6 Quillen's spectral sequence argument for homological stability

In this lecture, we want to revisit Quillen's argument for homological stability with twisted coefficients and look at a similar argument for representation stability.

Theorem 6.1. *Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. The map $\phi_n\colon M_n \to M_{n+1}$ induces a stabilization map on homology*

$$H_*(\mathrm{GL}_{n-1}(\mathbb{Z}); M_{n-1}) \longrightarrow H_*(\mathrm{GL}_n(\mathbb{Z}); M_n).$$

Assume that $HS_i(M) \cong 0$ for all $n > ki + a$ for some $k \ge 2$. Then the stabilization map is an isomorphism for all $n > k(i+1) + a + 1$ and surjective for all $n > k(i+1) + a$.

Proof. Consider the double complex

$$E_* \mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_*(M)_n.$$

There are two spectral sequences coming from this double complex, which both converge to the homology of the total complex of the double complex. The first spectral sequence takes central stability homology first:

$$E^1_{pq} \cong E_p \mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} HS_q(M)_n$$

Since $HS_i(M)_n \cong 0$ for all $n > ki + a$, we get in particular $E^1_{pq} \cong 0$ for $n > k(p+q) + a$. The spectral sequence therefore converges to zero in that range.

The second spectral sequence takes group homology first:

$$E^1_{pq} \cong H_q(\mathrm{GL}_n(\mathbb{Z}); CS_p(M)_n) \cong H_q(\mathrm{GL}_n(\mathbb{Z}); \mathrm{Ind}^{\mathrm{GL}_n(\mathbb{Z})}_{\mathrm{GL}_{n-p}(\mathbb{Z})} M_{n-p}) \cong H_q(\mathrm{GL}_{n-p}(\mathbb{Z}); M_{n-p})$$

This spectral sequence converges to zero in the same range. It turns out that $d^1\colon E^1_{pq} \to E^1_{p-1,q}$ is the stability map $H_*(\mathrm{GL}_{n-p}(\mathbb{Z}); M_{n-p}) \longrightarrow H_*(\mathrm{GL}_{n-p+1}(\mathbb{Z}); M_{n-p+1})$ if p is odd and zero if p is even.

We prove the theorem by induction over i. The statement is trivial for $i < 0$. Let us assume that the statement is true for $q < i$. Observe that $E^2_{pq} \cong 0$ for $q < i$ and $n > kq + p + a + 1$. This implies that $E^2_{0,i} \cong E^\infty_{0,i} \cong 0$ for $n > k(i+1) + a$ and thus the stability map is surjective in that range. Likewise, it implies that $E^2_{1,i} \cong E^\infty_{pq} \cong 0$ for $n > k(i+1) + a + 1$ and thus the stability map is injective in that range. ☐

For representation stability, let us consider the spectral sequence

$$E_* \mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \tilde{C}_{*-1}(Y(n)),$$

where $Y(n)$ is a semi-simplicial set similar to $W(n)$ but for $\mathrm{Aut}(F_n)$. In particular, $\tilde{C}_{p-1}(Y(n)) \cong \mathrm{Ind}^{\mathrm{Aut}\, F_n}_{\mathrm{Aut}\, F_{n-p}} \mathbb{Z}$ and the following connectivity result is true.

Theorem 6.2 (Hatcher-Vogtmann 1998, Randal-Williams–Wahl 2017)**.** *$\tilde{H}_{i-1}(Y(n)) \cong 0$ for $n > 2i$.*

Thus both spectral sequences associated to this double complex converge to zero for $n > 2(p+q)$. The spectral sequence taking group homology first turns out to be

$$E^2_{pq} \cong HS_p(H_q(\mathrm{IA}))_n,$$

where $H_q(\mathrm{IA})$ is the $\mathsf{VIC}(\mathbb{Z})$–module introduced in Lecture 4. Clearly, $H_0(\mathrm{IA}) \cong M(0)$ and therefore $E^2_{p,0} \cong 0$ for $n > 2p$. This implies that $E^2_{0,1} \cong HS_0(H_1(\mathrm{IA}))_n \cong 0$ for $n > 4$ and $E^2_{1,1} \cong HS_1(H_1(\mathrm{IA}))_n \cong 0$ for $n > 6$. This means that $H_1(\mathrm{IA})$ is generated in degrees ≤ 4 and related in degrees ≤ 6.

We know the first homology of IA_n exactly already, though. And it will turn out that these bounds are not sharp. Together with bounds on when $E^2_{p,1} \cong HS_p(H_1(\mathrm{IA}))_n$ vanishes for $p \in \{2,3\}$, we can make statements about $H_2(\mathrm{IA})$. These will likely also not be sharp, but very little is known about the second homology of IA_n.