TOPOLOGICAL NOETHERIANITY FOR ALGEBRAIC REPRESENTATIONS OF INFINITE RANK CLASSICAL GROUPS

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Abstract. Draisma recently proved that polynomial representations of $\mathbf{GL}_\infty$ are topologically noetherian. We generalize this result to algebraic representations of infinite rank classical groups.

1. Introduction

1.1. **Background.** In recent years, a number of novel noetherianity results have been discovered and exploited; for but a few examples, see [AH, Co, Co2, CEF, DE, DES, DK, Eg, HS, NSS, PSa, Sn, SS2]. It is not yet clear where the ultimate line between noetherianity and non-noetherianity lies: of the reasonable structures to consider, some are known to be noetherian and some not, and in between is a vast unknown territory.

Recently, Draisma [Dr] proved a breakthrough result claiming a large tract of the unknown for the noetherian side. To state it, we must recall some terminology. Fix an algebraically closed field $\mathbf{k}$ and let $\mathbf{GL} = \bigcup_{n \geq 1} \mathbf{GL}_n(\mathbf{k})$. A representation of $\mathbf{GL}$ is called **polynomial** if it is a subquotient of a finite[1] direct sum of tensor products of the standard representation $\mathbf{V} = \bigcup_{n \geq 1} \mathbf{k}^n$. If $\mathbf{E}$ is a polynomial representation then the dual space $\hat{\mathbf{E}}$ is canonically identified with the $\mathbf{k}$-points of the spectrum of the ring $\mathrm{Sym}(\mathbf{E})$, and in this way inherits a Zariski topology. Recall that if a group G acts on a space X (such as $\mathbf{GL}$ on $\hat{\mathbf{E}}$) then we say that X is **topologically G-noetherian** if every descending chain of G-stable closed subsets of X stabilizes. We can now state Draisma's theorem:

Theorem 1.1 (Draisma). *Let $\mathbf{E}$ be a polynomial representation of $\mathbf{GL}$. Then $\hat{\mathbf{E}}$ is topologically $\mathbf{GL}$-noetherian.*

This result has already found application: Erman–Sam–Snowden [ESS] have combined it with the resolution of Stillman's conjecture by Ananyan–Hochster [AH2] to establish a vast generalization of Stillman's conjecture.

1.2. **The main theorem.** Draisma's theorem elicits a few natural follow-up questions. Does noetherianity hold for any non-polynomial representations of $\mathbf{GL}$? And what about representations of similar groups, such as the infinite orthogonal group? The purpose of this paper is to provide some answers to these questions.

Let $\mathbf{O} = \bigcup_{n \geq 1} \mathbf{O}_n(\mathbf{k})$ be the infinite orthogonal group, and let $\mathbf{Sp} = \bigcup_{n \geq 1} \mathbf{Sp}_{2n}(\mathbf{k})$ be the infinite symplectic group. We say that a representation of $\mathbf{O}$ or $\mathbf{Sp}$ is **algebraic** if it appears as a subquotient of a finite direct sum of tensor powers of the standard representation $\mathbf{V}$. Let $\mathbf{V}_* = \bigcup_{n \geq 1} (\mathbf{k}^n)^*$, the so-called restricted dual of $\mathbf{V}$. The group $\mathbf{GL}$ acts on $\mathbf{V}_*$. A

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[1] In other settings, one allows infinite sums of tensor powers, but in this paper we restrict to finite sums.

representation of $\mathbf{GL}$ is called **algebraic** if it appears as a subquotient of a finite direct sum of representations of the form $\mathbf{V}^{\otimes n} \otimes \mathbf{V}_*^{\otimes m}$. We note that $\mathbf{V} \cong \mathbf{V}_*$ as representations of $\mathbf{O}$ and $\mathbf{Sp}$, so one does not get a larger category of representations by using $\mathbf{V}_*$. In characteristic 0, algebraic representations of $\mathbf{GL}$, $\mathbf{O}$, and $\mathbf{Sp}$ have been studied in [DPS, PSe, PSt, SS].

