3 The homology of $\text{IA}_n$

We first recall how group homology is constructed. Let $G$ be a group and $M$ be a $\mathbb{Z}G$–module. Let $E_\ast G$ be a projective $\mathbb{Z}G$–resolution of the trivial representation. This is unique up to chain homotopy (exercise). Group homology $H_\ast(G; M)$ is the homology of the chain complex

$$E_\ast G \otimes_G M.$$ 

Group homology is functorial in the following sense. Let $G$ and $H$ be two groups and $\phi: G \to H$ be group homomorphism. Let $M$ be a $\mathbb{Z}G$–module and $N$ a $\mathbb{Z}H$–module. Through $\phi$ the module $N$ can be considered as a $\mathbb{Z}G$–module that is denoted by $\phi^* N$. Let $\psi: M \to \phi^* N$ be a $G$–equivariant map. Let $\xi: E_\ast G \to \phi^* E_\ast H$ be a $G$–equivariant map that induces the identity map on the trivial representation. Such a map exists and is unique up to chain homotopy because $E_\ast G$ is a projective resolution. Then

$$\xi \psi: E_\ast G \otimes_G M \longrightarrow E_\ast H \otimes_H N$$

is a map of chain complexes and induces a homomorphism

$$H_\ast(G; M) \longrightarrow H_\ast(H; N).$$

Let

$$1 \to K \to G \to Q \to 1$$

be a short exact sequence of groups. Fix $g \in G$ and let $c_g \in \text{Aut}(K)$ be the conjugation $c_g(k) = gkg^{-1}$. Let $M$ be a $\mathbb{Z}G$–module. Then $\psi(m) = gm$ gives an $K$–equivariant map $\psi: \text{Res}_K^G M \to \text{Res}_K^G \phi^* M$. Similarly, $\xi(x) = xg^{-1}$ gives an $K$–equivariant map $\xi: \text{Res}_K^G E_\ast G \to \text{Res}_K^G \phi^* E_\ast G$. Therefore,

$$\text{Res}_K^G E_\ast G \otimes_K \text{Res}_K^G M \longrightarrow \text{Res}_K^G E_\ast G \otimes_K \text{Res}_K^G M$$

gives rise to an automorphism of $H_\ast(K; M)$. Because $K$ acts trivially by this action, $H_\ast(K; M)$ is in fact a $\mathbb{Z}Q$–module.

Now consider the sequence of short exact sequences

$$1 \to \text{IA}_n \to \text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z}) \to 1.$$ 

Then $(H_\ast(\text{IA}_n; \mathbb{Z}))_{n \in \mathbb{N}_0}$ is a sequence of $\mathbb{Z}\text{GL}_n(\mathbb{Z})$–modules. The inclusion $\text{IA}_n \subset \text{IA}_{n+1}$ induces a $\mathbb{Z}\text{GL}_n(\mathbb{Z})$–equivariant map

$$\phi_n: H_\ast(\text{IA}_n; \mathbb{Z}) \longrightarrow H_\ast(\text{IA}_{n+1}; \mathbb{Z}).$$

This data comes in fact from a VIC(\mathbb{Z})–module, that we will denote by $H_\ast(\text{IA})$ (exercise).

Let us concentrate now on $H_1(\text{IA})$.

**Theorem 3.1** (Andreadakis 1965 (for $n \leq 3$), Bachmuth 1966).

$$H_1(\text{IA}_n) \cong \text{Hom}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n) \cong (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^2$$

**Proposition 3.2.** There is a VIC(\mathbb{Z})–module $M$ given by $M_n = (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^2$ and

$$\phi_n: (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^{n+1} \longrightarrow (\mathbb{Z}^{n+1})^* \otimes \bigwedge^2 \mathbb{Z}^{n+1}$$

$$e_i^* \otimes (e_j \wedge e_k) \longrightarrow e_i^* \otimes (e_j \wedge e_k),$$

where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{Z}^m$ and $e_1^*, \ldots, e_m^*$ its dual basis of $(\mathbb{Z}^m)^*$.

**Corollary 3.3.** The VIC(\mathbb{Z})–module $H_1(\text{IA})$ is isomorphic to $M$ from the previous proposition.