4 Central stability homology

Central stability homology is supposed to detect in which degrees syzygies of $\mathsf{VIC}(\mathbb{Z})$–modules are generated. Let us first construct it:

Definition 4.1. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. For later notation, let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. Define

$$CS_p(M)_n := \bigoplus_{(f,C) \in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p, \mathbb{Z}^n)} M_C.$$

Let $d_i \colon CS_p(M)_n \to CS_{p-1}(M)_n$ map the summand M_C corresponding to (f, C) to the summand $M_{C \oplus \mathrm{span}(e_i)}$ corresponding to

$$(f, C) \circ (\mathrm{inc}_i, \mathrm{span}(e_i)) = (f \circ \mathrm{inc}_i, C \oplus \mathrm{span}(e_i)),$$

where $\mathrm{inc}_i \colon \mathbb{Z}^{p-1} \to \mathbb{Z}^p$ with

$$\mathrm{inc}_i(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}$$

Define $\partial := \sum_{i=1}^p (-1)^i d_i \colon CS_p(M)_n \to CS_{p-1}(M)_n$.

We call $CS_*(M)$ the *central stability chain complex* of M and its homology $HS_*(M) := H_*(CS_*(M))$ the *central stability homology* of M.

Theorem 4.2 (Maazen 1979, Randal-Williams–Wahl 2017)**.** *$HS_i(M(0))_n \cong 0$ for all $n > 2i$.*

This theorem can be reinterpreted to a connectivity statement. There is a semi-simplicial set (δ–complex) $W(n)$ whose p-simplicies $W_p(n) = \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n)$. Face maps are given by precomposition of $(\mathrm{inc}_i, \mathrm{span}(e_i)) \in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p, \mathbb{Z}^{p+1})$. The reduced homology of $W(n)$ is can be computed using its simplicial chain complex

$$\tilde{C}_p(W(n)) = \mathbb{Z} W_p(n) = \mathbb{Z}\,\mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n).$$

It is easy to observe that

$$\tilde{C}_*(W(n)) \cong CS_{*+1}(M(0))_n.$$

Theorem 4.3 (P.)**.** *Let M be a $\mathsf{VIC}(\mathbb{Z})$–module, $N \in \mathbb{N}_0$, and $d_0, \dots, d_N \in \mathbb{N}_0$ with $d_{i+1} - d_i \geq 2$. Then the following two statements are equivalent.*

1. *There is a partial resolution*
$$P_N \to \cdots \to P_0 \to M \to 0$$
of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$.

2. *$HS_i(M)_n \cong 0$ for all $n > d_i$.*

Proof. Exercise. □