The main theorem of this paper is:

Theorem 1.2. *Let $\mathbf{G} \in \{\mathbf{O}, \mathbf{Sp}, \mathbf{GL}\}$ and let $\mathbf{E}$ be an algebraic representation of $\mathbf{G}$. Then $\hat{\mathbf{E}}$ is topologically $\mathbf{G}$-noetherian.*

In §5, we show that certain very similar looking statements are false. It would be interesting to see if this theorem has any applications in commutative algebra along the lines of [ESS]. We note, however, that [ESS] makes use of the kind of statements found in §5 that are true in the case of polynomial representations, but false for algebraic representations.

1.3. **Possible generalizations.** There are two directions in which Theorem 1.2 might be generalized. First, one might hope for a statement at the level of ideals. That is, suppose that $\mathbf{E}$ is an algebraic representation of $\mathbf{G}$ (for $\mathbf{G}$ as in the theorem), and let $A = \mathrm{Sym}(\mathbf{E})$. The theorem is equivalent to the statement that any ascending chain of $\mathbf{G}$-stable radical ideals in A stabilizes. A plausible stronger statement is that every ascending chain of $\mathbf{G}$-stable ideals stabilizes. More generally, if $\mathbf{E}'$ is a second algebraic representation, one might hope that every ascending chain of $\mathbf{G}$-stable submodules of $A \otimes \mathbf{E}'$ stabilizes.

Second, one might hope to extend topological noetherianity to certain analogs of algebraic representations. Let $A = \mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty))$, equipped with its natural action of $\mathbf{GL}$. One can then consider the category Mod_A of A-modules equipped with a compatible polynomial action of $\mathbf{GL}$. Let $\mathrm{Mod}_A^{\mathrm{tors}}$ be the Serre subcategory of torsion modules. It is known [NSS] that the quotient category $\mathrm{Mod}_A / \mathrm{Mod}_A^{\mathrm{tors}}$ is equivalent to the category of algebraic representations of $\mathbf{O}$ (as a tensor category). Thus Theorem 1.2 in the case $\mathbf{G} = \mathbf{O}$ can be stated as: every finitely generated algebra in the category $\mathrm{Mod}_A / \mathrm{Mod}_A^{\mathrm{tors}}$ is topologically noetherian. It seems plausible that this statement might hold true for any A of the form $\mathrm{Sym}(\mathbf{E})$, with $\mathbf{E}$ a polynomial representation of $\mathbf{GL}$.

1.4. **Outline.** In §2, we show that it suffices to prove Theorem 1.2 for any one of the three groups. In §3, we go over some preliminary material. The main theorem (in the general linear case) is proved in §4. In §5, we discuss counterexamples to certain variants of the main theorem.

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2. Reduction to the general linear case

We now show that it suffices to prove Theorem 1.2 in any one of the three cases. Later, we will prove the theorem in the general linear case.

Lemma 2.1. *Suppose that $\mathbf{G}, \mathbf{H} \in \{\mathbf{GL}, \mathbf{Sp}, \mathbf{O}\}$ and that $\mathbf{H} \to \mathbf{G}$ is a homomorphism such that the standard representation of $\mathbf{G}$ pulls back to an algebraic representation of $\mathbf{H}$. Suppose also that Theorem 1.2 holds for $\mathbf{H}$. Then Theorem 1.2 holds for $\mathbf{G}$.*

Proof. The hypothesis implies that any algebraic representation of G pulls back to an algebraic representation of $\mathbf{H}$. Let $\mathbf{E}$ be an algebraic representation of $\mathbf{G}$. Suppose that $Z_\bullet \subset \hat{\mathbf{E}}$ is a descending chain of $\mathbf{G}$-stable closed subset. Then it is also a descending chain of $\mathbf{H}$-stable closed subsets, and thus stabilizes. Thus $\hat{\mathbf{E}}$ is topologically $\mathbf{G}$-noetherian. $\square$

