1 t_1, \ldots, \zeta_r t_r)$, where $\zeta_1, \ldots, \zeta_r$ are Nth roots of unity.

Lemma 3.1. *Let Λ be a finite abelian group of exponent N, let S be a subset of Λ, and let $\psi \colon \mathbf{Z}^r_{\ge 0} \to \Lambda$ be a monoid homomorphism. Suppose that $F(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^r} a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$ is a power series over $\mathbf{C}$. Let $G(\mathbf{t}) = \sum a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$, where the sum is extended over those $\mathbf{n} \in \mathbf{N}$ for which $\psi(\mathbf{n}) \in S$. Then G is a $\mathbf{Q}(\zeta_N)$-linear combination of N-cyclotomic translates of F.*

Proof. We have $G(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^r} \chi(\psi(\mathbf{n})) a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$, where $\chi \colon \Lambda \to \{0, 1\}$ is the characteristic function of S. To obtain the result, simply express χ as a $\mathbf{Q}(\zeta_N)$-linear combination of characters of Λ. □

Proposition 3.2. *Let $\mathcal{L}$ be a language on Σ equipped with a universal norm ν with values in $\mathbf{N}^I$, let $\mathcal{K}$ be a congruence language on Σ of modulus N, and let $\mathcal{L}' = \mathcal{L} \cap \mathcal{K}$. Then $\mathrm{H}_{\mathcal{L}',\nu}(\mathbf{t})$ is a $\mathbf{Q}(\zeta_N)$-linear combination of N-cyclotomic translates of $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$.*

Proof. Choose $\varphi \colon \Sigma \to \Lambda$ and $S \subset \Lambda$ so that $\mathcal{K} = \Sigma^\star_{\varphi,S}$. Since ν is universal, the map $\varphi \colon \Sigma^\star \to \mathcal{L}$ can be factored as $\psi \circ \nu$, where $\psi \colon \mathbf{N}^I \to \Lambda$ is a monoid homomorphism. Thus if $\mathrm{H}_{\mathcal{L}}(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^I} a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$, then $\mathrm{H}_{\mathcal{L}'}(\mathbf{t})$ is obtained by simply discarding the terms for which $\psi(\mathbf{n}) \notin S$. The result now follows from Lemma 3.1. □

A **quasi-ordered language** (of modulus N) is the intersection of an ordered language and a congruence language (of modulus N). Quasi-ordered languages are regular. The class of quasi-ordered languages is not closed under unions, intersections, or concatenations.

Our main result on quasi-ordered languages is the following theorem.

Theorem 3.3. *Let $\mathcal{L}$ be a quasi-ordered language of modulus N equipped with a norm valued in $\mathbf{N}^I$ adapted to a subpartition $\{I_1, \ldots, I_r\}$. Then $\mathrm{H}_{\mathcal{L}}(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$, where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials with coefficients in $\mathbf{Q}(\zeta_N)$, and $g(\mathbf{t})$ factors as $\prod_{i=1}^{n}(1-\lambda_k)$, where for each k there exists a j such that λ_k is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i, for $i \in I_j$.*

Proof. This follows immediately from [SS2, Theorem 3.3.7] and Proposition 3.2. □

Definition 3.4. Let $N \geq 1$ be an integer. We say that $h \in \mathbf{Q}[\![t_1, \ldots, t_r]\!]$ is of class $\mathcal{K}_N$ if it can be written as $f(\mathbf{t})/g(\mathbf{t})$ where f and g are polynomials in the t_i with coefficients in $\mathbf{Q}(\zeta_N)$ and g factors as $\prod_{k=1}^{n}(1-\lambda_i)$, where λ_i is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i.

Here is a coordinate-free version of the definition. Suppose Ξ is a finite free $\mathbf{Z}$-module and $f \in \widehat{\mathrm{Sym}}(\Xi_{\mathbf{Q}})$. We say that f is $\mathcal{K}_N$ if there is a $\mathbf{Z}$-basis $t_1, \ldots, t_r$ of Ξ so that f is $\mathcal{K}_N$ as a series in the t_i. This is independent of the choice of basis.

We drop the N from the notation if it is irrelevant. □

Lemma 3.5. *Let $i \colon \Xi \to \Xi'$ be a split injection of finite free $\mathbf{Z}$-modules, and let f be a series in $\widehat{\mathrm{Sym}}(\Xi_{\mathbf{Q}})$. Suppose that $i(f) \in \widehat{\mathrm{Sym}}(\Xi'_{\mathbf{Q}})$ is $\mathcal{K}_N$. Then f is $\mathcal{K}_N$.*

Proof. Let $j \colon \Xi' \to \Xi$ be a splitting of i. Then $f = j(i(f))$. Since j clearly takes $\mathcal{K}_N$ functions to $\mathcal{K}_N$ functions, it follows that f is $\mathcal{K}_N$. □

Suppose that $\mathcal{C}$ is directed and normed over $\mathbf{N}^I$ and pick an object x of $\mathcal{C}$. We define a norm on $|S_x|$ as follows: given $f \in |S_x|$, let $\widetilde{f} \in S_x(y)$ be a lift, and put $\nu(f) = \nu(y)$. This is well-defined because $\mathcal{C}$ is ordered: if $\widetilde{f}' \in S(z)$ is a second lift then y and z are isomorphic.

A QO_N**-lingual structure** on $|S_x|$ is a pair (Σ, i) consisting of a finite alphabet Σ normed over $\mathbf{N}^I$ and an injection $i \colon |S_x| \to \Sigma^\star$ compatible with the norms, i.e., such that $\nu(i(f)) = \nu(f)$ and such that for every poset ideal J of $|S_x|$, the language $i(J)$ is QO_N-lingual.

Theorem 3.6. *Let $\mathcal{C}$ be a QO_N-lingual Gröbner category and let M be a finitely generated representation of $\mathcal{C}$. Then $\mathrm{H}_M(\mathbf{t})$ is $\mathcal{K}_N$, i.e., is a rational function $f(\mathbf{t})/g(\mathbf{t})$, where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials with coefficients in $\mathbf{Q}(\zeta_N)$ and $g(\mathbf{t})$ factors as $\prod_{j=1}^{n}(1-\lambda_j)$ and each λ_j is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i.*

A normed category $\mathcal{C}$ is **strongly QO_N-lingual** if for each object x there exists a lingual structure $i \colon |S_x| \to \Sigma^\star$ and a congruence language $\mathcal{K}$ on Σ such that for every poset ideal I of $|S_x|$, the language $i(I)$ is the intersection of an ordered language with $\mathcal{K}$. (If we drop the adjective "strongly," then the congruence language $\mathcal{K}$ is allowed to depend on the ideal I.) We have the following variant of [SS2, Proposition 6.3.3]:

Proposition 3.7. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strongly QO_N-lingual normed categories. Suppose the posets $|\mathcal{C}_{1,x}|$ and $|\mathcal{C}_{2,y}|$ are noetherian for all x and y. Then $\mathcal{C}_1 \times \mathcal{C}_2$ is strongly QO_N-lingual.*

Proof. Let x_j be an object of $\mathcal{C}_j$, and let $i_j \colon |\mathcal{C}_{j,x_j}| \to \Sigma_j^\star$ be a strongly QO_N-lingual structure at x_j. Let $\Sigma = \Sigma_1 \amalg \Sigma_2$, normed over $\mathbf{N}^I$ in the obvious manner. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the given congruence languages of modulus N on Σ_1 and Σ_2, regarded as languages on Σ. Write $\mathcal{K}_i = (\Sigma_i^\star)_{\varphi_i, S_i}$, where $\varphi_i \colon \Sigma_i \to \Lambda_i$. Let $\varphi \colon \Sigma \to \Lambda_1 \oplus \Lambda_2$ be the map defined by $\varphi(x) = (\varphi_1(x), 0)$ for $x \in \Sigma_1$, and $\varphi(x) = (0, \varphi_2(x))$ for $x \in \Sigma_2$. Let $S = S_1 \times S_2$, and let $\mathcal{K} = \Sigma^\star_{\varphi,S}$. Then $\mathcal{K}$ is a congruence language on Σ of modulus N and has the following property: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are any languages on Σ_1 and Σ_2 then $(\mathcal{L}_1 \cap \mathcal{K}_1)(\mathcal{L}_2 \cap \mathcal{K}_2) = \mathcal{L}_1\mathcal{L}_2 \cap \mathcal{K}$.

The following diagram commutes:

$$\begin{tikzcd} {|\mathcal{C}_{(x_1,x_2)}|} \arrow[r, equal] & {|\mathcal{C}_{1,x_1}| \times |\mathcal{C}_{2,x_2}|} \arrow[r, "i_1 \times i_2"] \arrow[d] & \Sigma_1^\star \times \Sigma_2^\star \arrow[r] \arrow[d] & \Sigma^\star \arrow[d] \\ & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} \arrow[r, equal] & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} \arrow[r, equal] & \mathbf{N}^I \end{tikzcd}$$

The top right map is concatenation of words. We let $i\colon |\mathcal{C}_{(x_1,x_2)}| \to \Sigma^*$ be the composition along the first line, which is clearly injective. We claim that this is a strongly QO_N-lingual structure on $|\mathcal{C}_{(x_1,x_2)}|$. The commutativity of the above diagram shows that it is a lingual structure. Now suppose I is an ideal of $|\mathcal{C}_{(x_1,x_2)}|$. Since this poset is noetherian (being a direct product of noetherian posets), I is a finite union of principal ideals $I_1, \dots, I_n$. Each I_j is of the form $T_j \times T'_j$, where T_j is an ideal of $\mathcal{C}_{1,x_1}$ and T'_j is an ideal of $\mathcal{C}_{2,x_2}$. By assumption, the languages $i_1(T_j)$ and $i_2(T'_j)$ are strongly QO_N.

