# A SURVEY OF $\mathbf{FI}_d$-MODULES

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Contents



## 1. Introduction

1.1. **History.** In [CEF], Church, Ellenberg, and Farb introduced the notion of **FI**-module. In that paper, they studied **FI**-modules in characteristic 0, and proved two fundamental theorems about them: (a) finitely generated **FI**-modules over a noetherian coefficient ring are noetherian; and (b) finitely generated **FI**-modules over a field have eventually polynomial growth. Additionally, [CEF] gave an immense number of applications of **FI**-modules, though that will not concern us so much here. Shortly after [CEF] appeared, we released the paper [SS1] that studies modules over the twisted commutative algebra (tca) $\mathbf{A}(1)$ in characteristic 0.[1] In fact, modules over this tca are the same thing as **FI**-modules, so our paper can also be seen as a study of **FI**-modules, though the point of view is quite different. We gave an essentially complete description of the category of modules, and showed that the theory has an extremely rich internal structure.

A few months after the above two papers, Church, Ellenberg, Farb, and Nagpal released the paper [CEFN], which amounted to the first piece of progress on **FI**-modules over more general rings (i.e., not characteristic 0). They proved (a) for arbitrary noetherian coefficient rings and (b) for arbitrary coefficient fields. A few years later, Nagpal released his thesis [Nag], in which he proves a theorem that, in our opinion, is really the key to **FI**-modules over general rings: if $M$ is a finitely generated **FI**-module over a noetherian ring, then some shift of $M$ is $\sharp$-filtered. This theorem has allowed nearly all of the results from [SS1] to be extended to positive characteristic. These extensions have been carried out by a number of different authors (Church, Ellenberg, Gan, Li, Nagpal, Ramos, and probably others) in the last year or two. We remark that these extensions are not always a simple application of [Nag] and sometimes require serious new ideas, as is the case with [CE] for example. Our understanding of **FI**-modules in positive characteristic is still not quite as complete as it is in characteristic 0, but now comes reasonably close.

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[1] In [SS1], the notation $\mathrm{Sym}(\mathbf{C}\langle 1\rangle)$ was used in place of $\mathbf{A}(1)$.

The tca perspective shows that **FI**-modules are but the tip of an enormous iceberg: $\mathbf{A}(1)$ is the simplest tca—it is somewhat analogous to a polynomial ring in a single variable—and modules over more complicated tca's can be thought of as more complicated versions of **FI**-modules. In this document, we focus on the tca $\mathbf{A}(d)$ freely generated by $d$ indeterminates of degree 1; this is the next step up in difficulty from $\mathbf{A}(1)$. There is an "**FI**-perspective" on $\mathbf{A}(d)$. Let $\mathbf{FI}_d$ be the category whose objects are finite sets and where a morphism $f \colon S \to T$ is an injection together with a $d$-coloring on the complement of the image (i.e., a function $T \setminus f(S) \to [d]$). Then $\mathbf{A}(d)$-modules are equivalent to $\mathbf{FI}_d$-modules. However, we prefer to use the tca perspective in this paper for the most part.

In characteristic 0, the analogs of theorems (a) and (b) for $\mathbf{FI}_d$-modules were first proved in [Sn]. (In fact, this paper appeared well before [CEF], but the language is quite different.) Note that theorem (b) is different for $\mathbf{FI}_d$-modules: exponential growth is possible. (See Theorem 2.1 below for the exact statement.) These theorems were extended to arbitrary coefficient rings in [SS3] using Gröbner-theoretic techniques. While there are now several proofs of (a) and (b) for **FI**-modules, the proofs in [SS3] are the only one for $\mathbf{FI}_d$-modules with $d > 1$ (over general coefficient rings).

In a shortly forthcoming work [SS4], we extend our results on $\mathbf{A}(1)$-modules in [SS1] to $\mathbf{A}(d)$-modules for arbitrary $d$ (in characteristic 0). Like [SS1] did for $\mathbf{A}(1)$-modules, [SS4] gives an essentially complete picture of the category of $\mathbf{A}(d)$-modules. The purpose of this document is to give an overview of some of the results from [SS4]. None of them are known in positive characteristic (when $d > 1$), and we believe this will be a very fertile area of research in the near future.

1.2. **Summary.** The following table summarizes the state of affairs for **FI**-modules:

| | Characteristic 0 | Arbitrary |
|---|---|---|
| Theorems (a) and (b) | [CEF] | [CEFN] |
| Complete picture | [SS1] | [Nag] + others |

The corresponding picture for $\mathbf{FI}_d$-modules is:

| | Characteristic 0 | Arbitrary |
|---|---|---|
| Theorems (a) and (b) | [Sn] | [SS3] |
| Complete picture | [SS4] | Non-existent! |

This document summarizes [SS4], with the hope of stimulating research in the bottom right square.

1.3. **Basic definitions.** We let $\mathbf{A}(d)$ be the tca freely generated by $d$ degree 1 variables over the complex numbers. For the purposes of this paper, we identify $\mathbf{A}(d)$ with the polynomial ring $\mathrm{Sym}(\mathbf{C}^\infty \otimes \mathbf{C}^d)$, equipped with the natural action of $\mathbf{GL}_\infty$. We write $|\mathbf{A}(d)|$ when we want to think of $\mathbf{A}(d)$ simply as a $\mathbf{C}$-algebra and not a tca. An $\mathbf{A}(d)$-module is by definition an $|\mathbf{A}(d)|$-module equipped with a compatible polynomial action of $\mathbf{GL}_\infty$. (A representation of $\mathbf{GL}_\infty$ is **polynomial** if it decomposes into a direct sum of Schur functors.) As mentioned above, the category of $\mathbf{A}(d)$-modules is equivalent to the category of $\mathbf{FI}_d$-modules, though we usually prefer the $\mathbf{A}(d)$ perspective. (The equivalence between these two categories is induced by Schur–Weyl duality.) There is an obvious notion of finite generation for $\mathbf{A}(d)$- or $\mathbf{FI}_d$-modules; see [CEF, Definition 1.2] for details when $d = 1$.

1.4. **Outline.** The results of [SS4] can roughly be divided into two classes: one consists of rather technical structural results about $\mathbf{A}(d)$-modules, while the other consists of more accessible and perhaps interesting results. The latter logically depend on the former, and so [SS4] begins with more technical material. In the interest of making this paper more readable, we have taken a different tack. We begin each of §2 and §3 by stating one of the "second class" results. We then state the "first class" result that is used in the proof, and explain a bit about how the proof works. (Each of these sections also ends with a "bonus application" of the technical material.) We hope this motivates the more technical material and makes it easier to digest. In §4 we summarize the structure theory; this section is necessarily more technical. Finally, in §5 we state a few additional theorems.

## 2. Hilbert series, derived category generators, and regularity

In §2.1 we review the existing (i.e., prior to [SS4]) theory of Hilbert series for $\mathbf{A}(d)$-modules. In §2.2 we introduce an invariant called the formal character, which captures much more information than the Hilbert series, and state a rationality result for it. In §2.3, we state the technical result needed to prove the rationality of formal characters, and in §2.4 we explain how the proof of rationality works. Finally, in §2.5 we give a second application of the results of §2.3, to the regularity of $\mathbf{A}(d)$-modules.