Proposition 2.2. *If Theorem 1.2 holds for any one of the three groups then it holds for the other two.*

Proof. Suppose the theorem holds for $\mathbf{GL}$ and $\mathrm{char}(\mathbf{k}) \neq 2$. Consider the representation $\mathbf{V} \oplus \mathbf{V}_*$ of $\mathbf{GL}$. This representation carries both a symmetric form and a symplectic form, defined by the formulas $(v+\lambda, v'+\lambda') = \lambda(v') \pm \lambda'(v)$. These forms give group homomorphisms $\mathbf{GL} \to \mathbf{O}$ and $\mathbf{GL} \to \mathbf{Sp}$. These homomorphisms have the property that the standard representation pulls back to the representation $\mathbf{V} \oplus \mathbf{V}_*$. Thus Theorem 1.2 holds for $\mathbf{O}$ and $\mathbf{Sp}$ by Lemma 2.1.

If $\mathrm{char}(\mathbf{k}) = 2$ then a similar argument works. The bilinear form defined in the previous paragraph is alternating in characteristic 2 (i.e., it satisfies $(v+\lambda, v+\lambda) = 0$) and thus yields a map $\mathbf{GL} \to \mathbf{Sp}$ that again allows us to apply Lemma 2.1. The representation $\mathbf{V} \oplus \mathbf{V}_*$ also admits a quadratic form defined by $(v, \lambda) \mapsto \lambda(v)$, which affords a homomorphim $\mathbf{GL} \to \mathbf{O}$ to which we can apply Lemma 2.1.

Now suppose that Theorem 1.2 holds for $\mathbf{O}$. Applying Lemma 2.1 to the inclusion $\mathbf{O} \to \mathbf{GL}$, we see that Theorem 1.2 holds for $\mathbf{GL}$. Appealing to the previous two paragraphs, we thus see that Theorem 1.2 holds for $\mathbf{Sp}$ as well. Similarly, if Theorem 1.2 holds for $\mathbf{Sp}$ then we get it for $\mathbf{GL}$ and then $\mathbf{O}$. $\square$

A similar argument is used in the following proposition, which we also require. A representation of $\mathbf{GL} \times \mathbf{GL}$ is **polynomial** if it appears as a subquotient of a finite sum of representations of the form $\mathbf{V}^{\otimes n} \otimes \mathbf{V}^{\otimes m}$.

Proposition 2.3. *Let $\mathbf{E}$ be a polynomial representation of $\mathbf{GL} \times \mathbf{GL}$. Then $\hat{\mathbf{E}}$ is topologically $\mathbf{GL} \times \mathbf{GL}$ noetherian.*

Proof. Consider the diagonal copy of $\mathbf{GL}$ in $\mathbf{GL} \times \mathbf{GL}$. The restriction $\mathbf{E}|_{\mathbf{GL}}$ is then polynomial: indeed, if $\mathbf{E}$ is a subquotient of $\bigoplus_{i=1}^k \mathbf{V}^{\otimes n_i} \otimes \mathbf{V}^{\otimes m_i}$ then $\mathbf{E}|_{\mathbf{GL}}$ is a subquotient of $\bigoplus_{i=1}^k \mathbf{V}^{\otimes (n_i+m_i)}$. Since $\hat{\mathbf{E}}|_{\mathbf{GL}}$ is topologically noetherian by Draisma's theorem, the result follows. $\square$

