8 Central stability homology for polynomial \( \text{VIC}(\mathbb{Z}) \)-modules

**Proof of Theorem 7.5.** We prove the theorem by a double induction over \( r \) and \( i \). If \( r = -\infty \) or \( i < 0 \) the theorem is true. We thus may assume that if \( M \) has polynomial degree \( \leq s \) in ranks \( > d \),

\[
HS_q(M)_n \cong 0 \quad \text{for } n > \max(d + q, 2q + s)
\]
as long as \( s < r \) or \( q < i \).

Consider two double complexes:

\[
X_{pq} = \bigoplus_{(f,C) \in \text{Hom}_{\text{VIC}}(\mathbb{Z},Z^p,Z^n)} \bigoplus_{(g,D) \in \text{Hom}_{\text{VIC}}(\mathbb{Z},C)} \text{Im } f \oplus D \\
\cong CS_p(CS_q(\Sigma^p M))_n \\
\cong CS_q(CS_p(M(0)) \otimes M)_n
\]

and

\[
Y_{pq} = \bigoplus_{(f,C) \in \text{Hom}_{\text{VIC}}(\mathbb{Z},Z^p,Z^n)} \bigoplus_{(g,D) \in \text{Hom}_{\text{VIC}}(\mathbb{Z},C)} M_D \\
\cong CS_p(CS_q(M))_n \\
\cong CS_q(CS_p(M))_n
\]

Let

\[
E^1_{pq} = CS_p(HS_q(\Sigma^p M))_n
\]
denote the spectral sequence associated to \( X \). It converges to zero in the range \( n > 2(p + q) \).

Let us denote the spectral sequence associated to \( Y \) by \( \hat{E}^r_{pq} \). It turns out that \( d^1: \hat{E}^1_{1,q} \to \hat{E}^1_{0,q} \) is always the zero map.

The map of double complexes

\[
Y_{pq} \longrightarrow X_{pq}
\]

induces maps

\[
\hat{E}^1_{pq} \longrightarrow E^1_{pq}
\]

that are surjective for \( n > \max(d + p + q - 1, p + 2q + r - 1) \) and injective for \( n > \max(d + p + q, p + 2q + r + 1) \). This uses the induction hypothesis.

Therefore

\[
E^2_{0,i}(M)_n = E^1_{0,i} \cong HS_i(M)_n \quad \text{for } n > \max(d + i, 2i + r).
\]

The theorem follows because by induction

\[
E^1_{pq} \cong CS_p(HS_q(\Sigma^p M))_n \cong 0
\]

for \( q < i \) and \( n > \max(d + q, p + 2q + r) \). This implies that

\[
HS_i(M)_n \cong E^1_{0,1} \cong E^2_{0,i} \cong E^\infty_{0,i}
\]
in the given range, which vanishes for \( n > 2i \).