Then $i_1(T_j)$ is the intersection of an ordered language with $\mathcal{K}_1$, while $i_2(T'_j)$ is the intersection of an ordered language with $\mathcal{K}_2$. It follows that $i(I_j)$ is the intersection of an ordered language with $\mathcal{K}$. Thus $i(I)$ is the intersection of an ordered language with $\mathcal{K}$, completing the proof. $\square$

4. Categories of weighted surjections

In this section we study a generalization of the category of finite sets with surjective functions by considering weighted sets. This is preparatory material for the next section.

4.1. **The categories $\mathbf{OWS}_\Lambda$ and $\mathbf{FWS}_\Lambda$.** Let Λ be a finite abelian group. A **weighting** on a finite set S is a function $\varphi\colon S \to \Lambda$. A **weighted set** is a set equipped with a weighting; we write φ_S to denote the weighting. Suppose φ is a weighting on S, and let $f\colon S \to T$ be a map of sets. We define $f_*(\varphi)$ to be the weighting on T given by $f_*(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$. A **map** of weighted sets $S \to T$ is a surjective function $f\colon S \to T$ such that $f_*(\varphi_S) = \varphi_T$. We let $\mathbf{FWS}_\Lambda$ denote the category of weighted sets.

We require an ordered version of the category as well. Let $\mathbf{OWS}_\Lambda$ be the following category. The objects are totally ordered weighted sets. The order and weighting are not required to interact in any way. The morphisms are ordered maps of weighted sets, i.e., a morphism $S \to T$ is a surjective function $f\colon S \to T$ such that $f_*(\varphi_S) = \varphi_T$, and for all $x < y$ in T we have $\min f^{-1}(x) < \min f^{-1}(y)$.

Enumerate Λ as $\lambda_1, \dots, \lambda_r$, so that we can identify $\mathbf{N}^\Lambda$ with $\mathbf{N}^r$. We define a norm on $\mathbf{OWS}_\Lambda$ by $\nu(S) = (n_1, \dots, n_r)$, where $n_i = \#\varphi_S^{-1}(\lambda_i)$.

Our main result about $\mathbf{OWS}_\Lambda$ is:

Theorem 4.1.1. *The category $\mathbf{OWS}_\Lambda^{\mathrm{op}}$ is Gröbner and strongly QO_N-lingual, where N is the exponent of Λ.*

We prove this in the next section, and now use it to study $\mathbf{FWS}_\Lambda$.

Theorem 4.1.2. *The category $\mathbf{FWS}_\Lambda^{\mathrm{op}}$ is quasi-Gröbner.*

Proof. The forgetful functor $\Phi\colon \mathbf{OWS}_\Lambda \to \mathbf{FWS}_\Lambda$ is easily seen to satisfy property (F), and so the result follows from Theorem 4.1.1. $\square$

Corollary 4.1.3. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FWS}_\Lambda^{\mathrm{op}})$ is noetherian.*

Let M be a finitely generated $(\mathbf{FWS}^{\mathrm{op}}_{\Lambda_1} \times \cdots \times \mathbf{FWS}^{\mathrm{op}}_{\Lambda_r})$-module. Enumerate Λ_i as $\{\lambda_{i,j}\}$, and let $t_{i,j}$ be a formal variable corresponding to $\lambda_{i,j}$. Given $\mathbf{n} \in \mathbf{N}^{\#\Lambda_i}$, let $[\mathbf{n}]$ be the Λ_i-weighted set where n_j elements have weight $\lambda_{i,j}$. When $\mathbf{k}$ is a field, define the Hilbert series of M by

$$\mathrm{H}_M(\mathbf{t}) = \sum_{\mathbf{n}(1),\ldots,\mathbf{n}(r)} C_{\mathbf{n}(1)} \cdots C_{\mathbf{n}(r)} \dim_{\mathbf{k}} M([\mathbf{n}(1)], \ldots, [\mathbf{n}(r)]) \cdot \mathbf{t}_1^{\mathbf{n}(1)} \cdots \mathbf{t}_r^{\mathbf{n}(r)},$$

where for $\mathbf{n} \in \mathbf{N}^k$ we write $C_{\mathbf{n}}$ for the multinomial coefficient

$$C_{\mathbf{n}} = \frac{|\mathbf{n}|!}{\mathbf{n}!} = \frac{|\mathbf{n}|!}{n_1! \cdots n_k!}.$$

The following theorem is the main result we need in our applications, and follows immediately from Theorem 4.1.1 and the functor Φ used in the proof of the previous theorem. (Note: the reason the multinomial coefficient appears is that there are $C_{\mathbf{n}}$ isomorphism classes in $\mathbf{OWS}_{\Lambda_i}$ which map to the isomorphism class of $[\mathbf{n}]$ in $\mathbf{FWS}_{\Lambda_i}$.)

Theorem 4.1.4. *In the above situation, $\mathrm{H}_M(\mathbf{t})$ is $\mathcal{K}_N$ (see Definition 3.4).*

4.2. **Proof of Theorem 4.1.1.** Fix a finite set L and let $\Sigma = L \times \Lambda$. Given $a \in L$ and $\alpha \in \Lambda$, we write $\frac{a}{\alpha}$ for the corresponding element of Σ. We denote elements of $\Sigma^\star$ by $\frac{s}{\sigma}$, where $s \in L^\star$ and $\sigma \in \Lambda^\star$ are words of equal length. For $a \in L$, we define $w_a \colon \Sigma^\star \to \Lambda$ by

$$w_a\left(\frac{s_1 \cdots s_n}{\sigma_1 \cdots \sigma_n}\right) = \sum_{s_i = a} \sigma_i.$$

We let $\mathbf{w} \colon \Sigma^\star \to \Lambda^L$ be $(w_a)_{a \in L}$. For $\theta \in \Lambda^L$, we let $\mathcal{K}_\theta$ be the set of all $\frac{s}{\sigma} \in \Sigma^\star$ with $\mathbf{w}(\frac{s}{\sigma}) = \theta$. This is a congruence language of modulus N (the exponent of Λ).

We now define a partial order on $\Sigma^\star$. Let $\frac{s}{\sigma} \colon [n] \to \Sigma$ and $\frac{t}{\tau} \colon [m] \to \Sigma$ be two words. Define $\frac{s}{\sigma} \le \frac{t}{\tau}$ if there exists an ordered surjection $f \colon [m] \to [n]$ such that $t = f^*(s)$ and $\sigma = f_*(\tau)$. Note that if $\frac{s}{\sigma} \le \frac{t}{\tau}$ then $\mathbf{w}(\frac{s}{\sigma}) = \mathbf{w}(\frac{t}{\tau})$.

Lemma 4.2.1. *The poset $\Sigma^\star$ is noetherian.*

Proof. Given $\frac{s}{\sigma} \in \Sigma^\star$, call a value of s *exceptional* if it appears exactly once. If s has an exceptional value, then let $m(s)$ denote the index, counting from the end, of the first non-exceptional value.

Suppose that $\Sigma^\star$ is not noetherian and choose a minimal bad sequence $x_1 = \frac{s_1}{\sigma_1}, x_2 = \frac{s_2}{\sigma_2}, \ldots$. We can find an infinite subsequence $x_{i_1}, x_{i_2}, \ldots$ such that $m(s_{i_j})$ is defined and is constant, say equal to m_0 (assume we pick the subsequence maximal with this property, i.e., we do not leave out any of the x_i which satisfy the property). Let $w_{i_j} = \frac{t_{i_j}}{\tau_{i_j}}$ be the subword of x_{i_j} of elements (except for the first instance) whose numerator is m_0. For $j \gg 0$, we can find a subword w'_{i_j} of w_{i_j} such that the sum of the elements in the denominator sum to 0. These subwords cannot form a bad sequence, or else, denoting by k the first j where w'_{i_j} exists, the sequence $x_1, x_2, \ldots, x_{i_k - 1}, w'_{i_k}, \ldots$ is bad (we cannot have $w'_{i_{k'}} \ge x_i$ or else the numerator of x_i is the constant word m_0 and its denominator sums to 0 which means it would be one of the w_{i_j}) and contradicts minimality. So we pick a further subsequence of the x_{i_j} so that the w'_{i_j} form a weakly increasing sequence.