3. Preliminaries

3.1. **Spaces of matrices.** Let $\hat{\mathbf{M}}$ be the set of matrices $(a_{i,j})_{i,j \in \mathbf{N}}$ with $a_{i,j} \in \mathbf{k}$. Let $\mathbf{M} \subset \hat{\mathbf{M}}$ be the subset consisting of matrices with only finitely many non-zero entries. We have a trace pairing

$$\langle , \rangle \colon \hat{\mathbf{M}} \times \mathbf{M} \to \mathbf{k}, \qquad \langle A, B \rangle = \mathrm{tr}(A^t B)$$

that identifies $\hat{\mathbf{M}}$ with the linear dual of the space $\mathbf{M}$. We let $\mathbf{M}_n \subset \mathbf{M}$ be the set of matrices $(a_{i,j})$ with $a_{i,j} = 0$ for $i > n$ or $j > n$. We define $\hat{\mathbf{M}}_n$ similarly, but regard it as a quotient of $\hat{\mathbf{M}}$. Thus $\mathbf{M}$ is the union of the $\mathbf{M}_n$ and $\hat{\mathbf{M}}$ is the inverse limit of the $\hat{\mathbf{M}}_n$.

Let $\mathbf{U}, \mathbf{L} \subset \mathbf{M}$ and $\hat{\mathbf{U}}, \hat{\mathbf{L}} \subset \hat{\mathbf{M}}$ be the spaces of upper-triangular and lower-triangular matrices. The trace pairing identifies $\hat{\mathbf{L}}$ with the linear dual of $\mathbf{L}$ and $\hat{\mathbf{U}}$ with the dual of $\mathbf{U}$. One cannot multiply arbitrary elements of $\hat{\mathbf{M}}$, as this would typically involve infinite sums. However, the product AB is defined for $A \in \hat{\mathbf{L}}$ and $B \in \hat{\mathbf{M}}$, or for $A \in \hat{\mathbf{M}}$ and $B \in \hat{\mathbf{U}}$. We define $\mathbf{L}_n$, $\mathbf{U}_n$, $\hat{\mathbf{L}}_n$, and $\hat{\mathbf{U}}_n$ in the obvious ways.

We let $\mathbf{V}_n = \mathbf{k}^n$ and $\mathbf{V} = \bigcup_{n \geq 1} \mathbf{V}_n$. We let $\hat{\mathbf{V}}$ and $\hat{\mathbf{V}}_n$ be the dual spaces to $\mathbf{V}$ and $\mathbf{V}_n$, so that $\hat{\mathbf{V}}$ is the inverse limit of the spaces $\hat{\mathbf{V}}_n$.

3.2. **Polynomial representations.** Let $\mathbf{E}$ be a polynomial representation of $\mathbf{GL}$. Then the action of $\mathbf{GL}_n \subset \mathbf{GL}$ extends uniquely to an action of the monoid $\mathbf{M}_n$, and these assemble to an action of $\mathbf{M}$. Furthermore, the action of $\mathbf{U}$ on the dual $\hat{\mathbf{E}}$ extends uniquely to a continuous action of $\hat{\mathbf{U}}$, where continuous means that for fixed $v \in \hat{\mathbf{E}}$ the quantity uv depends only on the projection of u to $\hat{\mathbf{U}}_n$, for some n depending only on v. This is easy to see when $\mathbf{E} = \mathbf{V}$: the point is that, if e_i denotes the ith basis vector of $\mathbf{V}$, then ue_i^* only depends on the top left $i \times i$ block of u. The action of $\mathbf{L}$ on $\hat{\mathbf{E}}$ does *not* similarly extend to an action of $\hat{\mathbf{L}}$.

We can then equivalently think of the representation $\mathbf{E}$ as a polynomial functor $\underline{\mathbf{E}}$ on the category of $\mathbf{k}$-vector spaces. We let $\mathbf{E}_n$ be the value of the functor $\underline{\mathbf{E}}$ on $\mathbf{k}^n$, so that $\mathbf{E}$ itself is identified with the union of the $\mathbf{E}_n$. We let $\hat{\mathbf{E}}$ and $\hat{\mathbf{E}}_n$ be the linear duals of $\mathbf{E}$ and $\mathbf{E}_n$, so that $\hat{\mathbf{E}}$ is the inverse limit of the $\hat{\mathbf{E}}_n$. We note that $\mathbf{E}_n \subset \mathbf{E}$ is stable under the action of $\mathbf{U}$ and $\hat{\mathbf{U}}$. It follows that the projection map $\hat{\mathbf{E}} \to \hat{\mathbf{E}}_n$ is compatible with the action of $\hat{\mathbf{U}}$.