2.1. **Standard Hilbert series.** Let $M$ be an $\mathbf{FI}_d$-module over a field. We define its **Hilbert series** by

$$\mathrm{H}_M(t) = \sum_{n \geq 0} \dim(M_n) \frac{t^n}{n!},$$

where $M_n$ denotes the value of the functor $M$ on the set $[n] = \{1, \ldots, n\}$. The Hilbert series of an $\mathbf{A}(d)$-module $M$ is defined as the Hilbert series of the corresponding $\mathbf{FI}_d$-module. More directly, if $M = \bigoplus_\lambda M_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$ then

$$\mathrm{H}_M(t) = \sum_\lambda \dim(M_\lambda) \dim(\mathbf{M}_\lambda) \frac{t^{|\lambda|}}{|\lambda|!}$$

where $\mathbf{M}_\lambda$ is the Specht module. The main theorem about these series is:

**Theorem 2.1.** *If $M$ is finitely generated $\mathbf{FI}_d$- or $\mathbf{A}(d)$-module then $\mathrm{H}_M(t)$ has the form $\sum_{r=0}^d p_r(t) e^{rt}$ where $p_r(t)$ is a polynomial.*

This was proved in [Sn] in characteristic 0 using representation-theoretic techniques and [SS3] in general using techniques from Gröbner bases and formal languages.

**Remark 2.2.** When $d = 1$ this theorem recovers the fact that if $M$ is a finitely generated $\mathbf{FI}$-module then $\dim(M_n)$ is eventually a polynomial function of $n$. □

2.2. **Formal characters.** The standard Hilbert series records only the dimension of the $S_n$-representation $M_n$, and therefore forgets quite a bit of information. To remedy this, we introduce a more refined invariant called the **formal character**. Let $M$ be an $\mathbf{A}(d)$-module, decomposed as above. Then its formal character is defined as

$$\Theta_M = \sum_\lambda \dim(M_\lambda) s_\lambda$$

where $s_\lambda$ is the Schur function. Thus $\Theta_M$ is an infinite linear combination of symmetric functions. Since each symmetric function is a polynomial in the complete homogeneous

Schur functions $s_n$, one can think of $\Theta_M$ as a power series in the infinitely many variables $\{s_n\}_{n\geq 0}$. We note that $\Theta_M$ determines $M$ as a representation of $\mathbf{GL}_\infty$, up to isomorphism.

For $k \geq 0$, let $S_k = \sum_{n\geq k} \binom{n}{k} s_k$. Then in [SS4] we prove the following result:

**Theorem 2.3.** *If $M$ is a finitely generated $\mathbf{A}(d)$-module then $\Theta_M$ is a polynomial in the $s_n$'s and $S_n$'s.*

This is a vast improvement of Theorem 2.1. In fact, our results are much more precise and match various characteristics of $M$ and $\Theta_M$ (for instance, only those $S_n$ with $n \leq d$ appear). We have two proofs. One is representation theoretic, and boils down to a lengthy and detailed computation. The other is an elegant application of the structure theory of $\mathbf{A}(d)$-modules. We present the second proof here (or at least its main ideas).

There is a variant of the formal character, called the **enhanced Hilbert series**. It is easier to define from the $\mathbf{FI}_d$ point of view, so let $M$ be an $\mathbf{FI}_d$-module. Its enhanced Hilbert series is

$$\widetilde{\mathrm{H}}_M(t) = \sum_\lambda \mathrm{tr}(c_\lambda | M_{|\lambda|}) \frac{t^\lambda}{\lambda!},$$

where $c_\lambda$ is the conjugacy class in $S_{|\lambda|}$ corresponding to $\lambda$, $t^\lambda$ is $t_1^{m_1(\lambda)} t_2^{m_2(\lambda)} \cdots$, where $m_i(\lambda)$ is the multiplicity of $i$ in $\lambda$, and $\lambda! = m_1(\lambda)! m_2(\lambda)! \cdots$. It turns out that the enhanced Hilbert series is obtained from the formal character by applying a ring isomorphism $\mathbf{Q}[s_n] \to \mathbf{Q}[t_n]$, and so the two carry exactly the same information, just packaged differently. In particular, Theorem 2.3 implies a rationality result for $\widetilde{\mathrm{H}}_M$.

One advantage the enhanced Hilbert series has is that it admits a nice variant in positive characteristic: use the Brauer character instead of the normal character. (That is, replace $\mathrm{tr}(c_\lambda|-)$ with its Brauer version.) This leads to a natural problem:

**Problem 2.4.** *Prove a rationality result for the Brauer version of the enhanced Hilbert series of a finitely generated $\mathbf{FI}_d$-module over a field of positive characteristic.*

We have solved this with R. Nagpal for $d = 1$, but do not know how to do it for $d > 1$.

2.3. **Generators for the derived category.** We now introduce the somewhat technical structural result about $\mathbf{A}(d)$-modules that we will use to prove Theorem 2.3. Recall that $\mathbf{A}(d)$ is defined as $\mathrm{Sym}(\mathbf{C}^\infty \otimes \mathbf{C}^d)$. It is perhaps more canonical to let $E$ be a $d$-dimensional vector space and then define $\mathbf{A}(E)$ as $\mathrm{Sym}(\mathbf{C}^\infty \otimes E)$. From this point of view, there is no reason to restrict $E$ to simply being a vector space: one could just as well allow $E$ to be a free module over some $\mathbf{C}$-algebra, or even a locally free sheaf on a variety over $\mathbf{C}$. In fact, we will require exactly this.

Let $Y$ be the Grassmannian $\mathbf{Gr}_r(E)$ of rank $r$ quotients of $E = \mathbf{C}^d$, and write $\pi \colon \mathbf{Gr}_r(E) \to \mathrm{Spec}(\mathbf{C})$ for the structure map. We let $\mathcal{Q}$ be the tautological quotient bundle on $Y$. We consider $\mathbf{A}(\mathcal{Q})$ as a sort of sheaf of tca's on $Y$. We have $\pi_*(\mathbf{A}(\mathcal{Q})) = \mathbf{A}(E)$, so if $M$ is an $\mathbf{A}(\mathcal{Q})$-module then $\mathrm{R}^i\pi_*(M)$ is an $\mathbf{A}(E)$-module for all $i \geq 0$. It is not difficult to show that if $M$ is finitely generated then so is each $\mathrm{R}^i\pi_*(M)$. We define $\mathcal{C}_r \subset \mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$ to be the collection of objects of the form

$$\mathrm{R}\pi_*(V \otimes \mathcal{F} \otimes \mathbf{A}(\mathcal{Q}))$$

where $V$ is a finite length polynomial representation of $\mathbf{GL}_\infty$ and $\mathcal{F}$ is a coherent sheaf on $Y$. (It's enough to consider $V = \mathbf{S}_\lambda(\mathbf{C}^\infty)$, for variable $\lambda$, in fact.)

