Exercises

1.) Given an abstract simplicial complex, find an abstract $\Delta$-complex with the same realization.

2.) In this exercise, we want to show that $PB_n$ is weakly Cohen–Macaulay of dimension $n - 1$, following a proof by Church–Putman 2017.

(a) Let $V$ be a summand of $\mathbb{Z}^n$ and $v_0, \ldots, v_p$ a basis of $V$. Observe that $Lk_{PB_n}(\{v_0, \ldots, v_p\})$ is independent of the choice of basis of $V$. Denote this link by $Lk_{PB_n}(V)$.

(b) Let $PB_n^m = Lk_{PB_n}^{m+n}(\mathbb{Z}^m)$. Show that $Lk_{PB_n}^m(\sigma) \cong PB_n^{m+p}$ for every $(p - 1)$-simplex $\sigma$ of $PB_n^m$.

(c) Fix an $F: \mathbb{Z}^{m+n} \to \mathbb{Z}$ and $N > 0$. For a subcomplex $X$ of $PB_n^m$, define $X^{<N}$ to be the full subcomplex of $X$ spanned by the vertices $v$ with $|F(v)| < N$. Let $\sigma$ be a simplex of $PB_n^m$ that has a vertex $v$ with $F(v) = N$. Show that $Lk_{PB_n}(\sigma)$ can be retracted to $Lk_{PB_n}(\sigma)^{<N}$.

(d) We will prove that $PB_n^m$ is $(n - 2)$–connected by induction over $n$. For $n = 0$ there is nothing to show. For $n = 1$, prove that $PB_1^m$ is non-empty for $m \geq 0$.

(e) For the induction step, fix a map $\phi: S^p \to PB_n^m$ with $0 \leq p \leq n - 2$. (We may assume that there is a triangulation of $S^p$ such $\phi$ is simplicial.) We want to nullhomotope $\phi$. Let $F: \mathbb{Z}^{m+n} \to \mathbb{Z}$ be the map that returns the last coordinate and let

$$R(\phi) = \max(F(v) \mid v \text{ a vertex of } PB_n^m \text{ in the image of } \phi).$$

Show that the sphere can be coned off if $R(\phi) = 0$.

(f) If $R = R(\phi) > 0$ then there is a simplex $\sigma$ of $S^p$ of maximal dimension (with respect to the following condition) such that $F(\phi(x)) = R$ for all $x \in \sigma$. Check that $\phi$ maps $Lk_{S^p}(\sigma)$ into $Lk_{PB_n}(\phi(\sigma))^{<R}$.

(g) Assume that $\sigma$ is $k$–dimensional. Show that $Lk_{S^p}(\sigma)$ homeomorphic to $S^{p-k-1}$.

(h) Assume that $\phi(\sigma)$ is $\ell$–dimensional. (Note that $k \geq \ell$.) Prove that $Lk_{PB_n}(\phi(\sigma))^{<R}$ is $(n - \ell - 3)$–connected.

(i) Homotope $\phi$ to replace $\phi(\sigma)$ by a subcomplex in $Lk_{PB_n}(\sigma)^{<N}$.

(j) Observe that we can get reduce $R(\phi)$ this way. And finish the proof.

3.) Show that the link of a simplex in $PBC_n$ is isomorphic to a $PBC_m$ for some $m \leq n$.

4.) Show that $PBC_n$ is a join complex over $PB_n$ by the map $\pi$ that forgets the complement.

5.) Let $\tau$ be a $p$–simplex of $PBC_n$. Show that $Lk_{PB_n}(\phi(\tau))$ is weakly Cohen–Macaulay of dimension $n - p - 3$.

6.) Use Proposition 5.15 and the previous exercise to show that $HS_i(M(0))_n \cong 0$ for $n > 2i$.  

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