For the purposes of this paper, we really only need to consider $\mathbf{E}$'s that are finite sums of tensor powers of $\mathbf{V}$. In this case, one does not need to think about polynomial functors: the space $\mathbf{E}_n$ is then the corresponding sum of tensor powers of $\mathbf{V}_n$.

We identify $\hat{\mathbf{E}}$ with the spectrum of the ring $\mathrm{Sym}(\mathbf{E})$ (or more accurately, the $\mathbf{k}$-points of the spectrum), and equip it with the Zariski topology. By definition, a regular function f on $\hat{\mathbf{E}}$ is an element of $\mathrm{Sym}(\mathbf{E})$. Since $\mathbf{E}$ is the union of the $\mathbf{E}_n$, it follows that f belongs to $\mathrm{Sym}(\mathbf{E}_n)$ for some n; we say that f has **level** n. This implies that f factors through the projection $\hat{\mathbf{E}} \to \hat{\mathbf{E}}_n$.

4. The general linear case

Let $\mathbf{G} = \mathbf{GL} \times \mathbf{GL}$, and let $\mathbf{H}$ be the subgroup of elements of the form $(g, {}^t g^{-1})$. Of course, $\mathbf{H}$ is isomorphic to $\mathbf{GL}$. By definition, every algebraic representation of $\mathbf{H}$ is a subquotient of $\mathbf{E}|_{\mathbf{H}}$ for some polynomial representation $\mathbf{E}$ of $\mathbf{G}$, so it suffices to prove noetherianity of $\hat{\mathbf{E}}|_{\mathbf{H}}$ for all such $\mathbf{E}$. We therefore fix $\mathbf{E}$ for the rest of this section, and prove that $\hat{\mathbf{E}}|_{\mathbf{H}}$ is noetherian.

Let $\mathbf{G}$ act on $\mathbf{M}$ by the formula $(g, h) \cdot A = gA{}^t h$. The dual action of $\mathbf{G}$ on $\hat{\mathbf{M}}$ is given by the formula $(g, h) \cdot A = {}^t g^{-1} A h^{-1}$. Let $I \in \hat{\mathbf{M}}$ be the identity matrix. Note that the stabilizer of I in $\mathbf{G}$ is exactly $\mathbf{H}$. Given an $\mathbf{H}$-stable closed subset Z of $\hat{\mathbf{E}}$, let Z^+ be the closure of the $\mathbf{G}$-orbit of the set $\{I\} \times Z$ in $\hat{\mathbf{M}} \times \hat{\mathbf{E}}$. (The topology on the product is the Zariski topology on the product of schemes, which is not the product topology.) We will prove:

Proposition 4.1. *Let Z be an $\mathbf{H}$-stable closed subset of $\hat{\mathbf{E}}$. Then $\{I\} \times Z = (\{I\} \times \hat{\mathbf{E}}) \cap Z^+$.*

Before proving the proposition, we note an important consequence.

Corollary 4.2. *The function $Z \mapsto Z^+$ defines an order-preserving injection*

$$\{\mathbf{H}\text{-stable closed subsets of } \hat{\mathbf{E}}\} \to \{\mathbf{G}\text{-stable closed subsets of } \hat{\mathbf{M}} \times \hat{\mathbf{E}}\} \tag{4.3}$$

In particular, $\hat{\mathbf{E}}$ is topologically $\mathbf{H}$-noetherian.