If $\mathcal{T}$ is a triangulated category and $S$ is a collection of objects in $\mathcal{T}$ then there is a smallest triangulated subcategory of $\mathcal{T}$ containing $S$. We call this the subcategory **generated** by $S$.[2] We can now state the theorem of interest:

**Theorem 2.5.** *The category $\mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$ is generated by $\bigcup_{0 \le r \le d} \mathcal{C}_r$.*

Concretely, this means that any object of $\mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$ can be constructed from the objects in the $\mathcal{C}_r$'s in finitely many steps by taking shifts and cones. This theorem leads to the following axiomatic method for proving results about $\mathbf{A}(E)$-modules:

**Corollary 2.6.** *Let $\mathcal{P}$ be a property of objects of $\mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$ satisfying the following conditions:*

(a) *If $\mathcal{P}$ is true for $M$ then it is also true for any shift of $M$.*
(b) *If $\mathcal{P}$ is true for two members or an exact triangle then it is true for the third.*
(c) *$\mathcal{P}$ is true for the objects in $\mathcal{C}_r$, for all $0 \le r \le d$.*

*Then $\mathcal{P}$ is true for all objects in $\mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$.*

**Problem 2.7.** *Formulate and prove a version of Theorem 2.5 over arbitrary noetherian coefficient rings.*

**Remark 2.8.** Suppose $d = 1$, i.e., we are in the **FI**-module case. Then $\mathcal{C}_0$ consists of torsion objects while $\mathcal{C}_1$ consists of free objects. Theorem 2.5 is thus a consequence of the fact that every object $M$ of $\mathrm{D}^b_{\mathrm{fg}}(\mathbf{A}(E))$ fits into a triangle

$$T \to M \to F \to$$

where $T$ is a finite length complex of finitely generated torsion modules and $F$ is a finite length complex of finitely generated free modules. This result is known over arbitrary noetherian coefficient rings (see [Nag]), and so Problem 2.7 is known for $d = 1$. □

2.4. **Back to formal characters.** We now explain how to use Theorem 2.5 to prove Theorem 2.3. We apply Corollary 2.6, taking $\mathcal{P}(M)$ to be the statement "$\Theta_M$ is a polynomial in the $s_n$'s and $S_n$'s." (If $M$ is a complex then $\Theta_M$ is defined to be the alternating sum of the $\Theta$'s of the cohomology groups.) It is clear that $\mathcal{P}$ satisfies conditions (a) and (b) of Corollary 2.3. We must verify (c).

Fix $r$, and let $Y$, $\pi$, and $\mathcal{Q}$ be as in the previous section. Suppose that $M$ is a finitely generated $\mathbf{A}(\mathcal{Q})$-module. Then $M$ decomposes as $\bigoplus_\lambda M_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$ where $M_\lambda$ is a coherent sheaf on $Y$. We define the formal character of $M$ as

$$\Theta_M = \sum_\lambda [M_\lambda] s_\lambda,$$

where $[M_\lambda]$ denotes the class of $M_\lambda$ in the Grothendieck group $\mathrm{K}(Y)$ of $Y$. Thus $\Theta_M$ is a power series in the $s_n$'s with coefficients in the group $\mathrm{K}(Y)$. One now proves:

(d) If $M$ is an $\mathbf{A}(\mathcal{Q})$-module then $\mathrm{R}\pi_*(\Theta_M) = \Theta_{\mathrm{R}\pi_*(M)}$.
(e) $\Theta_{\mathbf{A}(\mathcal{Q})}$ is a polynomial in the $S_n$'s with coefficients in $\mathrm{K}(Y)$.

Statement (d) is immediate, while (e) is an exercise best done with the splitting principle. To finish up, we note that if $M = \mathbf{S}_\lambda(\mathbf{C}^\infty) \otimes \mathcal{F} \otimes \mathbf{A}(\mathcal{Q})$, with $\mathcal{F}$ a coherent sheaf on $Y$, then $\Theta_M = s_\lambda \cdot [\mathcal{F}] \cdot \Theta_{\mathbf{A}(\mathcal{Q})}$, and is thus a polynomial in the $s_n$'s and $S_n$'s by (e), and so $\Theta_{\mathrm{R}\pi_*(M)}$ is a polynomial in the $s_n$'s and $S_n$'s by (d). Therefore condition (c) holds, and so Theorem 2.3 follows.

[2]This definition of generation may differ slightly from other ones.

2.5. **Regularity.** We now give a second easy application of Theorem 2.5. For an $\mathbf{A}(E)$-module $M$, let $t_i(M)$ to be the maximum degree occurring in $\mathrm{Tor}_i^{\mathbf{A}(E)}(M, \mathbf{C})$. Define the **regularity** of $M$ to be the supremum of $t_i(M) - i$ over $i \geq 0$.

**Theorem 2.9.** *A finitely generated $\mathbf{A}(E)$-module has finite regularity.*

*Proof.* We define regularity for a complex of $\mathbf{A}(E)$-modules exactly as for an $\mathbf{A}(E)$-module. We apply Corollary 2.6, taking $\mathcal{P}(M)$ to be "$M$ has finite regularity." Again, (a) and (b) are clear and we must verify (c).

Suppose $M$ is an $\mathbf{A}(\pi^*(E))$-module on $Y$. We define the regularity of $M$ by looking at the Tor's with $\mathcal{O}_Y$. From the base change isomorphism

$$\mathrm{R}\pi_*(M \overset{\mathrm{L}}{\otimes}_{\mathbf{A}(\pi^*(E))} \mathcal{O}_Y) = \mathrm{R}\pi_*(M) \overset{\mathrm{L}}{\otimes}_{\mathbf{A}(E)} \mathbf{C},$$

one deduces that the regularity of $\mathrm{R}\pi_*(M)$ is at most the regularity of $M$ plus the dimension of $\mathbf{Gr}_r(E)$. In particular, if $M$ has finite regularity then so does $\mathrm{R}\pi_*(M)$. Thus to verify (c), it suffices to show that $V \otimes \mathcal{F} \otimes \mathbf{A}(\mathcal{Q})$ has finite regularity as an $\mathbf{A}(\pi^*(E))$-module. But one can explicitly compute the relevant Tor's using a Koszul complex, and so the result follows. $\square$

**Problem 2.10.** *Prove Theorem 2.9 over arbitrary noetherian coefficient rings.*

This is known for $d = 1$ (by [Nag]) but unknown for $d > 1$.

## 3. Depth, Fourier transform, and Poincaré series

In §3.1 we state a result on the asymptotic behavior of depth and projective dimension for $\mathbf{A}(d)$-modules. In §3.2 we introduce the technical tool needed to prove this theorem (the Fourier transform). In §3.3, we explain how to use the Fourier transform to prove the results from §3.1. Finally, in §3.4 we give another application of the Fourier transform, to Poincaré series.

3.1. **Depth.** Fix $d$ and put $A = \mathbf{A}(d)$. We write $A(\mathbf{C}^n)$ for the value of the Schur functor $A$ on $\mathbf{C}^n$; this is just $\mathrm{Sym}(\mathbf{C}^n \otimes \mathbf{C}^d)$, a polynomial ring in finitely many variables. Let $M$ be an $A$-module. Then $M(\mathbf{C}^n)$ is an $A(\mathbf{C}^n)$-module. We write $d_M(n)$ (resp. $\mathrm{pd}_M(n)$) for the depth (resp. projective dimension) of $M(\mathbf{C}^n)$ as an $A(\mathbf{C}^n)$-module. In [SS4], we prove the following result:

**Theorem 3.1.** *Let $M$ be a finitely generated $A$-module.*

(a) *There exist integers $a$ and $b$ such that $d_M(n) = an + b$ for $n \gg 0$.*
(b) *There exist integers $c$ and $d$ such that $\mathrm{pd}_M(n) = cn + d$ for $n \gg 0$.*

The two statements in the theorem are equivalent by the Auslander–Buchsbaum formula. We are more interested in statement (a), but will actually prove statement (b). We define the **depth** of an $A$-module $M$ to be the limiting value of $d_M(n)$ as $n \to \infty$; thus, in the notation of the theorem, $d_M(n) = \infty$ if $a > 0$ and $d_M(n) = b$ if $a = 0$.