Let $y_{i_j} = \frac{s'_{i_j}}{\sigma'_{i_j}}$ be the result of removing the subword w'_{i_j} from $\frac{s_{i_j}}{\sigma_{i_j}}$. Then $y_{i_j} \le x_{i_j}$ by construction of w'_{i_j}. If $x_1, \dots, x_{i_1-1}, y_{i_1}, y_{i_2}, \dots$ is a bad sequence, it violates minimality of $x_1, x_2, \dots$, so we must have $y_{i_j} \le y_{i_k}$ for some $j < k$ ($x_i \le y_{i_j}$ implies $x_i \le x_{i_j}$ from what we just said). This inequality is witnessed by an ordered surjection $f \colon [\ell(y_{i_k})] \to [\ell(y_{i_j})]$, i.e., $s'_{i_k} = f^*(s'_{i_j})$ and $\sigma'_{i_j} = f_*(\sigma'_{i_k})$. We also have an inequality $w'_{i_j} \le w'_{i_k}$ so we can take the disjoint union of the two ordered surjections to get a surjection (which is still ordered because we removed the initial instance of m_0 from w_{i_j} above) that witnesses the inequality $x_{i_j} \le x_{i_k}$. This is a contradiction, so we conclude that minimal bad sequences do not exist and that $\Sigma^\star$ is noetherian. □

Let $\frac{s}{\sigma} = \frac{s_1 \cdots s_n}{\sigma_1 \cdots \sigma_n}$ be a word in $\mathcal{K}_\theta$. Put

$$\Pi_i = \left\{ \tfrac{s_1}{*}, \cdots, \tfrac{s_i}{*} \right\},$$

where $*$ means any element of Λ. Define a language $\mathcal{L}(\frac{s}{\sigma})$ by

$$\mathcal{L}(\tfrac{s}{\sigma}) = (\tfrac{s_1}{\sigma_1})\Pi_1^\star \cdots (\tfrac{s_n}{\sigma_n})\Pi_n^\star.$$

It is clear that $\mathcal{L}(\frac{s}{\sigma})$ is an ordered language.

Lemma 4.2.2. *If $\frac{t}{\tau} \in \mathcal{L}(\frac{s}{\sigma}) \cap \mathcal{K}_\theta$ then $\frac{s}{\sigma} \le \frac{t}{\tau}$.*

Proof. Let $\frac{t}{\tau} \colon I \to \Sigma^\star$ in $\mathcal{L}(\frac{s}{\sigma}) \cap \mathcal{K}_\theta$ be given. Write $\frac{t}{\tau} = (\frac{s_1}{\sigma_1})w_1 \cdots (\frac{s_n}{\sigma_n})w_n$, with $w_i \in \Pi_i^\star$. Let $J \subset I$ be the indices occurring in the words $w_1, \dots, w_n$ and let K be the complement of J, so that $\frac{t}{\tau}|_K = \frac{s}{\sigma}$. We now define a map $f \colon I \to [n]$. On K, we let f be the unique order-preserving bijection. For $a \in \{s_1, \dots, s_n\}$, let $r(a) \in [n]$ be minimal so that $s_{r(a)} = a$. Now define f on J by $f(j) = r(t_j)$. It is clear that f is an ordered surjection and that $f^*(s) = t$. Since $\mathbf{w}(\frac{s}{\sigma}) = \mathbf{w}(\frac{t}{\tau}) = \theta$, it follows that $\mathbf{w}(\frac{t}{\tau}|_J) = 0$. From the way we defined f, it follows that $f_*(\tau|_J) = 0$. Thus $f_*(\tau) = \sigma$, which completes the proof. □

We say that a word $\sigma_1 \cdots \sigma_n \in \Lambda^\star$ is **minimal** if no non-empty subsequence of $\sigma_2 \cdots \sigma_n$ sums to 0. Note that we started with the second index. As any sufficiently long sequence in $\Lambda^\star$ contains a subsequence summing to 0, there are only finitely many minimal words. Let $\frac{s}{\sigma}$ in $\mathcal{K}_\theta$ be given. We say that $\frac{t}{\tau} \colon [m] \to \Sigma^\star$ is **minimal** over $\frac{s}{\sigma} \colon [n] \to \Sigma^\star$ if there is an ordered surjection $f \colon [m] \to [n]$ such that $t = f^*(s)$ and $\sigma = f_*(\tau)$ and for every $i \in [n]$ the word $\tau|_{f^{-1}(i)}$ is minimal. If $\frac{t}{\tau}$ is minimal over $\frac{s}{\sigma}$ then the length of $\frac{t}{\tau}$ is bounded, so there are only finitely many such minimal words.

Lemma 4.2.3. *Let $\frac{s}{\sigma} \le \frac{r}{\rho}$ be words in $\mathcal{K}_\theta$. There exists $\frac{t}{\tau}$ minimal over $\frac{s}{\sigma}$ such that $\frac{r}{\rho} \in \mathcal{L}(\frac{t}{\tau})$.*

Proof. Let $[n]$ and $[m]$ be the index sets of $\frac{s}{\sigma}$ and $\frac{r}{\rho}$, and choose a witness $f \colon [m] \to [n]$ to $\frac{s}{\sigma} \le \frac{r}{\rho}$. Let $I \subset [m]$ be the set of elements of the form $\min f^{-1}(i)$ for $i \in [n]$. Let $K \subset [m]$ be minimal subject to $I \subset K$ and $f_*(\rho|_K) = \sigma$. Then $\rho|_{f^{-1}(i) \cap K}$ is minimal for all $i \in [n]$. Indeed, if it were not then we could discard a subsequence summing to 0 and make K smaller. We thus see that $\frac{t}{\tau} = \frac{r}{\rho}|_K$ is minimal over $\frac{s}{\sigma}$. If $i \in [m] \setminus K$ then there exists $j < i$ in I with $t_i = t_j$, and so $\frac{r}{\rho} \in \mathcal{L}(\frac{t}{\tau})$. □

Lemma 4.2.4. *Every poset ideal of $\mathcal{K}_\theta$ is of the form $\mathcal{L} \cap \mathcal{K}_\theta$, where $\mathcal{L}$ is an ordered language on Σ.*

Proof. It suffices to treat the case of a principal ideal. Thus consider the ideal S generated by $\frac{s}{\sigma} \in \mathcal{K}_\theta$. Let $\frac{t_i}{\tau_i}$ for $1 \le i \le n$ be the words minimal over $\frac{s}{\sigma}$, and let $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}(\frac{t_i}{\tau_i})$. Then $\mathcal{L}$ is an ordered language, by construction. If $\frac{r}{\rho} \in \mathcal{L} \cap \mathcal{K}_\theta$ then $\frac{r}{\rho} \in \mathcal{L}(\frac{t_i}{\tau_i}) \cap \mathcal{K}_\theta$ for some i, and so $\frac{s}{\sigma} \le \frac{t_i}{\tau_i} \le \frac{r}{\rho}$ by Lemma 4.2.2, and so $\frac{r}{\rho} \in S$. Conversely, suppose $\frac{r}{\rho} \in S$. Then $\frac{r}{\rho} \in \mathcal{L}(\frac{t_i}{\tau_i})$ for some i by Lemma 4.2.3, and of course $\frac{r}{\rho} \in \mathcal{K}_\theta$, and so $\frac{r}{\rho} \in \mathcal{L} \cap \mathcal{K}_\theta$. □

Proof of Theorem 4.1.1. The category $\mathcal{C} = \mathbf{OWS}_\Lambda^{\mathrm{op}}$ is clearly directed. Let $x = ([n], \theta)$ be an object of $\mathcal{C}$. We apply the above theory with $L = [n]$. Suppose $f \colon x \to y$ is a map in $\mathcal{C}$, with $y = ([m], \varphi)$; note that this means that f is a surjection $[m] \to [n]$. We define a word $[m] \to \Sigma^\star$ by mapping $i \in [m]$ to $(f(i), \varphi(i))$. Obviously, one can reconstruct f from this word, and so this defines an injection $i \colon |\mathcal{C}_x| \to \Sigma^\star$. In fact, the image lands in $\mathcal{K}_\theta$. It is clear from the definition of the order on $\Sigma^\star$ that i is strictly order-preserving. Thus $|\mathcal{C}_x|$ is noetherian by Lemma 4.2.1. Lexicographic order on $\Sigma^\star$ induces an admissible order on $|\mathcal{C}_x|$. Finally, since i maps ideals to ideals, we see that it gives a strong QO_N-lingual structure on $|\mathcal{C}_x|$ by Lemma 4.2.4. □

5. Categories of G-injections

5.1. **Finite groups.** In this section, we assume that G is finite. We have the following basic property about representations of $\mathbf{FI}_G$:

Proposition 5.1.1. *There are natural functors $\mathbf{FA} \to \mathbf{FA}_G$ and $\mathbf{FI} \to \mathbf{FI}_G$ that satisfy property (F).*

Proof. Define a functor $\Phi \colon \mathbf{FA} \to \mathbf{FA}_G$ that sends a set to itself and a function $f \colon S \to T$ to $(f, 1)$ where $1 \colon S \to G$ is the constant map sending every element to the identity of G. To see that Φ satisfies property (F), pick a set x of size n, set $y_1, \ldots, y_{n^{|G|}}$ all equal to x and let $f_1, \ldots, f_{n^{|G|}}$ correspond to all automorphisms of x in $\mathbf{FA}_G$ under some enumeration.

For the second functor, note that Φ restricts to a functor $\mathbf{FI} \to \mathbf{FI}_G$. □

Corollary 5.1.2. *The categories $\mathbf{FA}_G$ and $\mathbf{FI}_G$ are quasi-Gröbner.*

Proof. This follows from Proposition 2.2.3 and [SS2, Theorems 7.1.2, 7.4.4]. □

Corollary 5.1.3. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA}_G)$ and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ are noetherian.*

Corollary 1.2.2 improves this result by allowing G to be any polycyclic-by-finite group.