Proof. It is clear that $Z \mapsto Z^+$ is order preserving. By the proposition, we can recover Z from Z^+, and so the map is injective. Since $\hat{\mathbf{M}} \times \hat{\mathbf{E}}$ is the dual of the polynomial representation $\mathbf{M} \oplus \mathbf{E}$ of $\mathbf{G}$, Proposition 2.3 shows that it is topologically $\mathbf{G}$-noetherian. In particular, the right side of (4.3) satisfies the descending chain condition. It follows that the left side does as well, and so $\hat{\mathbf{E}}$ is topologically $\mathbf{H}$-noetherian. $\square$

Let $\hat{\mathbf{U}}'$ be the subset of $\hat{\mathbf{U}}$ consisting of matrices where all diagonal entries are 1, and let $\hat{\mathbf{W}} = \hat{\mathbf{U}} \times \hat{\mathbf{U}}'$. Let $\varphi\colon \hat{\mathbf{W}} \to \hat{\mathbf{M}}$ be the function defined by $\varphi(u,v) = {}^t u I v$. Note that since ${}^t u \in \hat{\mathbf{L}}$ and $v \in \hat{\mathbf{U}}$, this matrix product is defined. Let $\varphi_n\colon \hat{\mathbf{W}}_n \to \hat{\mathbf{M}}_n$ be defined by the same formula. We note that the diagram

$$\begin{tikzcd} \hat{\mathbf{W}} \arrow[r, "\varphi"] \arrow[d] & \hat{\mathbf{M}} \arrow[d] \\ \hat{\mathbf{W}}_n \arrow[r, "\varphi_n"] & \hat{\mathbf{M}}_n \end{tikzcd}$$

commutes. That is, if $i, j \leq n$ then the (i,j) entry of $\varphi(u,v)$ only depends on the top left $n \times n$ blocks of u and v.

Lemma 4.4. *Let X be an affine variety over $\mathbf{k}$, and let $h\colon \hat{\mathbf{W}}_n \times X \to \mathbf{k}$ be a regular function. Then there is a monomial m in the diagonal entries of $\hat{\mathbf{U}}_n$ and a regular function $H\colon \hat{\mathbf{M}}_n \times X \to \mathbf{k}$ such that $H(\varphi(w), x) = m(u)h(w,x)$ holds for all $w = (u,v) \in \hat{\mathbf{W}}_n$ and $x \in X$.*

Proof. Consider the polynomial ring $R = \mathrm{Sym}(\mathbf{W}_n) = \mathbf{k}[u_{ij}, v_{kl}]_{1 \leq i \leq j \leq n, 1 \leq k < l \leq n} \otimes \mathcal{O}_X$. The morphism φ_n induces an injective ring homomorphism $\mathrm{Sym}(\mathbf{M}_n) \otimes \mathcal{O}_X \to R$, so we may view the former as a subring of the latter. Note that this subring is generated (as an $\mathcal{O}_X$-algebra) by terms of the form $\sum_{k=1}^{i} u_{ki} v_{kj}$ for $i < j$ and terms of the form $u_{ji} + \sum_{k=1}^{j-1} u_{ki} v_{kj}$ for $i \geq j$. We denote these terms by a_{ij}. It thus suffices to show that all u_{ij} and v_{kl} can be expressed as a quotient of a polynomial in the a_{ij} by a monomial in the u_{ii}. We do so by induction.

We have $u_{1i} = a_{i1}$ for any $i \geq 1$ and $v_{1j} = \frac{a_{1j}}{u_{11}}$ for any $j > 1$. Both of these are expressions of the desired form. Now suppose that u_{ki} and v_{kj} can be expressed in the desired form for all $k \leq i < j$ with $k < K$, for some K. For $i \geq K$ we see that a_{iK} is equal to u_{Ki} plus terms of the form $u_{ki} v_{kK}$ with $k < K$; since we have an expression of the desired form for each of these other terms, we find one for u_{Ki}. For $j > K$, we find that a_{Kj} is equal to $u_{KK} v_{Kj}$ plus terms of the form $u_{kK} v_{kj}$ with $k < K$; once again, this yields an expression of the desired form for v_{Kj}. This concludes the proof of the lemma. □