**Problem 3.2.** *Formulate and prove a version of Theorem 3.1 over arbitrary noetherian coefficient rings. (Note that it is not even clear how to define $d_M(n)$ in general.)*

**Remark 3.3.** Li–Ramos [LR] have formulated a theory of depth of $\mathbf{A}(1)$-modules in positive characteristic, but it does not immediately connect to the depth as defined in commutative algebra. $\square$

3.2. **The Fourier transform.** We now introduce the tool that will be used to prove Theorem 3.1. For an $A$-module $M$ and an integer $n$, the space

$$\bigoplus_{p \geq 0} \mathrm{Tor}_p^A(M, \mathbf{C})_{n+p}$$

is naturally a comodule over $B = \bigwedge(\mathbf{C}^\infty \otimes \mathbf{C}^d)$. In fact, there is a canonical complex $\mathcal{K}(M)$ of $B$-modules such that $\mathrm{H}^n(\mathcal{K}(M))$ is the above $B$-module. The functor $M \mapsto \mathcal{K}(M)$ defines an equivalence of categories between the derived category of $A$-modules and the derived category of $B$-comodules. This is an instance of Koszul duality.

For a polynomial representation $V = \bigoplus_\lambda V_\lambda \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$, define $V^\vee = \bigoplus_\lambda V_\lambda^* \otimes \mathbf{S}_\lambda(\mathbf{C}^\infty)$, where $(-)^*$ denotes the usual linear dual. One can show that $M \mapsto M^*$ defines an equivalence between the category of $B$-comodules and $B$-modules.[3] Thus the functor $M \mapsto \mathcal{K}(M)^\vee$ is an equivalence between the derived category of $A$-modules and the derived category of $B$-modules.

For a polynomial representation $V$ as above define $V^\dagger = \bigoplus_\lambda V_\lambda \otimes \mathbf{S}_{\lambda^\dagger}(\mathbf{C}^\infty)$, where $\lambda^\dagger$ is the transposed partition. We call $(-)^\dagger$ the **transpose functor**. It is a tensor functor, but not a symmetric tensor functor: indeed, $B^\dagger = A$. One easily shows that $M \mapsto M^\dagger$ is an equivalence between the categories of $B$-modules and $A$-modules.

We define the **Fourier transform** of $M \in \mathrm{D}(A)$, denoted $\mathcal{F}(M)$, to be $\mathcal{K}(M)^{\vee,\dagger}$. This again belongs to $\mathrm{D}(A)$, and $\mathcal{F}$ is an auto-equivalence of $\mathrm{D}(A)$.[4] By definition, we have

$$\mathrm{H}^{-n}(\mathcal{F}(M)) = \bigoplus_{p \geq 0} \mathrm{Tor}_p^A(M, \mathbf{C})_{p+n}^{\vee,\dagger}. \tag{3.4}$$

In [SS4], we prove the following result:

**Theorem 3.5.** *The Fourier transform carries $\mathrm{D}^b_{\mathrm{fg}}(A)$ into itself, and satisfies $\mathcal{F}^2 = \mathrm{id}$.*

This theorem is proved using Corollary 2.6. The fact that $\mathcal{F}$ preserves boundedness is equivalent to the finiteness of regularity, which has already been discussed (Theorem 2.9).

The above theorem is interesting because it says that if $M$ is a finitely generated $A$-module then each linear strand in its projective resolution can itself be naturally endowed with the structure of a finitely generated $A$-module (after minor modification). Thus the patterns one sees in projective resolutions of $A$-modules are the same patterns one sees in $A$-modules. This can be a very useful observation since the internal structure of a module is often easier to understand than its resolution.

**Problem 3.6.** *Prove Theorem 3.5 over an arbitrary noetherian coefficient ring.*

We have done this with R. Nagpal for $d = 1$, but do not know how to do it for $d > 1$.

3.3. **Back to depth.** We now explain how to use Theorem 3.5 to prove Theorem 3.1. Let $M$ be an $A$-module. Then $\mathrm{pd}_M(n) \leq m$ if and only if $\mathrm{Tor}_k^{A(\mathbf{C}^n)}(M(\mathbf{C}^n), \mathbf{C}) = 0$ for $k > m$. Now, Tor commutes with evaluation on $\mathbf{C}^n$, that is, we have

$$\mathrm{Tor}_k^{A(\mathbf{C}^n)}(M(\mathbf{C}^n), \mathbf{C}) = \mathrm{Tor}_k^A(M, \mathbf{C})(\mathbf{C}^n).$$

[3] Actually, some finiteness conditions must be imposed here, but we ignore them.

[4] More canonically, $\mathcal{F}$ is an equivalence between the derived categories of $\mathbf{A}(E)$ and $\mathbf{A}(E^*)$. In the main text we have implicitly identified $\mathbf{C}^d$ with its dual.

Thus we see that $\mathrm{pd}_M(n) \le m$ if and only if every partition in $\mathrm{Tor}_k^A(M, \mathbf{C})$ has $> n$ rows, for all $k > m$. For an $A$-module $M$, let $\gamma_M(n)$ be the maximum size of a partition with at most $n$ columns appearing in $M$. As transpose interchanges rows and columns, (3.4) now gives:

**Proposition 3.7.** *We have* $\mathrm{pd}_M(n) = \max_k \left(\gamma_{\mathrm{H}^{-k}(\mathcal{F}(M))}(n) - k\right)$.

Since $\mathrm{H}^{-k}(\mathcal{F}(M))$ is a finitely generated $A$-module for all $k$ and non-zero for only finitely many values of $k$, to prove Theorem 3.1 it is enough to prove:

**Proposition 3.8.** *Let $M$ be a finitely generated $A$-module. Then there exist integers $e$ and $f$ such that $\gamma_M(n) = en + f$ for $n \gg 0$.*

We prove this using a kind of Hilbert series argument. The proof is not worth including here. However, we do re-emphasize the main point. To prove Theorem 3.1(b), we needed to understand one aspect of the asymptotic behavior of projective resolutions of $A$-modules. The Fourier transform allowed us to convert this to a problem about a certain aspect of the asymptotic behavior of $A$-modules (namely, Proposition 3.8), which was easier to solve.

3.4. **Poincaré series.** We now give one more application of the Fourier transform. We define the **Poincaré series** of an $A$-module $M$ by

$$P_M(t, q) = \sum_{n \ge 0} (-q)^n \mathrm{H}_{\mathrm{Tor}_n^A(M, \mathbf{C})}(t).$$

One can recover the standard Hilbert series from the Poincaré series by setting $q = 1$ and multiplying by $\mathrm{H}_A(t) = e^{dt}$. The Poincaré series is a very subtle and difficult to study invariant because it does not factor through the Grothendieck group. Thus tools like Corollary 2.6, which we have so far relied upon, are of little use in its analysis. However, the Fourier transform saves the day: a simple calculation gives the identity

$$P_M(t, q) = \sum_{k \ge 0} (-q)^{-k} \mathrm{H}_{\mathrm{H}^{-k}(\mathcal{F}(M))}(-qt).$$

If $M$ is finitely generated then this is a finite sum since $\mathcal{F}(M)$ is a bounded complex. Furthermore, since each $\mathrm{H}^{-k}(\mathcal{F}(M))$ is a finitely generated $A$-module, we can appeal to Theorem 2.1 to understand their Hilbert series. We thus find:

**Theorem 3.9.** *If $M$ is a finitely generated $A$-module then $P_M(t, q)$ has the form*

$$\sum_{r=0}^{d} p_r(t, q) e^{-rqt}$$

*where $p_r(t, q) \in \mathbf{Q}[t, q, q^{-1}]$.*

**Problem 3.10.** *Prove Theorem 3.9 over an arbitrary field.*

We have done this with R. Nagpal when $d = 1$ but do not know how to do it for $d > 1$.

