Exercises

1.) Given an abstract simplicial complex, find an abstract Δ-complex with the same realization.

2.) In this exercise, we want to show that PB_n is weakly Cohen–Macaulay of dimension $n-1$, following a proof by Church–Putman 2017.

(a) Let V be a summand of $\mathbb{Z}^n$ and $v_0, \dots, v_p$ a basis of V. Observe that $\mathrm{Lk}_{\mathrm{PB}_n}(\{v_0, \dots, v_p\})$ is independent of the choice of basis of V. Denote this link by $\mathrm{Lk}_{\mathrm{PB}_n}(V)$.

(b) Let $\mathrm{PB}_n^m = \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\mathbb{Z}^m)$. Show that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma) \cong \mathrm{PB}_{n-p}^{m+p}$ for every $(p-1)$-simplex σ of PB_n^m.

(c) Fix an $F\colon \mathbb{Z}^{m+n} \to \mathbb{Z}$ and $N > 0$. For a subcomplex X of PB_n^m, define $X^{<N}$ to be the full subcomplex of X spanned by the vertices v with $|F(v)| < N$. Let σ be a simplex of PB_n^m that has a vertex v with $F(v) = N$. Show that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ can be retracted to $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$.

(d) We will prove that PB_n^m is $(n-2)$–connected by induction over n. For $n = 0$ there is nothing to show. For $n = 1$, prove that PB_1^m is non-empty for $m \geq 0$.

(e) For the induction step, fix a map $\phi\colon S^p \to \mathrm{PB}_n^m$ with $0 \leq p \leq n-2$. (We may assume that there is a triangulation of S^p such ϕ is simplicial.) We want to nullhomtope ϕ. Let $F\colon \mathbb{Z}^{m+n} \to \mathbb{Z}$ be the map that returns the last coordinate and let

$$R(\phi) = \max(F(v) \mid v \text{ a vertex of } \mathrm{PB}_n^m \text{ in the image of } \phi).$$

Show that the sphere can be coned off if $R(\phi) = 0$.

(f) If $R = R(\phi) > 0$ then there is a simplex σ of S^p of maximal dimension (with respect to the following condition) such that $F(\phi(x)) = R$ for all $x \in \sigma$. Check that ϕ maps $\mathrm{Lk}_{\S^p}(\sigma)$ into $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\sigma))^{<R}$.

(g) Assume that σ is k–dimensional. Show that $\mathrm{Lk}_{\S^p}(\sigma)$ homeomorphic to S^{p-k-1}.

(h) Assume that $\phi(\sigma)$ is ℓ–dimensional. (Note that $k \geq \ell$.) Prove that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\sigma))^{<R}$ is $(n-\ell-3)$–connected.

(i) Homotope ϕ to replace $\phi(\sigma)$ by a subcomplex in $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$.

(j) Observe that we can get reduce $R(\phi)$ this way. And finish the proof.

3.) Show that the link of a simplex in PBC_n is isomorphic to a PBC_m for some $m \leq n$.

4.) Show that PBC_n is a join complex over PB_n by the map π that forgets the complement.

5.) Let τ be a p–simplex of PBC_n. Show that $\mathrm{Lk}_{\mathrm{PB}_n}(\phi(\tau))$ is weakly Cohen–Macaulay of dimension $n-p-3$.

6.) Use Proposition 5.15 and the previous exercise to show that $HS_i(M(0))_n \cong 0$ for $n > 2i$.