Let f be a regular function on $\hat{\mathbf{E}}$ of level n that vanishes on Z. We define a function h_f on $\hat{\mathbf{W}} \times \hat{\mathbf{E}}$ by

$$h_f(w,x) = f(w \cdot x).$$

Note that the above formula makes use of the action map $\hat{\mathbf{U}} \times \hat{\mathbf{U}} \times \hat{\mathbf{E}} \to \hat{\mathbf{E}}$, which exists since $\hat{\mathbf{E}}$ is a polynomial representation of $\mathbf{G}$.

We claim that h_f is a regular function of level n. Consider the diagram

$$\begin{tikzcd} \hat{\mathbf{W}} \times \hat{\mathbf{E}} \arrow[r] \arrow[d] & \hat{\mathbf{E}} \arrow[r, "f"] \arrow[d] & \mathbf{k} \arrow[d, equal] \\ \hat{\mathbf{W}}_n \times \hat{\mathbf{E}}_n \arrow[r] & \hat{\mathbf{E}}_n \arrow[r, "f"] & \mathbf{k} \end{tikzcd}$$

The left horizontal maps are the action maps. Both squares commute, so the whole diagram does, and thus h_f factors through the left map and so has level n.

It follows from the Lemma 4.4 that there is a monomial m_f in the diagonal entries of $\hat{\mathbf{U}}_n$ such that $m_f h_f$ induces a regular function on $\hat{\mathbf{M}} \times \hat{\mathbf{E}}$ of level n; call this function H_f.

Lemma 4.5. *The function H_f vanishes on Z^+.*

Proof. Let $\mathbf{G}_m = \mathbf{GL}_m \times \mathbf{GL}_m$, and let $\mathbf{H}_m \subset \mathbf{G}_m$ be the set of matrices of the form $(g, {}^t g^{-1})$. It suffices to show that H_f vanishes on a dense subset of the $\mathbf{G}_m$-orbit of $\{I\} \times Z$ for all $m \gg 0$. Now, the set $\mathbf{W}_m \mathbf{H}_m \subset \mathbf{G}_m$ is dense, since a generic element of $\mathbf{GL}_m$ can be written as a product of an upper triangular and lower triangular matrix. Suppose that $g = wh$, with $m \geq n$. Writing $w = (u, v)$, we have $g = (uh, v^t h^{-1})$. Note that

$$g \cdot I = {}^t(uh)^{-1} I (v^t h^{-1})^{-1} = {}^t u^{-1} I v^{-1} = \varphi(w^{-1}).$$

Let $z \in Z$. We then have

$$H_f(g(I, z)) = H_f(\varphi(w^{-1}), gz) = m_f(u^{-1}) h_f(w^{-1}, gz).$$

Now, by definition, we have

$$h_f(w^{-1}, gz) = f(w^{-1} whz) = f(hz)$$

Since Z is $\mathbf{H}$-stable, we have $hz \in Z$. Since f vanishes on Z, we thus see that this expression vanishes. □

Lemma 4.6. *Suppose $z \in \hat{\mathbf{E}}$ does not belong to Z. Then there exists f vanishing on Z such that $H_f(I, z) \neq 0$.*

Proof. Let n be such that the projection $\overline{z}$ of z to $\hat{\mathbf{E}}_n$ does not belong to Z_n. We can thus find a regular function f on $\hat{\mathbf{E}}_n$ that vanishes on Z_n but does not vanish at $\overline{z}$. We have

$$H_f(I, z) = H_f(\varphi(I, I), z) = m_f(I) h_f((I, I), z).$$

Since $m_f(u)$ is monomial in the diagonal entries of u, it follows that $m_f(I) \neq 0$. Furthermore, we have $h_f(I, I, z) = f(z) \neq 0$. We thus see that $H_f(I, z) \neq 0$. □