Remark 5.1.4. Define a category $\mathbf{FI}_{d,G}$ of finite sets whose morphisms are pairs (f, σ) where f is a decorated injection as in the definition of $\mathbf{FI}_d$ and σ is as in the definition of $\mathbf{FI}_G$. As above, there is a natural functor $\mathbf{FI}_d \to \mathbf{FI}_{d,G}$ satisfying property (F). □

The category $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ only depends on $\mathrm{Rep}_{\mathbf{k}}(G)$ as an abelian category equipped with the extra structure of the invariants functor $\mathrm{Rep}_{\mathbf{k}}(G) \to \mathrm{Mod}_{\mathbf{k}}$. (See Proposition 5.2.4 for a precise statement.) Thus $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ "sees" very little of G. In good characteristic, we can be more explicit. Let $\mathbf{FB}$ be the groupoid of finite sets (maps are bijections).

Proposition 5.1.5. *Suppose $\mathbf{k}$ is a field in which the order of G is invertible. Then representations of $\mathbf{FI}_G$ are equivalent to representations of $\mathbf{FI} \times \mathbf{FB}^r$, where r is the number of non-trivial irreducible representations of G over $\mathbf{k}$.*

Proof. Let $V_1, \dots, V_r$ be the non-trivial irreducible representations of G, and let V_0 be the trivial representation. Suppose M is an $\mathbf{FI}_G$-module. We can then decompose $M(S)$ into isotypic pieces for the action of G^S:

$$M(S) = \bigoplus_{S=S_0 \amalg \cdots \amalg S_r} N(S_0, \dots, S_r) \otimes (V_0^{\boxtimes S_0} \boxtimes \cdots \boxtimes V_r^{\boxtimes S_r}), \tag{5.1.6}$$

where N is a multiplicity space. Suppose now that $f \colon S \to T$ is an injection. To build a morphism in $\mathbf{FI}_G$ we must also choose a function $\sigma \colon S \to G$. However, if σ and σ' are two choices then (f, σ) and (f, σ') differ by an element of $\mathrm{Aut}(T)$, namely an automorphism of the form (id_T, τ) where τ restricts to $\sigma' \sigma^{-1}$ on S. Thus it suffices to record the action of $(f, 1)$. Note that if $\tau \colon T \to G$ restricts to 1 on S then $(\mathrm{id}_T, \tau)(f, 1) = (f, 1)$. It follows that $(f, 1)$ must map $M(S)$ into the $G^{T \setminus S}$-invariants of $M(T)$. In other words, under the above decomposition, $(f, 1)$ induces a linear map

$$N(S_0, S_1, \dots, S_r) \to N(S_0 \amalg (T \setminus S), S_1, \dots, S_r).$$

Thus, associated to M we have built a representation N of $\mathbf{FI} \times \mathbf{FB}^r$. The above discussion makes clear that no information is lost in passing from M to N, and so this is a fully faithful construction. The inverse construction is defined by the formula (5.1.6). □

Remark 5.1.7. By the proposition, an $\mathbf{FI}_G$-module can be thought of as a sequence $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbf{N}^r}$, where each $M_{\mathbf{n}}$ is an $\mathbf{FI}$-module equipped with an action of $S_{\mathbf{n}}$. There are no transition maps, so in a finitely generated $\mathbf{FI}_G$-module, all but finitely many of the $M_{\mathbf{n}}$ are zero. Thus, at least in good characteristic, $\mathbf{FI}_G$-modules are not much different from $\mathbf{FI}$-modules, and essentially any result about $\mathbf{FI}$-modules (e.g., noetherianity) carries over to $\mathbf{FI}_G$-modules.

There are some similarities between the proposition and the results of [M, §6]. □

5.2. **$\mathbf{FI}_R$-modules.** We introduce a generalization of $\mathbf{FI}$-modules and prove a noetherianity result. As a special case, we improve Corollary 5.1.3 by allowing the group G to now be any polycyclic-by-finite group.

Let $\mathbf{k}$ be a ring. Let (R, ϵ) be an augmented $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra, that is, a $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra R equipped with a surjection of $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebras $\epsilon \colon R \to \mathbf{k}$. Note that if $\mathbf{k}$ is commutative, then a $\mathbf{k}$-algebra structure on R is the same as a $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra structure. We let $\mathfrak{a}$ be the kernel of ϵ, the **augmentation ideal**. For an R-module M we let $\Gamma(M)$ be the $\mathbf{k}$-submodule of M annihilated by $\mathfrak{a}$. For a finite set x, we write $R^{\otimes x}$ for the x-fold tensor product of R (over $\mathbf{k}$). If M is an $R^{\otimes y}$-module and x is a subset of y, we write $\Gamma_x(M)$ for the subspace of M annihilated by $\mathfrak{a}^{\otimes x}$.

An **$\mathbf{FI}_R$-module** is a rule M that attaches to every finite set x an $R^{\otimes x}$-module $M(x)$ and to every injection $f \colon x \to y$ of finite sets a map of $R^{\otimes x}$-modules $f_* \colon M(x) \to \Gamma_{y \setminus f(x)}(M(y))$ such that $(gf)_* = g_* f_*$ in the obvious sense. Here we regard $\Gamma_{y \setminus f(x)}(M(y))$ as an $R^{\otimes x}$-module via the homomorphism $f_* \colon R^{\otimes x} \to R^{\otimes y}$. Note that we have not actually defined a category $\mathbf{FI}_R$, but we still speak of $\mathbf{FI}_R$-modules. We write $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ for the category of $\mathbf{FI}_R$-modules. There are the usual notions of finite generation and noetherianity.

Let x be a finite set. We define the **principal projective $\mathbf{FI}_R$-module** at x by:

$$P_x(y) = \bigoplus_{f \colon x \to y} (R^{\otimes y} / \mathfrak{a}^{\otimes y \setminus f(x)}) \cdot e_f,$$

where the sum is over injections f. Thus $P_x(y)$ is spanned by the e_f, the annihilator of e_f is exactly $\mathfrak{a}^{\otimes y \setminus f(x)}$, and there are no other relations between the e_f. Note that $P_x(x)$ has a canonical element e_x, corresponding to the identity map $x \to x$. The following result justifies calling P_x the principal projective at x:

Lemma 5.2.1. *Let M be an $\mathbf{FI}_R$-module. Then the natural map $\mathrm{Hom}_{\mathbf{FI}_R}(P_x, M) \to M(x)$ given by evaluating on e_x is an isomorphism.*

Proof. It is clear that e_x generates P_x, and so the map is injective. Conversely, suppose that m is an element of $M(x)$. Given an injection of finite sets $f\colon x \to y$, the element $f_*(m)$ of $M(y)$ is annihilated by $\mathfrak{a}^{\otimes y \setminus f(x)}$, by definition. We therefore have a well-defined map of $R^{\otimes y}$ modules $P_x(y) \to M(y)$ given by $e_f \mapsto f_*(m)$. One readily verifies that this defines a map of $\mathbf{FI}_R$-modules taking e_x to m, which establishes surjectivity of the map in question. □

The above lemma shows that the P_x are projective, and that every finitely generated $\mathbf{FI}_R$-module is a quotient of a finite direct sum of the P_x's. We now come to our main result:

Theorem 5.2.2. *The category $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is noetherian if and only if $R^{\otimes n}$ is left-noetherian for all $n \geq 0$.*

Proof. Suppose first that $R^{\otimes n}$ is left-noetherian for all $n \geq 0$, and let us show that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is noetherian. It suffices to show that the principal projective P_x is noetherian. Let $P'_x\colon \mathbf{FI} \to \mathrm{Mod}_{R^{\otimes x}}$ be the principal projective representation of $\mathbf{FI}$ at x over the ring $R^{\otimes x}$. Thus $P'_x(y)$ is the free $R^{\otimes x}$-module with basis $\mathrm{Hom}_{\mathbf{FI}}(x, y)$. Suppose that $f\colon x \to y$ is an injection of finite sets. Then f_* induces an isomorphism $f_*\colon R^{\otimes x} \to R^{\otimes y}/\mathfrak{a}^{\otimes y \setminus f(x)}$. It follows that there is a natural isomorphism $\varphi_y\colon P'_x(y) \to P_x(y)$ given by $\lambda e_f \mapsto f_*(\lambda) e_f$. One readily verifies that if M is an $\mathbf{FI}_R$-submodule of P_x then $y \mapsto \varphi_y^{-1}(M(y))$ is a subobject of P'_x in the category $\mathrm{Rep}_{R^{\otimes x}}(\mathbf{FI})$. The noetherianity of this category ([SS2, Corollary 7.1.3]) now implies that P_x is noetherian, which completes the proof of this direction.