**Remark 3.11.** One can use the formal character or enhanced Hilbert series in the definition of Poincaré series to get a stronger invariant. There are versions of Theorem 3.9 for these variants. □

## 4. The structure theory of $\mathbf{A}(d)$-modules

In this section we give an overview of the structure theory of $\mathbf{A}(d)$-modules. The results here are somewhat technical, but, as we have seen above, are very powerful and give easy proofs of theorems of interest. We start in §4.1 by describing the "spectrum" of $\mathbf{A}(d)$, which gives a useful picture for understanding modules. Inspired by this, in §4.2 we describe a way to "cut up" the category $\mathrm{Mod}_A$ into various pieces that should be simpler to understand. In §4.3 and §4.4, we describe the structure of the pieces. Finally, in §4.5 and §4.6, we describe the "section functors," which control how the various pieces of $\mathrm{Mod}_A$ are glued together.

We fix $A = \mathbf{A}(E)$ in this section, where $E$ is a $d$-dimensional vector space.

**Remark 4.1.** Throughout this section, we use geometric constructions to describe various aspects of $A$-modules. These constructions invariably involve infinite dimensional schemes. We ignore various subtleties involved in working with such objects, to avoid being overly technical. In [SS4] we are much more careful, and in fact avoid working with infinite dimensional schemes to avoid these technicalities. However, the pictures sketched here are the intuition behind the more rigorous arguments. □

### 4.1. The spectrum of a tca.

Recall that an ideal of $A$ is simply an ideal of the underlying ring $|A|$ that is $\mathbf{GL}_\infty$ stable. We say that an ideal $\mathfrak{p}$ is **prime** if $|\mathfrak{p}|$ is prime. We define the **spectrum** of $A$, denoted $\mathrm{Spec}(A)$, to be the set of prime ideals with the Zariski topology. (That is, the closed sets are the $V(I)$ with $I$ an ideal of $A$.) The spectrum of $A$ gives us a coarse picture of the category of $A$-modules, so it is good to understand it before moving on to more subtle questions.

The spectrum of $|A|$ is the affine space $\mathrm{Hom}(E, (\mathbf{C}^\infty)^*)$. Given a point $f \colon E \to (\mathbf{C}^\infty)^*$ of this space, the subspace $\ker(f)$ of $E$ is an invariant of the $\mathbf{GL}_\infty$ orbit of $f$. This suggests that the spectrum of $A$ should be related to the Grassmannians on $E$. We now state the precise result. Define the **total Grassmannian** of $E$, denoted $\mathbf{Gr}(E)$, to be the following topological space. As a set, it is the disjoint union of the topological spaces underlying the schemes $\mathbf{Gr}_r(E)$ for $0 \le r \le d$. A subset $Z$ of $\mathbf{Gr}(E)$ is closed if and only if (a) $Z \cap \mathbf{Gr}_r(E)$ is Zariski closed for all $r$; and (b) $Z$ is downwards-closed in the sense that if $U \in Z$ and $V \subset U$ then $V \in Z$. We then have the following theorem:

**Theorem 4.2.** *The space* $\mathrm{Spec}(A)$ *is homeomorphic to* $\mathbf{Gr}(E)$.

This theorem suggests that the category of $A$-modules should be closely connected to the Grassmannians $\mathbf{Gr}_r(E)$. We will see that this is indeed the case.

**Remark 4.3.** In [SS1] we point out several analogies between modules over $\mathbf{A}(1)$ and graded modules over $\mathbf{C}[t]$. When $d = 1$, $\mathbf{Gr}(E)$ consists of two points, one closed, and one whose closure contains the other point. This has the same topology as the spectrum of a DVR, which we might think of as the localization of $\mathbf{C}[t]$ at $(t)$. □

**Problem 4.4.** *Determine the spectrum of the tca* $\mathbf{A}(d)$ *in positive characteristic.*

See §5.4 for the definition of $\mathbf{A}(d)$ in positive characteristic. The above problem seems very tractable and would be useful to solve, but we have not seriously worked on it yet.

**Remark 4.5.** In [SS4] we (together with R. Nagpal) show that the space $\mathbf{Gr}(E)$ has Krull dimension $\binom{d+1}{2}$, and conclude from this that the category of finitely generated $A$-modules has Krull–Gabriel dimension $\binom{d+1}{2}$. These results are not known in positive characteristic. □

4.2. **Cutting up the category of modules.** Let $\mathfrak{a}_r \subset A$ be the $r$th determinantal ideal. This is generated by

$$\bigwedge^{r+1}(\mathbf{C}^\infty) \otimes \bigwedge^{r+1}(E) \subset \mathrm{Sym}^{r+1}(\mathbf{C}^\infty \otimes E) \subset A$$

The set $V(\mathfrak{a}_r) \subset \mathrm{Spec}(A)$ is identified with $\bigcup_{k \le r} \mathbf{Gr}_r(E)$. We let $\mathrm{Mod}_{A,\le r}$ be the category of $A$-modules supported on $V(\mathfrak{a}_r)$. (An $A$-module is supported on $V(\mathfrak{a}_r)$ if every element is annihilated by some power of $\mathfrak{a}_r$.)

Suppose $X$ is a scheme and $Z$ is a closed subscheme, and let $U$ be the complement $X \setminus Z$. The one can describe the category $\mathrm{QCoh}(U)$ of quasi-coherent sheaves on $U$ as the quotient of $\mathrm{QCoh}(X)$ by the Serre subcategory $\mathrm{QCoh}(Z)$.[5] Taking this as our lead, we define $\mathrm{Mod}_{A,>r}$ to be the quotient of $\mathrm{Mod}_A$ by the Serre subcategory $\mathrm{Mod}_{A,\le r}$. The idea is that $\mathrm{Mod}_{A,>r}$ should correspond to modules on $\bigcup_{k>r} \mathbf{Gr}_r(E) \subset \mathrm{Spec}(A)$. We also let $\mathrm{Mod}_{A,r} \subset \mathrm{Mod}_{A,\ge r}$ be the image of $\mathrm{Mod}_{A,\le r}$. This category corresponds to modules on the single Grassmannian $\mathbf{Gr}_r(E)$.

We write $T_r \colon \mathrm{Mod}_A \to \mathrm{Mod}_{A,>r}$ for the localization functor. By general category theory, this has a right adjoint $S_r \colon \mathrm{Mod}_{A,>r} \to \mathrm{Mod}_A$. Intuitively, $T_r$ is like restricting a quasi-coherent sheaf to an open subscheme and $S_r$ is like pushing forward. We write $\Sigma_r$ for the composition $S_r \circ T_r$; this is the **saturation functor** with respect to $\mathfrak{a}_r$. There is one more important functor to introduce: $\Gamma_r \colon \mathrm{Mod}_A \to \mathrm{Mod}_{A,\le r}$ assigns to an $A$-module $M$ the maximal submodule supported on $V(\mathfrak{a}_r)$. We call the right-derived functors of $\Gamma_r$ **local cohomology**, and sometimes denote $\mathrm{R}^i\Gamma_r$ by $\mathrm{H}^i_{\mathfrak{a}_r}$. For an $A$-module $M$ (or complex of $A$-modules), there is a canonical exact triangle

$$\mathrm{R}\Gamma_r(M) \to M \to \mathrm{R}\Sigma_r M \to$$

Using the above functors, we define subcategories of the derived category analogous to the sub/quotients of $\mathrm{Mod}_A$ introduced above. We let $\mathrm{D}(A)_{\le r}$ be the subcategory of $\mathrm{D}(A)$ on objects $M$ satisfying $\mathrm{R}\Sigma_r(M) = 0$. Similarly, we let $\mathrm{D}(A)_{>r}$ be the subcategory on $M$'s satisfying $\mathrm{R}\Gamma_r(M) = 0$. Finally, we let $\mathrm{D}(A)_r$ be the intersection $\mathrm{D}(A)_{\le r} \cap \mathrm{D}(A)_{\ge r}$. It is not difficult to show that the we have a semi-orthogonal decomposition[6]

$$\mathrm{D}(A) = \langle \mathrm{D}(A)_0, \mathrm{D}(A)_1, \ldots, \mathrm{D}(A)_d \rangle.$$

Thus one can roughly think of an object of $\mathrm{D}(A)$ as being built out of $d+1$ pieces. The functor $\mathrm{R}\Gamma_r$ kills all the pieces in $\mathrm{D}(A)_k$ with $k > r$ and leaves the pieces with $k \le r$ alone, while the functor $\mathrm{R}\Sigma_k$ kills all the pieces with $k \le r$ and leaves the pieces with $k > r$ alone. We define $\mathrm{R}\Pi_r$ to be the composition $\mathrm{R}\Gamma_r \circ \mathrm{R}\Sigma_{r-1}$. This is the projection onto the $r$th piece of the semi-orthogonal decomposition.

