DERIVATIVE OF CONVOLUTION

Consider the initial-value problem

$$ay'' + by + cy = g(t), \qquad y(0) = b_0, \qquad y'(0) = b_1,$$

with g continuous (and of exponential order at ∞). The unique solution can be written as

$$y(t) = \mathcal{L}^{-1}\left\{\frac{(as+b)b_0 + ab_1}{as^2 + bs + c}\right\} + (h * g)(t),$$

where $h(t) = \mathcal{L}^{-1}\{1/(as^2 + bs + c)\}$ (cf. pg. 350). In particular, we have that $h * g$ is the unique solution of the initial-value problem

$$ay'' + by + cy = g(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

(cf. the function v in #2 Homework 5). It follows immediately from the definition of convolution that $(h * g)(0) = 0$. As for the second condition, we have

$$\begin{aligned}(h * g)'(t) &= \frac{d}{dt}\left[\int_0^t h(t-\xi)g(\xi)d\xi\right] \\ &= \left(h(t-\xi)g(\xi)\right)_{|\xi=t} + \int_0^t h'(t-\xi)g(\xi)d\xi \\ &= h(0)g(t) + (h' * g)(t),\end{aligned}$$

hence $(h*g)'(0) = h(0)g(0) + \int_0^0 h'(t-\xi)g(\xi)d\xi = 0$, since h is the unique solution of the initial-value problem

$$ay'' + by + cy = 0, \qquad y(0) = 0, \qquad y'(0) = 1/a.$$