Now suppose that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is noetherian. The group S_n acts on the ring $R^{\otimes n}$, by permuting the factors, and so we can form the twisted group algebra $R^{\otimes n}[S_n]$. Left modules over this ring are the same as $\mathbf{FI}_R$-modules supported in degree n (i.e., which vanish on sets of cardinality other than n). Thus the noetherianity of $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ implies the left noetherianity of $R^{\otimes n}[S_n]$. The left-noetherianity of $R^{\otimes n}$ results from the following lemma. □

Lemma 5.2.3. *Let R be a ring and let G be a finite group acting on R. Suppose that the twisted group algebra $R[G]$ is left noetherian. Then R is left noetherian.*

Proof. Let M be a left R-module. For $g \in G$, let M^g be the left R-module with underlying abelian group M on which R acts by $x \cdot m = x^g m$. Let $\widetilde{M} = \bigoplus_{g \in G} M^g$. Then $\widetilde{M}$ is naturally a left $R[G]$-module. Suppose now that $I_1 \subset I_2 \subset \cdots$ is an ascending chain of left ideals of R. Then $\widetilde{I}_1 \subset \widetilde{I}_2 \subset \cdots$ is an ascending chain of left ideals of $R[G]$, and therefore stabilizes. This clearly implies that the original chain stabilizes as well, and so R is left noetherian. □

We now relate $\mathbf{FI}_R$-modules to $\mathbf{FI}_G$-modules:

Proposition 5.2.4. *Let G be a group and let $R = \mathbf{k}[G]$ be its group algebra (augmented in the usual manner). Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is canonically equivalent to $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$.*

Proof. Let M be an $\mathbf{FI}_G$-module. Then for each finite set x, we have a representation $M(x)$ of G^x, which can be thought of as an $\mathbf{k}[G^x] = R^{\otimes x}$-module. Given an injection $f\colon x \to y$ of finite sets, we have a map $f_*\colon M(x) \to M(y)$ of $\mathbf{k}$-modules. This map lands in the $G^{y \setminus f(x)}$ invariants of $M(y)$, and is G^x equivariant when G^x acts on the target via the homomorphism

$G^x \to G^y$ induced by f. We therefore have a map $M(x) \to \Gamma_{y \setminus f(x)}(M(y))$ of $R^{\otimes x}$-modules. This shows that giving an $\mathbf{FI}_G$-module is exactly the same as giving an $\mathbf{FI}_R$-module. □

Corollary 5.2.5. *The category* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ *is noetherian if and only if the group algebra* $\mathbf{k}[G^n]$ *is left-noetherian for all* $n \geq 0$.

Remark 5.2.6. Everything in this section also applies to $\mathbf{FI}_d$-versions of the categories. □

5.3. **Twisted homological stability for wreath products.** Let G be a group and $\mathbf{k}$ be a commutative ring and let $M \colon \mathbf{FI}_G \to \mathrm{Mod}_{\mathbf{k}}$ be an $\mathbf{FI}_G$-module. Let $M_n = M([n])$. Consider the morphism $[n] \to [n+1]$ consisting of the injection defined by $i \mapsto i$ and the trivial G-map $[n] \to G$ sending every element to the identity. This gives a map $f_n \colon M_n \to M_{n+1}$.

Theorem 5.3.1. *Assume that* G *is a polycyclic-by-finite group and that* $\mathbf{k}$ *is noetherian, and let* M *be a finitely generated* $\mathbf{FI}_G$*-module. Then for each* $i \geq 0$*, the map*

$$f_{n*} \colon \mathrm{H}_i(S_n \ltimes G^n; M_n) \to \mathrm{H}_i(S_{n+1} \ltimes G^{n+1}; M_{n+1})$$

is an isomorphism for $n \gg 0$.

Proof. We apply [PS, Theorem 4.2]. First, $\mathbf{FI}_G$ is a complemented category where the monoidal structure is given by disjoint union of sets, and the generator is the one element set. The two conditions of that theorem are that the category of $\mathbf{FI}_G$-modules is noetherian, and that the isomorphism above holds for $n \gg 0$ when M is the trivial $\mathbf{FI}_G$-module. The first point is Corollary 1.2.2 and the second point is [HW, Proposition 1.6]. □

Define a shift functor Σ on $\mathbf{FI}_G$-modules by $(\Sigma M)(S) = M(S \amalg \{*\})$. In [Wa, Definition 5.1], the following definition is introduced (to avoid ambiguity, we use the phrase Σ-degree instead of just degree). First, the 0 functor has Σ-degree -1 and in general, F has Σ-degree $\leq r$ if the kernel and cokernel of the natural map $F \to \Sigma F$ have Σ-degree $\leq r-1$. Then [Wa, Theorem 5.6] proves Theorem 5.3.1 for any G under the assumption that M has finite Σ-degree (and gives bounds for when stability occurs in terms of the Σ-degree).

So it is natural to ask if a finitely generated $\mathbf{FI}_G$-module necessarily has finite Σ-degree. We do not know if this is true, but will now prove it when G is finite.

Proposition 5.3.2. *If* G *is finite, then a finitely generated* $\mathbf{FI}_G$*-module has finite* Σ*-degree.*

Proof. Let M be an $\mathbf{FI}_G$-module. The kernel of $M \to \Sigma M$ is torsion, i.e., all morphisms act trivially. If M is finitely generated, then the same is true for the kernel (Corollary 5.1.3), and hence it is supported in finitely many degrees. It is easy to see that a torsion module concentrated in degrees $\leq d$ has Σ-degree $\leq d$, so from now on, we only need to focus on the cokernel of $M \to \Sigma M$, which we denote ΔM.

Note that Δ is right-exact. Also, $\Sigma P_S = \bigoplus_{S' \subseteq S} P_S^{\oplus G^{S \setminus S'}}$ and $\Delta P_S = \bigoplus_{S' \subsetneq S} P_S^{\oplus G^{S \setminus S'}}$. Since G is finite, we observe that if M is finitely generated in degree $\leq d$, then ΔM is finitely generated in degree $\leq d-1$ (here we use that G is finite). By induction, we are done. □

Remark 5.3.3. Proposition 5.3.2 and the discussion above shows that Theorem 5.3.1 follows from [Wa, Theorem 5.6] when G is finite. When G is trivial, we have been informed that Theorem 5.3.1 is contained in work of Betley [B] and Church [Ch] also with bounds for when stability occurs. By Proposition 5.1.1, when G is finite, the result also follows from this work. Our result is new when G is infinite. □

5.4. **Homotopy groups of configuration spaces.** Let $\mathbf{FI}(G)$ be the category of sets with a free G-action with finitely many orbits and injective G-equivariant maps. There is an equivalence $\mathbf{FI}_G \to \mathbf{FI}(G)$ defined by $S \mapsto S \times G$ (the G-action is by $h \cdot (s, g) = (s, gh^{-1})$) and sending $(f, \sigma)\colon S \to T$ to the morphism $S \times G \to T \times G$ given by $(s, g) \mapsto (f(s), \sigma(s)g)$.

Let M be a connected manifold with $\dim(M) \geq 3$. Fix $k \geq 2$ and set $G = \pi_1(M)$. Let $\widetilde{M}$ be the universal cover of M, so G acts on $\widetilde{M}$ by deck transformations. Given a set X with a free G-action, let $\mathrm{Conf}^G_X(\widetilde{M})$ be the space of injective G-equivariant maps. Also, for any set S, let $\mathrm{Conf}_S(M)$ be the space of injective maps. There is a natural map $\mathrm{Conf}^G_X(\widetilde{M}) \to \mathrm{Conf}_{X/G}(M)$ with fiber $G^{X/G}$. Since $\dim(M) \geq 3$, $G^{X/G} = \pi_1(\mathrm{Conf}_{X/G}(M))$ and hence $\mathrm{Conf}^G_X(\widetilde{M})$ is simply-connected.

Given an equivariant injective map of sets $X \to Y$ with free G-action, we have a forgetful map $\mathrm{Conf}^G_Y(\widetilde{M}) \to \mathrm{Conf}^G_X(\widetilde{M})$ and hence a map on homotopy groups $\pi_k(\mathrm{Conf}^G_Y(\widetilde{M})) \to \pi_k(\mathrm{Conf}^G_X(\widetilde{M}))$ (we have not chosen basepoints: each $\mathrm{Conf}^G_Z(\widetilde{M})$ is simply-connected, so there is a canonical isomorphism between the homotopy groups for any two choices of basepoint).

So we can define a functor $\mathbf{FI}(G)^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ by $X \mapsto \pi_k(\mathrm{Conf}^G_Y(\widetilde{M}))$. Using the equivalence above, this is also a functor on $\mathbf{FI}_G^{\mathrm{op}}$. Now let A be any abelian group. We define an $\mathbf{FI}_G$-module by $S \mapsto \mathrm{Hom}(\pi_k(\mathrm{Conf}^G_{S\times G}(\widetilde{M})), A)$.

Remark 5.4.1. When $G = \pi_1(M)$ is trivial and $A = \mathbf{Q}$, we get an $\mathbf{FI}$-module defined over $\mathbf{Q}$, which is studied in [KM]. In particular, they show that these are finitely generated $\mathbf{FI}$-modules. They prove a similar result when $A = \mathbf{Z}$ and also for the functor $S \mapsto \mathrm{Ext}^1_{\mathbf{Z}}(\pi_k(\mathrm{Conf}^G_{S\times G}(\widetilde{M})), \mathbf{Z})$. □

It would be interesting to find more general conditions on M, A, and k under which the above $\mathbf{FI}_G$-module is finitely generated.

6. Categories of G-surjections

In this section we study the surjective version of G-sets. The noetherian property is deduced in §6.1. In §6.2, we explain the connection between $\mathbf{FS}_G^{\mathrm{op}}$-modules and Δ-modules. In §6.3, we give some examples of $\mathbf{FS}_G^{\mathrm{op}}$-modules. The results on Hilbert series are stated in §6.4. The remainder of the section is devoted to proving the Hilbert series results.