One of the fundamental results of [SS4] is the following theorem:

**Theorem 4.6.** *The functors $\mathrm{R}\Gamma_r$ and $\mathrm{R}\Sigma_r$ take $\mathrm{D}^b_{\mathrm{fg}}(A)$ into itself. Equivalently, if $M$ is a finitely generated $A$-module then $\mathrm{R}^i\Gamma_r(M)$ is finitely generated for all $i$ and vanishes for $i \gg 0$, and similarly for $\mathrm{R}^i\Sigma_r(M)$.*

This theorem implies that $\mathrm{D}^b_{\mathrm{fg}}(A)$ admits a semi-orthogonal decomposition into pieces $\mathrm{D}^b_{\mathrm{fg}}(A)_r$. This is an extremely useful structural result about $A$-modules.

[5] This assumes some very mild finiteness properties of $X$.

[6] The only slightly non-trivial fact one needs to prove this is that injective objects of $\mathrm{Mod}_{A,\le r}$ remain injective in $\mathrm{Mod}_A$.

**Problem 4.7.** *Prove a version of Theorem 4.6 over arbitrary noetherian coefficient rings.*

This has been carried out for $d = 1$ in [LR], but is unknown for $d > 1$.

4.3. **The category** $\mathrm{Mod}_A^{\mathrm{gen}}$**.** We now want to better understand the structure of the categories $\mathrm{Mod}_{A,r}$ introduced above. The case $r = 0$ is easy: the finitely generated objects of $\mathrm{Mod}_{A,0}$ are just the finite length $A$-modules. The simple objects in this category are just the irreducible $\mathbf{GL}_\infty$ representations $\mathbf{S}_\lambda(\mathbf{C}^\infty)$ with the positive degree elements of $A$ acting by 0. In this subsection, we discuss the next easiest case, namely $\mathrm{Mod}_{A,d}$, where $d = \dim(E)$. We denote this category by $\mathrm{Mod}_A^{\mathrm{gen}}$ to emphasize that it is the top piece of the filtration (we think of $\mathrm{Mod}_A^{\mathrm{gen}}$ as "generic $A$-modules").

An $A$-module can be thought of as a $\mathbf{GL}_\infty$-equivariant sheaf on $\mathrm{Hom}(\mathbf{C}^d, (\mathbf{C}^\infty)^*)$ (ignoring various technical issues). The complement of $V(\mathfrak{a}_{d-1})$ is the locus where this map is injective. The group $\mathbf{GL}_\infty$ acts transitively on this locus (again, ignoring technicalities), and the stabilizer of a point is the subgroup $G \subset \mathbf{GL}_\infty$ consisting of matrices of the form

$$\begin{pmatrix} 1 & X \\ 0 & Y \end{pmatrix}$$

where 1 is the $d \times d$ identity matrix. We thus see that $\mathrm{Mod}_A^{\mathrm{gen}}$ is equivalent to a category of representations of $G$. Now, $G$ decomposes as a semi-direct product. The normal subgroup (where $Y = 1$) is isomorphic to the additive group $\mathbf{C}^d \otimes \mathbf{C}^\infty$, and the other factor (where $X = 0$) is isomorphic to $\mathbf{GL}_\infty$. (Note that these $\infty$'s are actually "smaller" than the original ones!) We thus see that giving a representation of $G$ is the same as giving a $\mathbf{GL}_\infty$-equivariant representation of $\mathbf{C}^d \otimes \mathbf{C}^\infty$. Furthermore, giving a representation of the abelian Lie algebra $\mathbf{C}^d \otimes \mathbf{C}^\infty$ is the same as giving a module over its universal enveloping algebra, which is just $\mathrm{Sym}(\mathbf{C}^d \otimes \mathbf{C}^\infty)$. Thus objects of $\mathrm{Mod}_A^{\mathrm{gen}}$ are looking a lot like $\mathbf{A}(d)$-modules! In fact:

**Theorem 4.8.** *The category* $\mathrm{Mod}_A^{\mathrm{gen}}$ *is equivalent to* $\mathrm{Mod}_{A,0}$.

The basic reason one gets $\mathrm{Mod}_{A,0}$ here is that, if one starts with a finitely generated $A$-module, then the $G$-representation one gets has finite length, and thus, when converted to an $A$-module, is also finite length. Since $\mathrm{Mod}_{A,0}$ has such a simple structure, this gives a very complete picture of $\mathrm{Mod}_A^{\mathrm{gen}}$. For instance, it shows:

**Corollary 4.9.** *The Grothendieck group of* $\mathrm{Mod}_A^{\mathrm{gen,fg}}$ *is* $\Lambda$*, the ring of symmetric functions.*

**Problem 4.10.** *Describe* $\mathrm{Mod}_A^{\mathrm{gen}}$ *over a field of positive characteristic. For instance, compute its Grothendieck group.*

Theorem 4.8 is false in positive characteristic. When $d = 1$, we have some understanding of the generic category in positive characteristic (e.g., we know its Grothendieck group), but not a complete picture. For $d > 1$, there are no results in positive characteristic.

4.4. **The category** $\mathrm{Mod}_{A,r}$**.** By definition, every object of $\mathrm{Mod}_{A,r}$ is represented by an $A$-module supported on $V(\mathfrak{a}_r)$. It will be easier to first study the subcategory of $\mathrm{Mod}_{A,r}$ represented by modules annihilated by $\mathfrak{a}_r$. We denote this category by $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$. We note that every finitely generated object of $\mathrm{Mod}_{A,r}$ admits a finite length filtration where the graded pieces belong to $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$.

To understand $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$, we again return to the geometric picture. Intuitively, objects of this category correspond to $\mathbf{GL}_\infty$-equivariant sheaves on the rank $r$ locus in $\mathrm{Hom}(\mathbf{C}^d, (\mathbf{C}^\infty)^*)$. Let $Y = \mathbf{Gr}_r(E)$, let $\mathcal{Q}$ be tautological rank $r$ quotient on $Y$, and let $\pi \colon Y \to \mathrm{Spec}(\mathbf{C})$ be the structure map. There is an isomorphism of schemes between the locus of injections in $\mathrm{Hom}(\mathcal{Q}, (\mathbf{C}^\infty)^*)$ and the rank $r$ locus in $\mathrm{Hom}(\mathbf{C}^d, (\mathbf{C}^\infty)^*)$. A $\mathbf{GL}_\infty$-equivariant sheaf on the locus of injections in $\mathrm{Hom}(\mathcal{Q}, (\mathbf{C}^\infty)^*)$ can be thought of as an object of the category $\mathrm{Mod}_B^{\mathrm{gen}}$, where $B = \mathbf{A}(\mathcal{Q})$, by a mild generalization of the previous section. We thus see that $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$ should be equivalent to $\mathrm{Mod}_B^{\mathrm{gen}}$. In fact, this is true, and it's easy to directly write down an equivalence. There is a natural surjection $\pi^*(A) \to B$, induced by the surjection $\pi^*(E) \to \mathcal{Q}$, and so if $M$ is an $A$-module then $\pi^*(M) \otimes_{\pi^*(A)} B$ is a $B$-module.

