1 Free groups and their automorphisms

Definition 1.1. Let $T$ be a set. A word from the alphabet $T$ is a map $[n] := \{1, \ldots, n\} \to T$ for some $n \in \mathbb{N}_0$. We denote the empty word $[0] \to T$ by $\varepsilon$. We call $n$ the length of a word $[n] \to T$.

Example 1.2. For an alphabet $T = \{a, b, c\}$, words are $\varepsilon, ab, aaa, abaca$. They have the lengths 0, 2, 3, 5, respectively.

Definition 1.3. Let $S$ be a set. Let $T = S \times \{-1, 1\}$, where by abuse of notation we identify $S \cong S \times \{1\} \subset T$ and write simply $s$ for $(s, 1) \in T$. We also write $s^{-1}$ for $(s, -1) \in T$. We call $s^{-1}$ the inverse of $s$.

Define an equivalence relation on the set of words from the alphabet $T$ by adding/removing a pair of adjacent $s$ and $s^{-1}$.

The free group $F_S$ is the set of all equivalence classes of words from $T$. Group multiplication is given by concatenation.

For $S = \{x_1, \ldots, x_n\}$ denote $F_S$ by $F_n$.

Exercise 1.4. Show that this describes a well defined group.

Theorem 1.5. Let $G$ be a group. A set map $S \to G$ uniquely determines a group homomorphism $F_S \to G$ extending the set map via $S \subset F_S$.

Proof. Exercise.

Definition 1.6. Let $S$ be a subset of a group $G$. We say that $S$ generates the smallest subgroup of $G$ that contains $S$.

Observe that $S \subset G$ generates the group $G$ if and only if the induced map $F_S \to G$ is surjective. More generally, the image of $F_S \to G$ is the subgroup generated by $S$.

Definition 1.7. Let $G$ be a group. An automorphism of $G$ is a group homomorphism $f : G \to G$, i.e. $f(gh) = f(g)f(h)$, that is bijective. The set of automorphisms of $G$ is denoted by $\text{Aut}(G)$ and it forms a group.

Note that an element of $f \in \text{Aut}(F_n)$ is determined by the images $f(x_1), \ldots, f(x_n)$.

Example 1.8. There is an inclusion of the symmetric group $S_n$ into $\text{Aut}(F_n)$ by sending $\sigma \in S_n$ to the automorphism defined by $x_i \mapsto x_{\sigma(i)}$.

Inverting the $i$th generator is an automorphism:

$$\text{inv}_i : x_j \mapsto \begin{cases} x_j^{-1} & j = i \\ x_j & j \neq i \end{cases}$$

Multiplying the $i$th generator to the $j$th (from the left or the right) is an automorphism:

$$\text{leftmul}_{ij} : x_k \mapsto \begin{cases} x_ix_j & k = j \\ x_k & k \neq j \end{cases}$$

$$\text{rightmul}_{ij} : x_k \mapsto \begin{cases} x_jx_i & k = j \\ x_k & k \neq j \end{cases}$$
Theorem 1.9 (Nielsen, 1924). \( \text{Aut}(F_n) \) is generated by permutations, \( \text{inv}_1 \), and \( \text{leftmul}_{12} \).

Definition 1.10. Let \( G \) be a group. For \( a, b \in G \), the commutator is \( [a, b] = aba^{-1}b^{-1} \). The commutator subgroup \( G' \) of \( G \) is generated by all commutators. More generally, let \( H \) be a subgroup of \( G \), denote \( [G, H] \) to be the subgroup generated by commutators \( [g, h] \) with \( g \in G \) and \( h \in H \).

The lower central series of \( G \) is a series of subgroups \( \gamma_i G \) of \( G \), defined recursively by \( \gamma_1 G = G \) and \( \gamma_{i+1} G = [G, \gamma_i G] \).

We call \( G^{ab} := G/G' \) the abelianization of \( G \).

Proposition 1.11. There is a surjective group homomorphism

\[
\text{Aut}(F_n) \longrightarrow \text{GL}_n(\mathbb{Z}).
\]

Proof. Note that \( F_n^{ab} \cong \mathbb{Z}^n \) (exercise). Because abelianizing is functorial (exercise), we get a group homomorphism

\[
\text{Aut}(F_n) \longrightarrow \text{Aut}(\mathbb{Z}^n).
\]

Observe that \( \text{Aut}(\mathbb{Z}^n) \cong \text{GL}_n(\mathbb{Z}) \) is the group of invertible integral \( n \times n \) matrices (exercise). \( \text{GL}_n(\mathbb{Z}) \) is generated by

\[
\begin{pmatrix}
-1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

and all matrices with ones on the diagonal and one one off the diagonal (exercise). All of these generators are in the image of the map \( \text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z}) \). More precisely, they are images of \( \text{inv}_1 \) and \( \text{leftmul}_{ij} \). \( \square \)

Definition 1.12. The Torelli subgroup of \( \text{Aut}(F_n) \) is the kernel

\[
\text{IA}_n := \ker(\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})).
\]