Exercises

1.) Fill in the details about X_{pq} and Y_{pq}:

(a) Show that

$$Y_{pq} \cong CS_p(CS_q(M))_n \cong CS_q(CS_p(M))_n.$$

(b) Those isomorphisms describe the differential in both p and q direction. Show that Y_{pq} is a double complex, i.e. that the two differentials commute.

(c) Show that

$$X_{pq} \cong CS_p(CS_q(\Sigma^p M))_n \cong CS_q(CS_p(M(0)) \otimes M)_n.$$

(d) Those isomorphisms describe the differential in both p and q direction. Show that X_{pq} is a double complex, i.e. that the two differentials commute.

2.) We want to show that $d^1 \colon \widehat{E}^1_{1,q} \to \widehat{E}^1_{0,q}$ is always zero. Find an isomorphism $\psi \colon \widehat{E}^0_{1,q} \to \widehat{E}^0_{0,q+1}$ that is a chain homotopy from the map of chain complexes $\widehat{E}^0_{1,*} \to \widehat{E}^0_{0,*}$ to the zero map.

For the remainder of the exercises, fix $r \in \mathbb{N}_0$, $d \in \mathbb{N}_0 \cup \{-\infty\}$, and $i \in \mathbb{N}_0$. Assume that

$$HS_q(N)_n \cong 0 \qquad \text{for all } n > \max(e+q, 2q+s)$$

if N is a $\mathsf{VIC}(\mathbb{Z})$–module with polynomial degree $\le s$ in ranks $> e$ as long as $s < r$ or $q < i$. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module with polynomial degree $\le r$ in ranks $> d$.

3. We want to prove that

$$\widehat{E}^1_{pq} \longrightarrow E^1_{pq}$$

is surjective for $n > \max(d+p+q-1, p+2q+r-1)$ and injective for $n > \max(d+p+q, p+2q+r+1)$.

(a) Prove that $\Sigma^p M$ has polynomial degree $\le r$ in ranks $> d-p$.

(b) For $p \ge 1$, prove that $\ker(M \to \Sigma^p M)_n \cong 0$ for all $n > d$ and $\operatorname{coker}(M \to \Sigma^p M)$ has polynomial degree $r-1$ in ranks $> d-1$. (Hint: Use Exercise 7.2a)

(c) Use this information to show that

$$HS_q(M)_n \longrightarrow HS_q(\Sigma^p M)_n$$

is surjective for $n > \max(d+q-1, 2q+r-1)$ and injective for $n > \max(d+q, 2q+r+1)$.

(d) Finish the proof.

4. Let us finish the proof of Theorem 7.5:

(a) Observe that $E^1_{0,i} \cong HS_i(M)_n$.

(b) Use the previous exercises to show that $E^1_{0,i} = E^2_{0,i}$ for $n > \max(d+i, 2i+r)$.

(c) Use the induction hypothesis to show that $E^2_{0,i} = E^\infty_{0,i}$ for $n > \max(d+i-1, 2i+r)$. [Hint: Consider E^1_{pq} for $q < i$, and $p+q = i+1$.]

(d) Show that $E^\infty_{pq} \cong 0$ for $n > 2(p+q)$ to finish the proof.