6.1. Basic properties.

Proposition 6.1.1. *There is a natural functor $\mathbf{FS}^{\mathrm{op}} \to \mathbf{FS}_G^{\mathrm{op}}$ that satisfies property (F).*

Proof. Let $\Phi\colon \mathbf{FS} \to \mathbf{FS}_G$ be the functor taking a function $f\colon S \to T$ to the G-function $(f, \sigma)\colon S \to T$ where $\sigma = 1$. The "natural functor" in the statement of the proposition is Φ^{op}. Let $x \in \mathbf{FS}_G$ be given. Say that a morphism $(f, \sigma)\colon y \to x$ is minimal if the function $(f, \sigma)\colon y \to x \times G$ is injective. There are finitely many minimal maps up to isomorphism. Now consider a map $(f, \sigma)\colon y \to x$ in $\mathbf{FS}_G$. Define an equivalence relation on y by $a \sim b$ if $f(a) = f(b)$ and $\sigma(a) = \sigma(b)$, and let $g\colon y \to y'$ be the quotient. Then the induced map $(f', \sigma)\colon y' \to x$ is minimal. Furthermore, $(f, \sigma) = (g, 1)(f', \sigma) = \Phi(g)(f', \sigma)$. Flipping all the arrows, we see that Φ^{op} satisfies property (F). □

Given a finite collection $\underline{G} = (G_i)_{i\in I}$ of finite groups, we write $\mathbf{FS}_{\underline{G}}$ for the product category $\prod_{i\in I} \mathbf{FS}_{G_i}$.

Corollary 6.1.2. *The category $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ is quasi-Gröbner.*

Proof. This follows from Propositions 2.2.4, 2.2.3 and [SS2, Theorem 8.1.2]. □

Corollary 6.1.3. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}})$ *is noetherian.*

6.2. **Generalized Δ-modules.** Let $\mathcal{A}$ be an abelian category equipped with a symmetric "cotensor" structure, i.e., a functor $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, and analogous data opposite to that of a tensor structure. (Here we are using the Deligne tensor product of abelian categories [De].) Given a surjection $f \colon T \to S$ of finite sets, there is an induced functor $f^* \colon \mathcal{A}^{\otimes S} \to \mathcal{A}^{\otimes T}$ by cotensoring along the fibers of f. A **Δ-module** over $\mathcal{A}$ is a rule M that assigns to each finite set S an object M_S of $\mathcal{A}^{\otimes S}$ and to each surjection $f \colon T \to S$ of finite sets a morphism $M_f \colon f^*(M_S) \to M_T$, such that if $f \colon T \to S$ and $g \colon S \to R$ are surjections, then the diagram

$$\begin{tikzcd} & g^*(M_S) \arrow[dr, "M_g"] & \\ (gf)^*(M_R) \arrow[ur, "f^*(M_g)"] \arrow[rr, "M_{gf}"] & & M_T \end{tikzcd}$$

commutes. There are two main examples relevant to this paper:

- Let $\mathcal{A}$ be the category of polynomial functors $\mathrm{Vec} \to \mathrm{Vec}$. Then $\mathcal{A}^{\otimes 2}$ is identified with the category of polynomial functors $\mathrm{Vec}^2 \to \mathrm{Vec}$. There is a comultiplication $\mathcal{A} \to \mathcal{A}^{\otimes 2}$ taking a functor F to the functor $(U, V) \mapsto F(U \otimes V)$, and this gives $\mathcal{A}$ the structure of a symmetric cotensor category. Δ-modules over $\mathcal{A}$ are Δ-modules as defined in [SS2, §9.1].
- Let $\mathcal{A}$ be the category of representations of a finite group G. Then $\mathcal{A}^{\otimes 2}$ is identified with the category of representations of $G \times G$. There is a comultiplication $\mathcal{A} \to \mathcal{A}^{\otimes 2}$ taking a representation V of G to the representation $\mathrm{Ind}_G^{G \times G}(V)$ of $G \times G$, where G is included in $G \times G$ via the diagonal map. This gives $\mathcal{A}$ the structure of a symmetric cotensor category. Δ-modules over $\mathcal{A}$ are representations of $\mathbf{FS}_G^{\mathrm{op}}$.

If $n!$ is invertible in the base field then the category of polynomial functors of degree $\le n$ is equivalent, as a cotensor category, to the category $\prod_{k=0}^n \mathrm{Rep}(S_n)$. We thus find that Δ-modules of degree $\le n$ (in the sense of [SS2, §9.3]) coincide with representations of $\prod_{k=0}^n \mathbf{FS}_{S_k}^{\mathrm{op}}$. Thus our results on $\mathbf{FS}_G^{\mathrm{op}}$ can be loosely viewed as a generalization of our results on Δ-modules from [SS2] ("loosely" because in bad characteristic the results are independent of each other). It seems possible that our results could generalize to Δ-modules over any "finite" abelian cotensor category.

6.3. **Examples: Segre products of simplicial complexes.** We now give a source of $\mathbf{FS}_G^{\mathrm{op}}$-modules. Let X and Y be simplicial complexes on finite vertex sets X_0 and Y_0. Define a simplicial complex $X * Y$ on the vertex set $X_0 \times Y_0$ as follows. Let $p_1 \colon X_0 \times Y_0 \to X_0$ be the projection map, and similarly define p_2. Then $S \subset X_0 \times Y_0$ is a simplex if and only if $p_1(S)$ and $p_2(S)$ are simplices of X and Y and have the same cardinality as S. We call $X * Y$ the **Segre product** of X and Y. It is functorial for maps of simplicial complexes. It is not a topological construction, and depends in an essential way on the simplicial structure.

Fix a finite simplicial complex X, equipped with an action of a group G. The diagonal map $X_0 \to X_0 \times X_0$ induces a map of simplicial complexes $X \to X * X$. We thus obtain a functor from $\mathbf{FS}_G^{\mathrm{op}}$ to the category of simplicial complexes by $S \mapsto X^{*S}$. Fixing i, we obtain a representation M_i of $\mathbf{FS}_G^{\mathrm{op}}$ by $S \mapsto \mathrm{H}_i(X^{*S}; \mathbf{k})$. It is not difficult to directly show that $S \mapsto \mathrm{C}_i(X^{*S}; \mathbf{k})$ is a finitely generated representation of $\mathbf{FS}_G^{\mathrm{op}}$, where C_i denotes the space of simplicial i-chains. Thus by Corollary 6.1.3, M_i is a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$. Theorem 6.4.1 below gives information about the Hilbert series of M_i.

The case where X is just a single simplex is already extremely complicated and interesting, and is closely related to syzygies of the Segre embedding. In fact, if X has d vertices then the $\mathbf{FS}_{S_d}^{\mathrm{op}}$-module given by $\mathrm{H}_{p-1}(X^{*\bullet}; \mathbf{k})$ coincides with the degree d piece of the Δ-module F_p of p-syzygies of the Segre embedding (as defined in [Sn]) under the equivalence in the previous section, at least when $d!$ is invertible in $\mathbf{k}$.

6.4. **Hilbert series.** Let M be a finitely generated $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$-module over a field $\mathbf{k}$. Let $\mathbf{n} \in \mathbf{N}^I$, and write $[\mathbf{n}]$ for $([n_i])_{i \in I}$. Then $M([\mathbf{n}])$ is a finite dimensional representation of $\underline{G}^{\mathbf{n}}$. Let $[M]_{\mathbf{n}}$ denote the image of the class of this representation under the map

$$\mathcal{R}_{\mathbf{k}}(\underline{G}^{\mathbf{n}}) = \bigotimes_{i \in I} \mathcal{R}_{\mathbf{k}}(G_i)^{\otimes n_i} \to \mathrm{Sym}^{|\mathbf{n}|}(\mathcal{R}_{\mathbf{k}}(\underline{G})),$$

where

$$\mathcal{R}_{\mathbf{k}}(\underline{G}) = \bigoplus_{i \in I} \mathcal{R}_{\mathbf{k}}(G_i).$$

Note that one can recover the isomorphism class of $M([\mathbf{n}])$ as a representation of $\underline{G}^{\mathbf{n}}$ from $[M]_{\mathbf{n}}$ due to the $S_{\mathbf{n}}$-equivariance. If $\{L_{i,j}\}$ are the irreducible representations of the G_i, then $[M]_{\mathbf{n}}$ can be thought of as a polynomial in corresponding variables $\{t_{i,j}\}$. Define the **Hilbert series** of M by

$$\mathrm{H}_M(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^I} [M]_{\mathbf{n}}.$$

This is an element of the ring $\widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \cong \mathbf{Q}[\![t_{i,j}]\!]$. This definition does not fit into our framework of Hilbert series of normed categories, though it can be seen as an enhanced Hilbert series.

The following is a simplified version of our main theorem on Hilbert series. Recall the definition of $\mathcal{K}_N$ from Definition 3.4.