**Theorem 4.11.** *The above functor induces an equivalence* $\mathrm{Mod}_{A,r}[\mathfrak{a}_r] \cong \mathrm{Mod}_B^{\mathrm{gen}}$.

Theorem 4.8 remains true in relative situations, and so $\mathrm{Mod}_B^{\mathrm{gen}}$ is equivalent to $\mathrm{Mod}_{B,0}$. This category is easy to understand (every finitely generated object has a finite length filtration where the graded pieces have trivial $B$-action), and so gives a very clear picture of $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$. For example, we have the following corollary:

**Corollary 4.12.** *The Grothendieck group of* $\mathrm{Mod}_{A,r}^{\mathrm{fg}}$ *is canonically isomorphic to* $\Lambda \otimes \mathrm{K}(\mathbf{Gr}_r(E))$, *where* $\Lambda$ *is the ring of symmetric functions and* $\mathrm{K}(\mathbf{Gr}_r(E))$ *is the Grothendieck group of* $\mathbf{Gr}_r(E)$, *which is a free* $\mathbf{Z}$*-module of rank* $\binom{d}{r}$.

We have thus described the Grothendieck groups of the graded pieces of $\mathrm{Mod}_A$ under some filtration. In general, this is not enough to describe the Grothendieck group of $\mathrm{Mod}_A^{\mathrm{fg}}$, since the Grothendieck group is only a right-exact functor of the category. However, in our case, Theorem 4.6 implies that $\mathrm{K}(\mathrm{Mod}_A^{\mathrm{gen}})$ is the direct sum of the $\mathrm{K}(\mathrm{Mod}_{A,r}^{\mathrm{fg}})$. We thus find:

**Corollary 4.13.** *The Grothendieck group of* $\mathrm{Mod}_A^{\mathrm{fg}}$ *is free as a* $\Lambda$*-module of rank* $2^d$.

**Problem 4.14.** *Describe* $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$ *in positive characteristic.*

**Problem 4.15.** *Compute the Grothendieck group of* $\mathrm{Mod}_A^{\mathrm{fg}}$ *in positive characteristic.*

We know of no progress on these problems for $d > 1$.

4.5. **The section functor on** $\mathrm{Mod}_A^{\mathrm{gen}}$**.** Recall that the section functor $S_{d-1} \colon \mathrm{Mod}_A^{\mathrm{gen}} \to \mathrm{Mod}_A$ is the right adjoint to the localization functor $T_{d-1} \colon \mathrm{Mod}_A \to \mathrm{Mod}_A^{\mathrm{gen}}$. In this section, we put $S = S_{d-1}$ and $T = T_{d-1}$ for simplicity. Let $H = \mathrm{Hom}(\mathbf{C}^d, (\mathbf{C}^\infty)^*)$, thought of as an infinite dimensional affine space, and let $U$ be the open subscheme where the map is injective. Write $j \colon U \to H$ for the open immersion. We have explained that $A$-modules are $\mathbf{GL}_\infty$-equivariant sheaves on $H$, while objects of $\mathrm{Mod}_A^{\mathrm{gen}}$ are $\mathbf{GL}_\infty$-equivariant sheaves on $U$. It is thus natural to guess that the section functor should correspond to $j_*$, and this is indeed the case. In fact, this is true even at the derived level:

**Theorem 4.16.** *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Mod}_A^{\mathrm{gen}} \arrow[r, "\mathrm{R}^i S"] \arrow[d, equal] & \mathrm{Mod}_A \arrow[d, equal] \\ \mathrm{QCoh}_{\mathbf{GL}_\infty}(U) \arrow[r, "\mathrm{R}^i j_*"] & \mathrm{QCoh}_{\mathbf{GL}_\infty}(H) \end{tikzcd}$$

**Remark 4.17.** In [SS4], there is a version of this theorem stated at finite level (i.e., after evaluating on $\mathbf{C}^n$), and it is not $\mathrm{R}^i j_*$ that appears but $(\mathrm{R}^i j_*)^{\mathrm{pol}}$, the "polynomial piece" of $\mathrm{R}^i j_*$. It turns out that at infinite level it is unnecessary to take the polynomial piece. $\square$

The above theorem is useful because one can actually compute $\mathrm{R}j_*$. We have a natural map $\pi\colon U \to \mathbf{Gr}_d((\mathbf{C}^\infty)^*)$ defined by taking an injection $f\colon \mathbf{C}^d \to (\mathbf{C}^\infty)^*$ to its image. Let $\mathcal{R}$ be the tautological rank $d$ subbundle on $\mathbf{Gr}_d((\mathbf{C}^\infty)^*)$. Then $\pi$ identifies $U$ with the scheme $\underline{\mathrm{Isom}}(\mathbf{C}^d, \mathcal{R})$ over $\mathbf{Gr}_d((\mathbf{C}^\infty)^*)$. In particular, the map $\pi$ is affine. We can thus compute $\mathrm{R}^i j_*$ by first applying $\pi_*$ and then computing $\mathrm{H}^i(\mathbf{Gr}_d((\mathbf{C}^\infty)^*), -)$. One can use the Borel–Weil–Bott theorem to compute these latter cohomology groups. By actually carrying out the details of these computations, we find:

**Theorem 4.18.** *We have the following:*

(a) *If $M \in \mathrm{Mod}_A^{\mathrm{gen}}$ is finitely generated then $(\mathrm{R}^i S)(M)$ is a finitely generated $A$-module for all $i$, and vanishes for $i \gg 0$.*
(b) *If $M = T(V \otimes A)$ for a polynomial representation $V$ of $\mathbf{GL}_\infty$ then $S(M) = V \otimes A$ and $(\mathrm{R}^i S)(M) = 0$ for $i > 0$.*

**Corollary 4.19.** *Projective $A$-modules are injective.*

*Proof.* This follows from the fact that $T(V \otimes A)$ is injective in $\mathrm{Mod}_A^{\mathrm{gen}}$, which drops out of our analysis of this category, and the completely formal fact that $S$ takes injective to injectives. $\square$

**Problem 4.20.** *Generalize Theorem 4.18 to arbitrary noetherian coefficients rings.*

This has been done for $d = 1$ in [LR], but is unknown for $d > 1$.