Proof of Proposition 4.1. It is clear that $\{I\} \times Z$ is contained in the intersection of $\{I\} \times \hat{\mathbf{E}}$ and Z^+. Suppose that $z \in \hat{\mathbf{E}}$ does not belong to Z. Let f be as in Lemma 4.6. By Lemma 4.5, we see that H_f belongs to the ideal of Z^+. Since $H_f(I, z) \neq 0$, it follows that $(I, z) \notin Z^+$, which proves the result. □

5. Some counterexamples

Draisma's theorem states that if $\mathbf{E}$ is a polynomial representation of $\mathbf{G}$ then $\hat{\mathbf{E}}$ is topologically $\mathbf{G}$-noetherian. One can identify $\mathbf{E}_n$ with the spectrum of $\mathrm{Sym}(\hat{\mathbf{E}}_n)$, and in this way regard $\mathbf{E} = \varinjlim \mathbf{E}_n$ as an ind-scheme. (The elements of $\mathrm{Sym}(\hat{\mathbf{E}})$ define functions on $\mathbf{E}$, and their zero loci define the closed sets.) It is therefore sensible to ask if $\mathbf{E}$ is topologically noetherian. In [ESS] it is proved that this is the case: in fact, the $\mathbf{G}$-stable closed sets of $\mathbf{E}$ and $\hat{\mathbf{E}}$ are shown to be in bijection, and so noetherianity of $\mathbf{E}$ follows from Draisma's theorem.

We now explain that ind- version of Theorem 1.2 fails. In fact, noetherianity fails due to an obvious obstruction in each case.

First consider the $\mathbf{GL}$ case. If m is an element of $\mathbf{M}$ then one can make sense of the determinant $\chi(m) = \det(1 - tm)$, a polynomial in t. The action of $\mathbf{GL}$ on $\mathbf{M}$ by conjugation defines an algebraic representation, and leaves χ invariant. Let $c_i(m)$ be the coefficient of t^i in $\chi(m)$. Then $c_i \colon \mathbf{M} \to \mathbf{A}^1$ is a $\mathbf{GL}$-invariant function. Taken together, the c_i define an $\mathbf{GL}$-invariant function $c \colon \mathbf{M} \to \mathbf{A}^\infty$, where the target is the ind-scheme $\varinjlim \mathbf{A}^n$. It is easy to see that c is surjective, from which it follows that $\mathbf{M}$ is not topologically $\mathbf{H}$-noetherian. (If

$Z_\bullet$ is an infinite strictly descending chain of closed subsets of $\mathbf{A}^\infty$ then $c^{-1}(Z_\bullet)$ is an infinite strictly descending chain of $\mathbf{H}$-stable closed subsets of $\mathbf{S}$.)

The other cases are similar. In the symplectic case, one considers the characteristic polynomial of anti-symmetric matrices, while in the orthogonal case one uses symmetric matrices. We remark that $\hat{\mathbf{V}}$ (and more generally $\hat{\mathbf{V}}^d$ for any $d \geq 1$) is known to be topologically $\mathfrak{S}$-noetherian [Co, Co2, AH, HS], where $\mathfrak{S}$ is the infinite symmetric group, and for similar reasons $\mathbf{V}$ is not topologically $\mathfrak{S}$-noetherian (one makes an invariant polynomial by using the coordinates as roots, so that the c_i's are elementary symmetric functions).

It is interesting to observe that these counterexamples do not apply in the pro- setting since the invariants they use no longer make sense: one cannot take the characteristic polynomial of an element of $\hat{\mathbf{M}}$, as this would involve infinite sums. Similarly, the proof of noetherianity in the pro- setting does not apply in the ind-setting since the element $I \in \hat{\mathbf{M}}$ does not belong to $\mathbf{M}$.

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