Theorem 6.4.1. *Let M be a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ over an algebraically closed field $\mathbf{k}$. Then $\mathrm{H}_M(\mathbf{t})$ is a $\mathcal{K}_N$ function of the $\mathbf{t}$, where N is the least common multiple of the exponents of the G_i.*

Stating the full result requires some additional notions. Let G be a finite group and let $\mathbf{k}$ be an arbitrary field. Let $\{H_j\}_{j \in J}$ be a collection of subgroups such that the orders of the commutator subgroups $[H_j, H_j]$ are invertible in $\mathbf{k}$, and let H_j^{ab} be the abelianization of H_j. There is a functor $\mathrm{Rep}_{\mathbf{k}}(H_j) \to \mathrm{Rep}_{\mathbf{k}}(H_j^{\mathrm{ab}})$ given by taking coinvariants under $[H_j, H_j]$. This functor is exact since the order of $[H_j, H_j]$ is invertible in $\mathbf{k}$, and thus induces a homomorphism $\mathcal{R}_{\mathbf{k}}(H_j) \to \mathcal{R}_{\mathbf{k}}(H_j^{\mathrm{ab}})$. There are also homomorphisms $\mathcal{R}_{\mathbf{k}}(G) \to \mathcal{R}_{\mathbf{k}}(H_j)$ given by restriction. We say that the family $\{H_j\}$ is **good** if the composite

$$\mathcal{R}_{\mathbf{k}}(G) \to \bigoplus_{j \in J} \mathcal{R}_{\mathbf{k}}(H_j) \to \bigoplus_{j \in J} \mathcal{R}_{\mathbf{k}}(H_j^{\mathrm{ab}})$$

is a split injection (i.e., an injection with torsion-free cokernel). We say that G is N**-good** if it admits a good family $\{H_j\}$ such that the exponent of each H_j^{ab} divides N. We say that a family $\underline{G}$ of finite groups is N-good if each member is. These notions depend on $\mathbf{k}$.

The following is our main theorem on Hilbert series. The proof is given in §6.6.

Theorem 6.4.2. *Suppose that $\underline{G}$ is N-good and $\mathbf{k}$ contains the Nth roots of unity. Let M be a finitely generated $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$-module over $\mathbf{k}$. Then $\mathrm{H}_M(\mathbf{t})$ is a $\mathcal{K}_N$ function of the $t_{i,j}$.*

Using Brauer's theorem, we show that over an algebraically closed field, every group is N-good where N is the exponent of the group (Proposition 6.5.2), and so Theorem 6.4.1 follows from Theorem 6.4.2. We show that symmetric groups are 2-good if $n!$ is invertible in $\mathbf{k}$, which recovers some of our results on Hilbert series of Δ-modules (see [SS2, §9.1]) in good characteristic. For general groups, we know little about the optimal value of N. Finding some results could be an interesting group theory problem.

Example 6.4.3. Let G be a finite group and let $\{V_i\}_{i\in I}$ be the set of irreducible representations of G over $\mathbf{C}$. Define an $\mathbf{FS}_G^{\mathrm{op}}$-module M_i by

$$M_i(S) = \mathrm{Ind}_G^{G^S}(V_i),$$

where $G \to G^S$ is the diagonal map. Let C be the set of conjugacy classes in G, χ_i be the character of V_i, and t_i be an indeterminate corresponding to V_i. A computation similar to that in [Sn, Lem. 5.7] gives

$$\mathrm{H}_{M_i}(\mathbf{t}) = \frac{1}{\#G} \sum_{c\in C} \frac{\#c \cdot \chi_i(c)}{1 - (\sum_{j\in I} \chi_j(c) t_j)}.$$

This is a $\mathcal{K}_N$ function of the t_i, as predicted by Theorem 6.4.1, where N is such that all characters of G take values in $\mathbf{Q}(\zeta_N)$. $\square$

6.5. **Group theory.** Let $p = \mathrm{char}(\mathbf{k})$. If $p = 0$ then every group has order prime to p, and the only p-group is the trivial group. We say that a collection $\{H_i\}_{i\in I}$ of subgroups of G is a **covering** if the map on Grothendieck groups

$$\mathcal{R}_{\mathbf{k}}(G) \to \bigoplus_{i\in I} \mathcal{R}_{\mathbf{k}}(H_i)$$

is a split injection. Recall that if ℓ is a prime, then an **ℓ-elementary group** is one that is the direct product of an ℓ-group and a cyclic group of order prime to ℓ. An **elementary group** is a group which is ℓ-elementary for some prime ℓ.

Lemma 6.5.1. *The following result holds over any field $\mathbf{k}$:*

(a) *Let $\{H_i\}_{i\in I}$ be a covering of G, and for each i let $\{K_j\}_{j\in J_i}$ be a covering of H_i, and let $J = \amalg_{i\in I} J_i$. Then $\{K_j\}_{j\in J}$ is a covering of G.*
(b) *Let $\{H_i\}_{i\in I}$ be a covering of G, and suppose each H_i is N-good. Then G is N-good.*
(c) *Suppose that H is a p-elementary group and write $H = H_1 \times H_2$, where H_1 is cyclic of order prime to p and H_2 is a p-group. Then $\{H_1\}$ is a covering of H.*

The following hold if $\mathbf{k}$ is algebraically closed:

(d) *The collection of elementary subgroups $\{H_i\}_{i\in I}$ of G is a covering of G.*
(e) *Let H be a group of order prime to p. Then H is N-good where N is the exponent of H.*

Proof. (a) and (b) are clear.

(c) The only simple $\mathbf{k}[H_2]$-module is trivial, so $\mathcal{R}_{\mathbf{k}}(H) \to \mathcal{R}_{\mathbf{k}}(H_1)$ is an isomorphism.

(d) Let $\alpha\colon \mathcal{R}_{\mathbf{k}}(G) \to \bigoplus \mathcal{R}_{\mathbf{k}}(H_i)$ be the restriction map. Let $\mathcal{P}_{\mathbf{k}}(G)$ be the Grothendieck group of finite-dimensional projective $\mathbf{k}[G]$-modules. The map $\mathcal{R}_{\mathbf{k}}(G) \times \mathcal{P}_{\mathbf{k}}(G) \to \mathbf{Z}$ given by $(V, W) \mapsto \dim_{\mathbf{k}} \mathrm{Hom}_G(V, W)$ is a perfect pairing [Se, §14.5]. Combining this with Frobenius reciprocity, it follows that the dual of α can be identified with the induction map $\bigoplus \mathcal{P}_{\mathbf{k}}(H_i) \to \mathcal{P}_{\mathbf{k}}(G)$. This map is surjective by Brauer's theorem [Se, §17.2, Thm. 39]. Since the dual of α is surjective, it follows that α is a split injection, which proves the claim.

(e) Indeed, arguing with duals and Frobenius reciprocity again, it is enough to find subgroups $\{K_i\}_{i\in I}$ of H such that the induction map $\bigoplus_{i\in I} \mathcal{R}_{\mathbf{k}}(K_i^{\mathrm{ab}}) \to \mathcal{R}_{\mathbf{k}}(H)$ is surjective. (Note that $\mathcal{R}_{\mathbf{k}} = \mathcal{P}_{\mathbf{k}}$ for groups of order prime to p.) This follows from Brauer's theorem [Se, §10.5, Thm. 20]. □

Proposition 6.5.2. *Suppose* $\mathbf{k}$ *is algebraically closed and* G *is a finite group. Then* G *is* N*-good where* N *is the exponent of* G.

Proof. By parts (a), (c), and (d) of Lemma 6.5.1, G has a covering by its subgroups of order prime to p. For each of these groups, its set of subgroups is good by part (e). Now finish by applying (b). □

We now construct a good collection of subgroups for the symmetric group S_n in good characteristic. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\sum_i \lambda_i = n$, let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ be the corresponding Young subgroup of S_n.

Proposition 6.5.3. *Suppose* $n!$ *is invertible in* $\mathbf{k}$. *Then* $\{S_\lambda\}$ *is a good collection of subgroups of the symmetric group* S_n.

Proof. Under the assumption on $\mathrm{char}(\mathbf{k})$, the representations of S_n are semisimple. Using Frobenius reciprocity, the restriction map on representation rings is dual to induction. We claim that each irreducible character of S_n is a $\mathbf{Z}$-linear combination of the permutation representations of S_n/S_λ (this implies $\{S_\lambda\}$ is a good collection of subgroups). Recall that the irreducible representations of S_n are indexed by partitions of n (we will denote them $\mathbf{M}_\lambda$). Also, recall the dominance order on partitions: $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. An immediate consequence of Pieri's rule [SS1, (2.10)] is that the permutation representation S_n/S_λ contains $\mathbf{M}_\lambda$ with multiplicity 1 and the remaining representations $\mathbf{M}_\mu$ that appear satisfy $\mu \geq \lambda$. This proves the claim. □

Corollary 6.5.4. *Let* $\mathbf{k}$ *be a field in which* $n!$ *is invertible. Then* S_n *is* 2*-good.*

Proof. The group S_λ^{ab} has exponent 1 or 2 for any λ. □

6.6. **Proof of Theorem 6.4.2.** The idea is to first use the good family of subgroups to reduce to the case where each G_i is abelian of invertible order. For such G, we identify $\mathbf{FS}_G^{\mathrm{op}}$-modules $\mathbf{FWS}_\Lambda^{\mathrm{op}}$-modules, where Λ is the group of characters of G. The theorem then follows from our results for Hilbert series of $\mathbf{FWS}_\Lambda^{\mathrm{op}}$-modules. We now go through the details.