4.6. **The section functor on $\mathrm{Mod}_{A,r}$.** Let $Y = \mathbf{Gr}_r(E)$, let $\mathcal{Q}$ be the tautological quotient bundle on $Y$, and let $\pi\colon Y \to \mathrm{Spec}(\mathbf{C})$ be the structure map. Recall that $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$ is equivalent to $\mathrm{Mod}_B^{\mathrm{gen}}$; write $\Psi\colon \mathrm{Mod}_{A,r}[\mathfrak{a}_r] \to \mathrm{Mod}_B^{\mathrm{gen}}$ for this equivalence. Let $S'\colon \mathrm{Mod}_B^{\mathrm{gen}} \to \mathrm{Mod}_B$ be the section functor. The following result is not difficult to guess, but the proof has some technical points:

**Theorem 4.21.** *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Mod}_A & \mathrm{Mod}_B \arrow[l, "\pi_*"'] \\
\mathrm{Mod}_{A,r}[\mathfrak{a}_r] \arrow[u, "S_{r-1}"] & \mathrm{Mod}_B^{\mathrm{gen}} \arrow[l, "\Psi^{-1}"'] \arrow[u, "S'"']
\end{tikzcd}$$

*Moreover, it continues to commute at the derived level: that is, if $M$ is an object of $\mathrm{Mod}_{A,r}[\mathfrak{a}_r]$ and $N = \Psi(M)$ is the corresponding object of $\mathrm{Mod}_B^{\mathrm{gen}}$, then there is a canonical isomorphism $\mathrm{R}S_{r-1}(M) = \mathrm{R}\pi_*(\mathrm{R}S'(N))$.*

**Corollary 4.22.** *If $M \in \mathrm{Mod}_{A,r}$ is finitely generated then $\mathrm{R}^i S_{r-1}(M)$ is finitely generated for all $i$ and vanishes for $i \gg 0$.*

*Proof.* By dévissage, one can reduce to the case $M \in \mathrm{Mod}_{A,r}[\mathfrak{a}_r]$. Let $N = \Psi(M)$. Then $\mathrm{R}S'(N)$ is a bounded complex of finitely generated $B$-modules by a relative version of Theorem 4.18, and so $\mathrm{R}\pi_*(\mathrm{R}S'(N))$ is a bounded complex of finitely generated $A$-modules. $\square$

The above corollary implies the following: if $M$ is a finitely generated $A$-module supported on $V(\mathfrak{a}_r)$, then $\mathrm{R}\Sigma_{r-1}(M)$ is a bounded complex of finitely generated $A$-modules. (Recall that $\Sigma_{r-1} = S_{r-1} \circ T_{r-1}$, and $T_{r-1}$ is an exact functor.) This is a weaker than Theorem 4.6, since that theorem has no support condition on $M$. However, it turns out that there is a completely formal inductive argument that allows one to deduce Theorem 4.6 from Corollary 4.22. Once these results are in hand, it is not difficult to prove Theorem 2.5.

## 5. Additional results and comments

5.1. **Depth and local cohomology.** In §4.2, we defined the local cohomology $\mathrm{H}^i_{\mathfrak{a}_r}(M)$ of an $\mathbf{A}(d)$-module along the determinantal ideal $\mathfrak{a}_r$. Here we will be most interested in the case $r = 0$, where we write $\mathfrak{m}$ in place of $\mathfrak{a}_r$ since it is the maximal ideal of $\mathbf{A}(d)$. In §3.1, we defined the depth of an $A$-module. We prove that the usual relation between these two invariants holds:

**Theorem 5.1.** *Let $M$ be a finitely generated $A$-module of depth $d$. Then $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < d$ and, if $d$ is finite, $\mathrm{H}^d_{\mathfrak{m}}(M) \neq 0$.*

There is a corresponding theorem for $\mathrm{H}^i_{\mathfrak{a}_r}$ for any $r$, where depth is replaced by $\mathfrak{a}_r$-depth.

**Problem 5.2.** *Theorem 5.1 shows that $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i$ if and only if $a > 0$ in Theorem 3.1. Is there a way to detect the actual value of $a$ using something like local cohomology?*

5.2. **Bounds on regularity.** Recall that for an $\mathbf{A}(d)$-module $M$, we let $t_i(M)$ be the maximum degree occurring in $\mathrm{Tor}_i^{\mathbf{A}(d)}(M, \mathbf{C})$, and define the regularity of $M$ as the supremum of $t_i(M) - i$ over $i \geq 0$. We have shown (Theorem 2.9) that the regularity of a finitely generated module is always finite. In fact, we have the following result:

**Theorem 5.3.** *Let $M$ be a finitely generated $\mathbf{A}(d)$-module. Then the regularity of $M$ can be bounded as a function of $t_0(M), \dots, t_n(M)$, where $n = \lceil \frac{1}{4}d^2 \rceil + d + 1$.*

This theorem, which is one of the most difficult in [SS4], is inspired by the main theorem of [CE], which proves the result for $d = 1$ (and has $n = 1$).

**Problem 5.4.** *Generalize Theorem 5.3 to arbitrary noetherian coefficient rings.*

This is known for $d = 1$ by [CE], but is not known in general.

**Problem 5.5.** *Determine the optimal bound for regularity in terms of the $t$'s.*

For $d = 1$, the bound in [CE] is optimal, but for $d > 1$ the optimal bounds are unknown, even in characteristic 0. In fact, it is not even known what the optimal value of $n$ in Theorem 5.3 is when $d > 1$.

5.3. **The duality theorem.** Recall that the category $\mathrm{D}^b_{\mathrm{fg}}(A)$ admits a semi-orthogonal decomposition $\langle \mathrm{D}^b_{\mathrm{fg}}(A)_0, \dots, \mathrm{D}^b_{\mathrm{fg}}(A)_d \rangle$ and that $\mathrm{R}\Pi_r$ is the functor that projects onto the $r$th piece of this semi-orthogonal decomposition.

**Theorem 5.6** (Duality theorem)**.** *Let $M \in \mathrm{D}^b_{\mathrm{fg}}(A)$. Then there is a canonical isomorphism*

$$\mathcal{F}(\Pi_r(M)) = \Pi_{d-r}(\mathcal{F}(M)),$$

*where $\mathcal{F}$ is the Fourier transform (§3.2).*

In other words, the Fourier transform "reverses" the semi-orthogonal decomposition. There are various other manifestations of this duality. We mention one more:

**Theorem 5.7.** *Let $M$ be a finitely generated $A$-module, and write $\mathrm{H}_M(t) = \sum_{r=0}^d p_r(t) e^{rt}$ with $p_r(t)$ a polynomial. Then $\mathrm{H}_{\mathcal{F}(M)}(t) = \sum_{r=0}^d p_{d-r}(-t) e^{rt}$.*

5.4. **The definition of tca.** We have not given the actual definition of tca yet, so we include it here. A **twisted commutative algebra** (tca) over a commutative ring $\mathbf{k}$ is a unital associative graded $\mathbf{k}$-algebra $A = \bigoplus_{n\geq 0} A_n$ equipped with a $\mathbf{k}$-linear action of the symmetric group $S_n$ on $A_n$ such that

(a) The multiplication map $A_n \times A_m \to A_{n+m}$ is $S_n \times S_m$ equivariant, and
(b) Given $x \in A_n$ and $y \in A_m$ we have $xy = (yx)^\tau$, where $\tau = \tau_{m,n} \in S_{n+m}$ is defined by

$$\tau(i) = \begin{cases} i+n & \text{if } 1 \leq i \leq m, \\ i-m & \text{if } m+1 \leq i \leq n+m. \end{cases}$$

This is the "twisted commutativity" condition.

The twisted commutative algebra freely generated by $d$ elements of degree 1 is identified with the tensor algebra on $\mathbf{k}^d$, that is, $A_n = (\mathbf{k}^d)^{\otimes n}$ with the obvious $S_n$ action and multiplication. The equivalence between modules over this tca and representations of $\mathbf{FI}_d$ over $\mathbf{k}$ is explained in [SS3, Proposition 7.2.5]. If $\mathbf{k}$ has characteristic 0 the Schur–Weyl gives an equivalence between symmetric group representations and polynomial representations of $\mathbf{GL}_\infty$, and under this equivalence the tensor algebra tca on $\mathbf{C}^d$ corresponds to the algebra $\mathbf{A}(d)$ we have been using in this paper.

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