3 The homology of IA_n

We first recall how group homology is constructed. Let G be a group and M be a $\mathbb{Z}G$–module. Let E_*G be a projective $\mathbb{Z}G$–resolution of the trivial representation. This is unique up to chain homotopy (exercise). Group homology $H_i(G;M)$ is the homology of the chain complex

$$E_*G \otimes_G M.$$

Group homology is functorial in the following sense. Let G and H be two groups and $\phi\colon G \to H$ be group homomorphism. Let M be a $\mathbb{Z}G$–module and N a $\mathbb{Z}H$–module. Through ϕ the module N can be considered as a $\mathbb{Z}G$–module that is denoted by ϕ^*N. Let $\psi\colon M \to \phi^*N$ be a G–equivariant map. Let $\xi\colon E_*G \to \phi * E_*H$ be a G-equivariant map that induces the identity map on the trivial representation. Such a map exists and is unique up to chain homotopy because E_*G is a projective resolution. Then

$$E_*G \otimes_G M \xrightarrow{\xi\otimes\psi} E_*H \otimes_H N$$

is a map of chain complexes and induces a homomorphism

$$H_*(G;M) \longrightarrow H_*(H;N).$$

Let

$$1 \to K \to G \to Q \to 1$$

be a short exact sequence of groups. Fix $g \in G$ and let $c_g \in \mathrm{Aut}(K)$ be the conjugation $c_g(k) = gkg^{-1}$. Let M be a $\mathbb{Z}G$–module. Then $\psi(m) = gm$ gives an K–equivariant map $\psi\colon \mathrm{Res}^G_K M \to \mathrm{Res}^G_K \phi^*M$. Similarly, $\xi(x) = xg^{-1}$ gives an K–equivariant map $\xi\colon \mathrm{Res}^G_K E_*G \to \mathrm{Res}^G_K \phi^*E_*G$. Therefore,

$$\mathrm{Res}^G_K E_*G \otimes_K \mathrm{Res}^G_K M \xrightarrow{\xi\otimes\psi} \mathrm{Res}^G_K E_*G \otimes_K \mathrm{Res}^G_K M$$

gives rise to an automorphism of $H_*(K;M)$. Because K acts trivially by this action, $H_*(K;M)$ is in fact a $\mathbb{Z}Q$–module.

Now consider the sequence of short exact sequences

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to 1.$$

Then $(H_*(\mathrm{IA}_n;\mathbb{Z}))_{n\in\mathbb{N}_0}$ is a sequence of $\mathbb{Z}\,\mathrm{GL}_n(\mathbb{Z})$–modules. The inclusion $\mathrm{IA}_n \subset \mathrm{IA}_{n+1}$ induces a $\mathrm{GL}_n(\mathbb{Z})$–equivariant map

$$\phi_n\colon H_*(\mathrm{IA}_n;\mathbb{Z}) \longrightarrow H_*(\mathrm{IA}_{n+1};\mathbb{Z}).$$

This data comes in fact from a $\mathsf{VIC}(\mathbb{Z})$–module, that we will denote by $H_i(\mathrm{IA})$ (exercise).

Let us concentrate now on $H_1(\mathrm{IA})$.

Theorem 3.1 (Andreadakis 1965 (for $n \le 3$), Bachmuth 1966)**.**

$$H_1(\mathrm{IA}_n) \cong \mathrm{Hom}(\mathbb{Z}^n, \textstyle\bigwedge^2 \mathbb{Z}^n) \cong (\mathbb{Z}^n)^* \otimes \textstyle\bigwedge^2 \mathbb{Z}^2$$

Proposition 3.2. *There is a $\mathsf{VIC}(\mathbb{Z})$–module M given by $M_n = (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^2$ and*

$$\phi_n\colon (\mathbb{Z}^n)^* \otimes \textstyle\bigwedge^2 \mathbb{Z}^{n+1} \longrightarrow (\mathbb{Z}^{n+1})^* \otimes \textstyle\bigwedge^2 \mathbb{Z}^{n+1}$$

$$e_i^* \otimes (e_j \wedge e_k) \longmapsto e_i^* \otimes (e_j \wedge e_k),$$

where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{Z}^m$ and $e_1^*, \ldots, e_m^*$ its dual basis of $(\mathbb{Z}^m)^*$.

Corollary 3.3. *The $\mathsf{VIC}(\mathbb{Z})$–module $H_1(\mathrm{IA})$ is isomorphic to M from the previous proposition.*