Lemma 6.6.1. *Let* $\underline{G} = (G_i)_{i\in I}$ *be finite groups. For each* i*, let* H_i *be a subgroup of* G_i *such that the order of* $[H_i, H_i]$ *is invertible in* $\mathbf{k}$. *Define a functor*

$$\Phi\colon \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{H}^{\mathrm{ab}}}^{\mathrm{op}})$$

by letting $\Phi(M)(\underline{S})$ *be the* $[\underline{H}, \underline{H}]^{\underline{S}}$*-coinvariants of* $M(\underline{S})$. *Then we have the following:*

(a) $\Phi(M)$ *is a well-defined object of* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{H}^{\mathrm{ab}}}^{\mathrm{op}})$.
(b) *If* M *is finitely generated then so is* $\Phi(M)$.
(c) *Let* $\varphi_i\colon \mathcal{R}_{\mathbf{k}}(G_i) \to \mathcal{R}_{\mathbf{k}}(H_i^{\mathrm{ab}})$ *be the map induced by restricting to* H_i *followed by taking* $[H_i, H_i]$*-coinvariants, and let* $\varphi\colon \mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})$ *be the sum of the* φ_i. *Then* $\mathrm{H}_{\Phi(M)}$

is the image of H_M *under the ring homomorphism* $\widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \to \widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})_{\mathbf{Q}})$ *induced by* φ.

Proof. (a) For a tuple $\underline{S} = (S_i)_{i \in I}$ of sets, let $K(\underline{S})$ be the $\mathbf{k}$-subspace of $M(\underline{S})$ spanned by elements of the form $gm - m$ with $g \in [\underline{H}, \underline{H}]^{\underline{S}}$ and $m \in M(\underline{S})$. If $f \colon \underline{S} \to \underline{T}$ is a morphism in $\mathbf{FS}^I$ then the induced map $f^* \colon M(\underline{T}) \to M(\underline{S})$ carries $gm - m$ to $f^*(g)f^*(m) - f^*(m)$. Thus K is a $(\mathbf{FS}^{\mathrm{op}})^I$-submodule of M, and so M/K is a well-defined $(\mathbf{FS}^{\mathrm{op}})^I$-module. The group actions clearly carry through, and so $\Phi(M)$ is well-defined.

(b) Suppose $M \in \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}})$ is finitely generated. Then the restriction of M to $(\mathbf{FS}^{\mathrm{op}})^I$ is finitely generated by Propositions 6.1.1 and 2.1.3. The restriction of $\Phi(M)$ to $(\mathbf{FS}^{\mathrm{op}})^I$ is a quotient of the restriction of M, and is therefore finitely generated. Thus $\Phi(M)$ is finitely generated, by Proposition 2.1.2.

(c) This is clear. $\square$

Proposition 6.6.2. *The functor* $\Phi \colon \mathbf{FS}^{\mathrm{op}} \times \mathbf{FS}^{\mathrm{op}} \to \mathbf{FS}^{\mathrm{op}}$ *given by disjoint union satisfies property (F).*

Proof. Pick a finite set S. Let $f \colon T_1 \amalg T_2 \to S$ be a surjection. Then this can be factored as $T_1 \amalg T_2 \to f(T_1) \amalg f(T_2) \to S$ where the first map is the image of a morphism $(T_1, T_2) \to (f(T_1), f(T_2))$ under Φ^{op}. So for the $y_1, y_2, \ldots$ in the definition of property (F), we take the pairs (T, T') of subsets of S whose union is all of S. $\square$

Lemma 6.6.3. *Let* $\underline{G} = (G_i)_{i \in I}$ *be groups, let* $f \colon J \to I$ *be a surjection, and let* $f^*(\underline{G})$ *be the resulting family of groups indexed by* J. *Let* $\Phi \colon \mathbf{FS}_{f^*(\underline{G})}^{\mathrm{op}} \to \mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ *be the functor induced by disjoint union, i.e.,* $\Phi(\{S_j\}_{j \in J}) = \{T_i\}_{i \in I}$ *where* $T_i = \coprod_{j \in f^{-1}(i)} S_j$.

(a) Φ *satisfies property (F); in particular, if* M *is finitely generated then so is* $\Phi^*(M)$.
(b) *Let* $\varphi_i \colon \mathcal{R}_{\mathbf{k}}(G_i) \to \bigoplus_{j \in f^{-1}(i)} \mathcal{R}_{\mathbf{k}}(G_j)$ *be the diagonal map, and let* $\varphi \colon \mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(f^*(\underline{G}))$ *be the sum of the* φ_i. *Then* $\mathrm{H}_{\Phi^*(M)}$ *is the image of* H_M *under the ring homomorphism* $\widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \to \widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(f^*(\underline{G}))_{\mathbf{Q}})$ *induced by* φ.

Proof. (a) Consider the commutative diagram of categories

$$\begin{tikzcd} \mathbf{FS}_{f^*(\underline{G})}^{\mathrm{op}} \arrow[r, "\Phi"] & \mathbf{FS}_{\underline{G}}^{\mathrm{op}} \\ (\mathbf{FS}^{\mathrm{op}})^J \arrow[u] \arrow[r, "\Phi'"] & (\mathbf{FS}^{\mathrm{op}})^I \arrow[u] \end{tikzcd}$$

The functor Φ' is defined just like Φ; it satisfies property (F) by Proposition 6.6.2. The vertical maps satisfy property (F) by Proposition 6.1.1. Thus Φ satisfies property (F) by Propositions 2.1.4.

(b) This is clear. $\square$

Lemma 6.6.4. *Suppose that* $\underline{G} = (G_i)_{i \in I}$ *is a family of commutative groups of exponents dividing* N. *Suppose that* N *is invertible in* $\mathbf{k}$ *and that* $\mathbf{k}$ *contains the* N*th roots of unity. Let* $\Lambda_i = \mathrm{Hom}(G_i, \mathbf{k}^\times)$ *be the group of characters of* G_i, *and let* $\underline{\Lambda} = (\Lambda_i)_{i \in I}$. *Then there is an equivalence* $\Phi \colon \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FWS}_{\underline{\Lambda}}^{\mathrm{op}})$ *respecting Hilbert series, i.e.,* $\mathrm{H}_M = \mathrm{H}_{\Phi(M)}$ *for* $M \in \mathrm{Rep}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}})$.

Before giving the proof, we offer two clarifications. First, $\mathbf{FWS}_{\underline{\Lambda}}$ denotes the category $\prod_{i \in I} \mathbf{FWS}_{\Lambda_i}$. An object of this category is a tuple of sets $\underline{S} = (S_i)_{i \in I}$ equipped with a weight function $\varphi_i \colon S_i \to \Lambda_i$ for each i. Second, H_M and $\mathrm{H}_{\Phi(M)}$ are both series in variables indexed by the characters of the G_i. This is why they are comparable.

Proof. Let M be a representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$. Let $\underline{S} = (S_i)_{i \in I}$ be a tuple of sets. Then we have a decomposition

$$M_{\underline{S}} = \bigoplus M_{\underline{S}, \underline{\varphi}},$$

where the sum is over weightings $\underline{\varphi}$ of $\underline{S}$, and $M_{\underline{S},\underline{\varphi}}$ is the subspace of $M_{\underline{S}}$ on which $\underline{G}^{\underline{S}}$ acts through $\underline{\varphi}$. If $\underline{f} \colon \underline{S} \to \underline{T}$ is a morphism in $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ then the map $\underline{f}_* \colon M_{\underline{S}} \to M_{\underline{T}}$ carries $M_{\underline{S},\underline{\varphi}}$ into $M_{\underline{T}, \underline{f}_*(\underline{\varphi})}$. We define $\Phi(M)$ to be the functor on $\mathbf{FWS}_{\underline{\Lambda}}^{\mathrm{op}}$ which assigns to a weighted set $(\underline{S}, \underline{\varphi})$ the space $M_{\underline{S},\underline{\varphi}}$. This construction can be reversed: given a representation M of $\mathbf{FWS}_{\underline{\Lambda}}^{\mathrm{op}}$, we can build a representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ by defining $M_{\underline{S}}$ to be the sum of the $M_{\underline{S},\underline{\varphi}}$. We leave to the reader the verification that these constructions are quasi-inverse to each other. This shows that Φ is an equivalence. It is clear that it preserves Hilbert series: we note that the multinomial coefficients in the definition of $\mathrm{H}_{\Phi(M)}$ count, for each $M_{\underline{S},\underline{\varphi}}$, the number of $M_{\underline{S},\underline{\varphi}'}$ where $\underline{\varphi}'$ is a permutation of $\underline{\varphi}$. $\square$

Proof of Theorem 6.4.2. Let M be a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$, where $\underline{G} = (G_i)_{i \in I}$. For each $i \in I$, let $\{H_j\}_{j \in J_i}$ be a good collection of subgroups of $\underline{G}$ such that the exponent of each H_j^{ab} divides N. Let $J = \coprod_{i \in I} J_i$ and let $f \colon J \to I$ be the projection map. Then we have functors

$$\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{f^*(\underline{G})}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{H}^{\mathrm{ab}}}^{\mathrm{op}}).$$

Let M' be the image of M under the composition. By Lemmas 6.6.1 and 6.6.3, M' is finitely generated and $\mathrm{H}_{M'}$ is the image of H_M under the ring homomorphism corresponding to the natural additive map $\mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})$. By Lemma 6.6.4 and Theorem 4.1.4, $\mathrm{H}_{M'}$ is $\mathcal{K}_N$. Thus by Lemma 3.5, H_M is $\mathcal{K}_N$. This completes the proof. $\square$

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