MATH 286 – WINTER 2008
HOMEWORK SET 6

What you need to know:

- Power series: ratio test, radius of convergence
- $\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}$ for $|q| < 1$
- Taylor series expansions
- Power series solutions near an ordinary point
- Sections 5.1, 5.2, 5.3

What you shouldn't forget:

- Existence and Uniqueness Theorems
- Linear, separable, exact, and constant-coefficient ODEs
- Reduction of order
- How to solve a non-homogeneous ODE

Ex # 1. Find the Taylor series expansion of

$$f(x) = \ln(1 + x^2)$$

about $x_0 = 0$. (*Hint: Multiply the series for* $\frac{1}{1+x^2}$ *by* $2x$ *and integrate ....*)

Ex # 2. Consider the two series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} 2^{-n} x^{n-1}.$$

Compute $f + g$ if possible. What is the radius of convergence of $f + g$?

Ex # 3. Let f and g be two analytic function at x_0. Are the following statements always true or sometimes false? Explain.

i) $3f(x) + g(x)$ is analytic at x_0.
ii) $\frac{f(x)}{g(x)}$ is analytic at x_0.
iii) $f'(x)$ is analytic at x_0.
iv) $[f(x)]^3 - \int_{x_0}^{x} g(t)dt$ is analytic at x_0.

Ex # 4. ($\star$) Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that $f^{(j)}(0) = 0$ for all $j \geq 0$. Hence the Taylor series of $f(x)$ is $0 + 0 + \ldots$, which converges for all x, but is equal to $f(x)$ only when $x = 0$. (*This is an example of an infinitely differentiable function which is not analytic! because its Taylor series does not converge to the original function.*)

Ex # 5. Find a power series expansion about $x = 0$ for a general solution of

$$(x^2 + 1)y'' - xy' + y = 0.$$

Include the general formula for the coefficients.

Ex # 6. ($\star$)

i) Find a power series solution of

$$y'' - 2xy' - 2y = 0$$

and write it as $y(x) = c_1 e^{x^2} + c_2 y_2(x)$.
ii) In particular, check that $y_1(x) = e^{x^2}$ is a solution.
iii) Use the method of reduction of order to find an explicit form for y_2.
iv) Deduce that

$$\int_0^x e^{-\eta^2} d\eta = e^{-x^2} \sum_{n=0}^{\infty} \frac{n!}{2(2n+1)!} (2x)^{2n+1}.$$

Ex # 7.

i) Determine the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}.$$

ii) Compute the sum of the series (*Hint: Compute the series of the derivatives and then integrate ....*)

Ex # 8. Compute the first four non-zero terms in the power series expansion of the solution of the initial-value problem

$$y'' + ty' + e^t y = 0, \qquad y(0) = 0, \qquad y'(0) = -1.$$